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SECOND NOTE ON JERABEK'S HYPERBOLA

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Abstract. In this article we study the concurrence on a point of the Jerabek hyperbola, of a triangle ABC, of three lines defined by a point P on the circumcircle of the triangle. These lines are the Steiner line of P, the trilinear polar of P and the line whose orthopole is a point D on the Euler circle, such that the line DP passes through the orthocenter.

1 The Jerabek hyperbola

This article complements a recently published article [4], bringing in connection the results proved there with the result published in [1], describing the Jerabek hyperbola as the geometric locus of the intersection point of the Steiner line and a line passing through the circumcenter of the triangle of reference. In fact, in [4] it was shown that the Jerabek's rectangular hyperbola of the triangle ABC is generated by the intersections $X = (s_P, t_P)$ of the Steiner line s_P and the trilinear polar t_P of a point P moving on the circumcircle of ABC

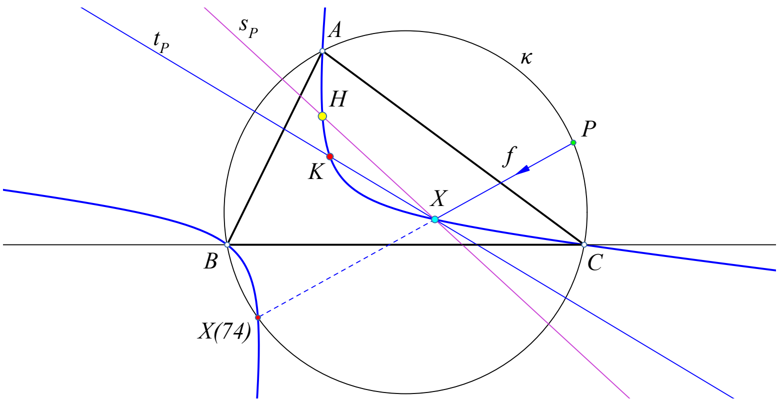

Figure 1: Jerabek's hyperbola as a geometric locus

(See Figure 1). In addition the map $X = f(P)$ was seen to be a projectivity, mapping the circumcircle κ onto the Jerabek hyperbola, in such a way, that the lines PX pass always through the triangle center $X(74)$.

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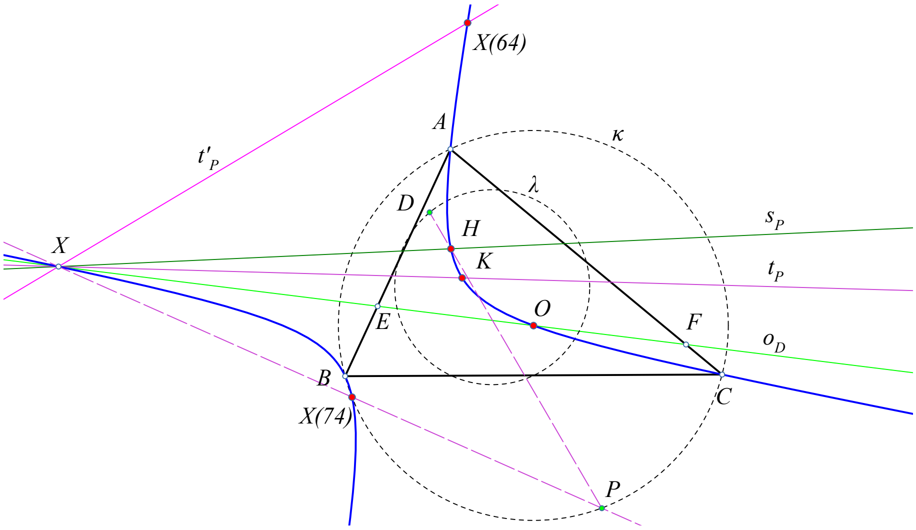

Figure 2: Jerabek's hyperbola as a geometric locus II

On the other side, in [1] it was shown that the Jerabek hyperbola can be described as the geometric locus of intersection points $X' = (s_P, o_D)$, where o_D is a line, whose orthopole is the point D on the Euler circle, such that DP passes through the orthocenter H of the triangle (See Figure 2). In view of the aforementioned results it is obvious, that the two points concur, $X = X'$, and the three lines $\{s_P, t_P, o_D\}$ meet at a point X of Jerabek's hyperbola.

