INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 8 (2019), No. 1, 22 - 31

INTEGER SEQUENCES, PYTHAGOREAN TRIPLETS AND CIRCLE CHAINS INSCRIBED INSIDE A PARABOLA

GIOVANNI LUCCA

Abstract. In this paper we consider the infinite chains of mutually tangent circles that can be inscribed inside a parabola and we derive the expressions for the radii and centres coordinates; moreover, we establish the conditions that relate the circle chains to Pythagorean triplets and to certain integer sequences.

1. Introduction

Let us consider a generic parabola, having axis coincident with the ordinates axis y and vertex coincident with the origin O, expressed by equation:

$$y = ax^2 \qquad a \neq 0 \tag{1}$$

Inside this parabola we inscribe a chain of mutually tangent circles so that the generic n-th circle is tangent to the preceding and succeeding ones and to the parabola itself (see Fig.1).
As far as we know this problem has been touched only in [1] by considering the particular case for $a = 1$ inside equation (1); here, we want to study the problem from a more general point of view and relate the characteristics of the circle chains to Pythagorean triplets and to certain integer sequences.

Keywords and phrases: Circle chains, Integer sequences, Pythagorean triplets
(2010)Mathematics Subject Classification: 51M04, 51M15
Received: 04.12.2018. In revised form: 13.03.2019. Accepted: 03.03.2019.

We also point out that, in the context of the problem we are going to face, it is not necessary to consider the more general form of the equation of the parabola i.e.:

$$y = ax^2 + bx + c \tag{2}$$

because the *width* of the parabola depends only on the parameter a. (To this aim, we remark that, by means of a simple axes translation, it is always possible to transform a parabola expressed by formula (2) into another parabola expressed as in (1)). Thus, without any loss of generality, one can consider equation (1) for deriving the properties of the circle chains we are going to show.

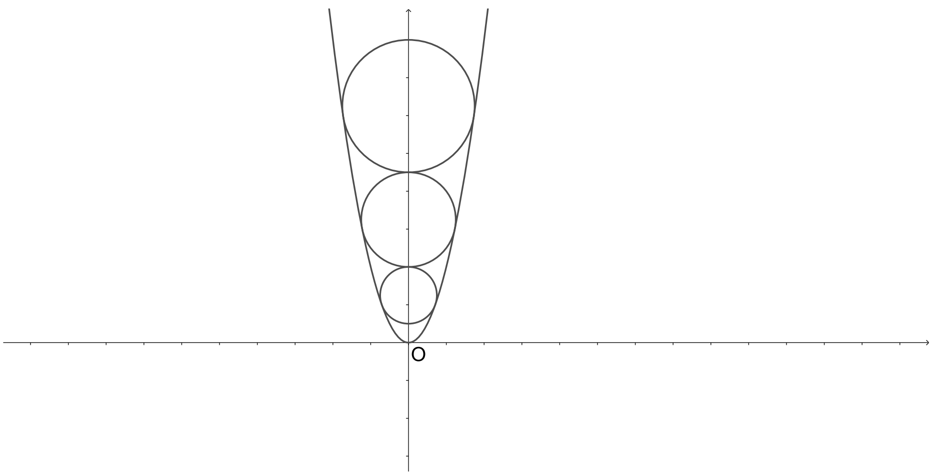

FIGURE 1. Example of circle chain inscribed inside a parabola.

2. Formulas for radii and centres of the circles

In order to construct the chain we have first to remark that, for symmetry reasons the centre of each circle, having radius r_n, has to lie on the y axis so that its coordinates are given by the pair $(0, Y_n)$.
Hence, the generic n-th circle of the chain has equation:

$$x^2 + (y - Y_n)^2 = r_n^2 \qquad n = 0, 1 \ldots \tag{3}$$

In order to determine the tangency conditions for each circle of the chain, one has to consider the following system:

$$\begin{cases} y = ax^2 \\ x^2 + (y - Y_n)^2 = r_n^2 \end{cases} \tag{4}$$

By substituting the first equation into the second one, we obtain:

$$ay^2 + y\left(1 - 2aY_n^2\right) + a\left(Y_n^2 - r_n^2\right) = 0 \tag{5}$$

The tangency conditions between the circles and the parabola requests that the discriminant of equation (5) is zero; therefore one has:

$$\left(1 - 2aY_n\right)^2 - 4a^2\left(