Next section deals with some details concerning orthopoles. The motivation for this came from the reference in [1] saying that "*EF* orthopolar of *P* w.r to *ABC*", which is misleading. Actually it is D the orthopole of $EF = o_D$ and next section supplies a synthetic proof of this, which could be known, but I have not found a reference for it. In the last section we show that the harmonic conjugate t'_P of t_P with respect to $\{s_P, o_D\}$ passes through the triangle center $X(64)$, lying also on the hyperbola.

2 Orthopoles in short

Given a line ε, draw orthogonals $\{AA', BB', CC'\}$ to ε from the vertices of the triangle ABC, and subsequently draw orthogonals $\{A'A'', B'B'', C'C''\}$ to the opposite sides (See Figure

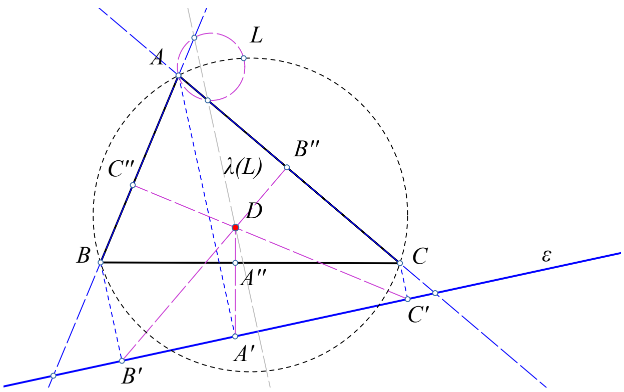

Figure 3: Orthopole D of the line ε w.r. to triangle ABC

3). It is proved that the last three lines concur at a point D, called the *orthopole* of ε

w.r. to *ABC* ([3, p.12]). It is also proved that D is contained in the (Wallace-Simson) WS-line $\lambda(L)$, which is orthogonal to ε. Thus, if ε moves parallel to itself, its orthopole D moves on the fixed line λ. When ε intersects the circumcircle of *ABC*, it is proved that the orthopole is the common point of the three WS-lines $\{\lambda(L), \mu(M), \nu(N)\}$, where λ is as before and $\{\mu, \nu\}$ are the WS-lines of the intersection points $\{M, N\}$ of the circumcircle

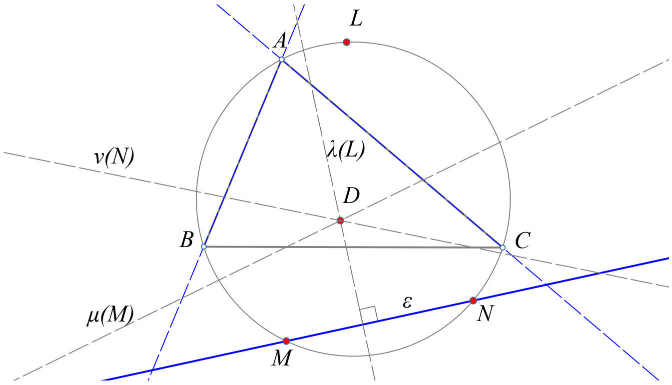

Figure 4: Orthopole D of ε as intersection of $\{\lambda(L), \mu(M), \nu(N)\}$

with ε (See Figure 4). Since the WS-lines of diametral points of the circumcircle intersect orthogonally on points of the Euler circle, we conclude that the orthopoles of diameters are on the Euler circle of the triangle. The two next lemmata supply the details of the constructions performed in [1].

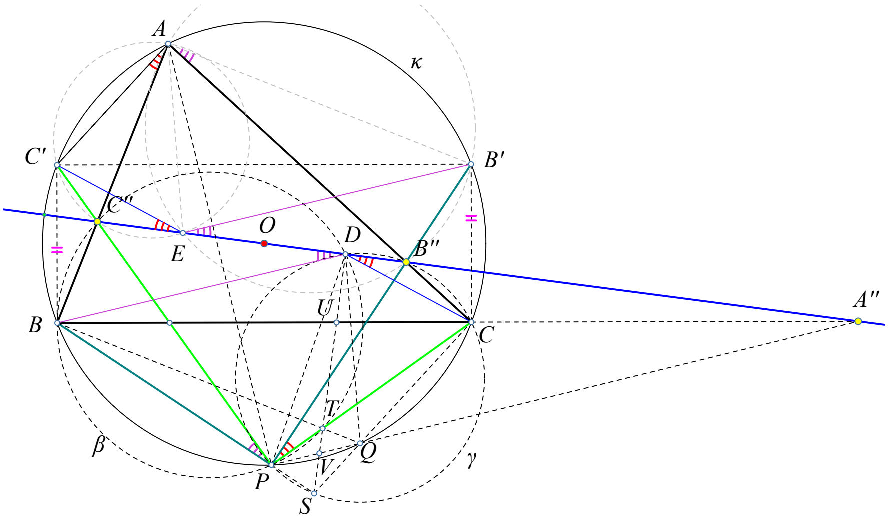

Figure 5: Diameter related to point P on the circumcircle

Lemma 1. *From a point P of the circumcircle $\kappa(O)$ of triangle ABC draw the orthogonal lines $\{PB', PC'\}$ to $\{PB, PC\}$, intersecting the sides $\{AC, AB\}$ at points $\{B'', C''\}$. Then, the line $B''C''$ passes through the center O of κ and intersects BC at A'' forming a right angle $\widehat{APA''}$.*

Proof. To show this, notice first that the pairs $\{(B, B'), (C, C')\}$ consisting of diametral points of the circumcircle, define a rectangle (See Figure 5). Then, it is easy to see that circles $\{\beta = (PBC''), \gamma = (PCB'')\}$ intersect at a point D on $B''C''$ and similarly the two circles $\{(AC'B), (ACB')\}$ intersect also at a point E on $B''C''$ and the triangles $\{DBC, EC'B'\}$

have parallel sides and are congruent. From this follows that line ED or $B''C''$ passes through O.

To show the other claim about the right angle, consider the orthogonal ε to $B''C''$ at D, which is a bisector of the angle $\widehat{BDC}$, hence intersects BC at a point U, which is harmonic conjugate to A'' w.r. to $\{B,C\}$. This implies that PA'' intersects κ at a point Q, such that its intersection point V with ε is also harmonic conjgugate to A'' w.r. to $\{P,Q\}$. Consider now the points $\{T=(BQ,PC), S=(PB,CQ)\}$. In the complete quadrilateral defined by the quadrangle $TPSQ$ line TS is the polar of A'' w.r. to the lines $\{SB,SC\}$, hence TS coincides with $UV=\varepsilon$. Point T is on circle β. This, because $\widehat{C''DT}$ and $\widehat{C''PT}$ are right angles, hence $C''DTP$ is cyclic. Also S is on γ, since $\{\widehat{PCQ}=\widehat{PBQ}=\widehat{PDT}\}$, hence $PDCS$ is cyclic. The angle $\widehat{QCA}$ is right since it is opposite to $\{\widehat{QBA}=\widehat{TBC''}=\widehat{C''DT}\}$, which is right. This completes the proof, since $\widehat{APQ}$, being opposite to $\widehat{ACQ}$ is also right. □

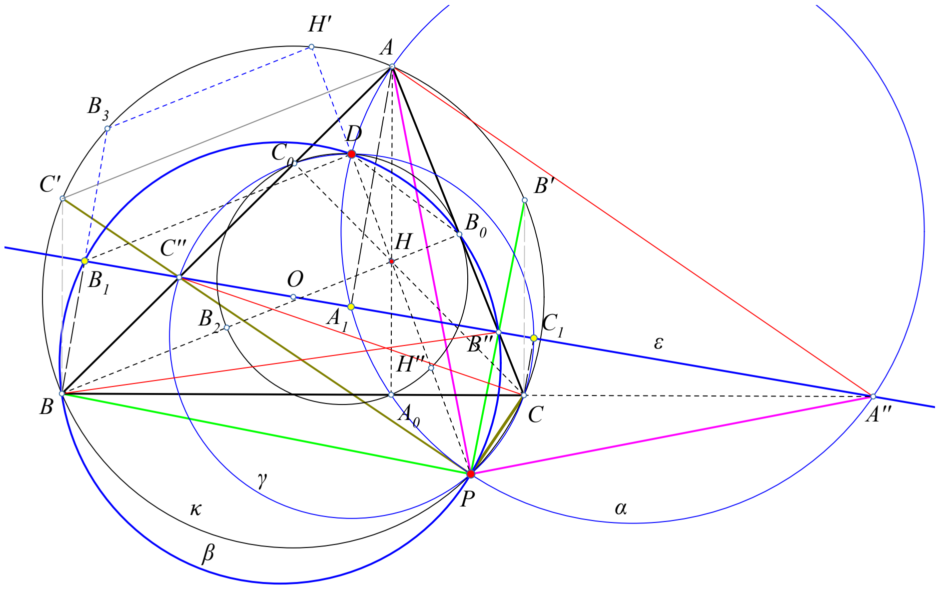

Figure 6: The orthopole D of the diameter ε

Lemma 2. *Let $\varepsilon = B''C''$ be the diameter constructed from a point P as in the previous lemma and $\{A_1, B_1, C_1\}$ be the projections of the vertices of the triangle on ε. Then the following are valid properties.*

1. *The circles $\{\alpha=(PAA_1), \beta=(PBB_1), \gamma=(PCC_1)\}$ define an intersecting pencil.*

2. *The other than P intersection point of the pencil lies on line PH, where H is the orthocenter of the triangle.*

Proof. For *nr-1* consider such a circle, β say. Notice then that B_1BPB'' is cyclic with right angles at $\{B_1,P\}$, hence BB'' is a diameter of the circle β. Similarly CC'' is a diameter of γ. By lemma 1, AA'' is also a diameter of α. Since the three diameters $\{AA'', BB'', CC''\}$ are diagonals of the complete quadrilateral defined by the quadrangle $BCB''C''$, the claim follows from the well known property that the circles with diameters these diagonals define a pencil of circles, which is of intersecting type, since P is on all three circles.

Nr-2 follows by observing that each one of the circles $\{\alpha,\beta,\gamma\}$ contains correspondingly the altitude $\{AA_0, BB_0, CC_0\}$ of the triangle as a chord. Hence, the orthocenter H of the triangle satisfying $HA \cdot HA_0 = HB \cdot HB_0 = HC \cdot HC_0$ has the same power w.r. to all three circles, consequently is on the radical axis of the pencil, thereby proving the claim. $\square$

Theorem 1. *With the notation and the conventions adopted so far, the other common point D of the three circles $\{\alpha, \beta, \gamma\}$ is on the Euler circle of ABC and coincides with the orthopole of line ε.*

Proof. To prove the first claim, consider the product $HD \cdot HP$ for the circle β. Let $\{H'', B_2\}$ be the intersection points of $\{HD, HB\}$ with the Euler circle (See Figure 6). Since H is the homothety center of the homothety with ratio 1/2, transforming the circumcircle κ to the Euler circle, this product is equal to

$$HD \cdot HP = 2HD \cdot HH'' = 2HB_0 \cdot HB_2.$$

Since $\{B_0, B_2, H''\}$ are on the Euler circle, this equation expresses the power of H w.r. to the Euler circle and implies that D is on that circle.

To show the second claim it suffices to show that the lines $\{B_1D, C_1D, A_1D\}$ are respectively orthogonal to the sides $\{AC, BA, BC\}$. We show this for B_1D, the other cases being similar. For this, extend BB_1 to its double to point B_3, which is on the circumcircle $\kappa = (ABC)$, since the medial line of BB_3 is ε containing the center O of κ. Let also H' be the intersection of PD with κ. Both quadrangles $\{BB_3H'P, BB_1DP\}$ are cyclic, hence their respective angles at $\{H', D\}$, being opposite to $\widehat{B_1BP}$ are equal, consequently lines $\{B_3H', B_1D\}$ are parallel. Since D, being on the Euler circle, is the middle of HH' line B_1D is also parallel to BH, which is orthogonal to AC. $\square$

3 X(64)

This triangle center, defined in Kimberling's list [2] as the *"isogonal conjugate of $X(20)$"* is on Jerabek's hyperbola, since this is the isogonal conjugate of the Euler line and $X(20)$, called also *"de Longchamps point"* is a point of that line. The following theorem results from a typical calculation.

Theorem 2. *The four points $\{X(3), X(4), X(6), X(64)\}$ on the Jerabek hyperbola form a harmonic quadruple of conic points.*

Proof. Since the Jerabek hyperbola passes through the vertices of the triangle of reference ABC, the proof reduces to showing that the pencil of the lines through these points and a vertex of the triangle, B say, is harmonic. Using barycentric coordinates, the four points are described as follows ([2]):

$$\begin{aligned}
\text{(circumcenter)} \quad & X(3) \ : \ (a^2S_A \ : \ b^2S_B \ : \ c^2S_C),\\
\text{(orthocenter)} \quad & X(4) \ : \ (S_BS_C \ : \ S_CS_A \ : \ S_AS_B),\\
\text{(symmedian point)} \quad & X(6) \ : \ (a^2 \ : \ b^2 \ : \ c^2),\\
& X(64) \ : \ \left(\frac{a^2}{a^2S_A - S_BS_C} \ : \ \frac{b^2}{b^2S_B - S_CS_A} \ : \ \frac{c^2}{c^2S_C - S_AS_B}\right).
\end{aligned}$$

The four lines of the pencil are then described by equations of the form $\{\alpha x + \gamma z = 0\}$:

$$\begin{aligned}
BX(3) \ : \ \alpha_1 x + \gamma_1 z &= (c^2S_C)x - (a^2S_A)z = 0,\\
BX(4) \ : \ \alpha_2 x + \gamma_2 z &= (S_AS_B)x - (S_BS_C)z = 0,\\
BX(6) \ : \ \alpha_3 x + \gamma_3 z &= (c^2)x - (a^2)z = 0,\\
BX(64) \ : \ \alpha_4 x + \gamma_4 z &= \left(\frac{c^2}{c^2S_C - S_AS_B}\right)x - \left(\frac{a^2}{a^2S_A - S_BS_C}\right)z = 0.
\end{aligned}$$

Here we used the standard notation $\{a = |BC|, b = |CA|, c = |AB|\}$ and $S_A = (b^2 + c^2 - a^2)/2$, $S_B = (c^2 + a^2 - b^2)/2$ and $S_C = (a^2 + b^2 - c^2)/2$. Having that, it suffices to show that the cross ratio

$$\frac{\left(\frac{\alpha_1}{\gamma_1} - \frac{\alpha_3}{\gamma_3}\right)}{\left(\frac{\alpha_2}{\gamma_2} - \frac{\alpha_3}{\gamma_3}\right)} : \frac{\left(\frac{\alpha_1}{\gamma_1} - \frac{\alpha_4}{\gamma_4}\right)}{\left(\frac{\alpha_2}{\gamma_2} - \frac{\alpha_4}{\gamma_4}\right)} = -1,$$

which is a bit tedious, but otherwise unproblematic calculation. □

References

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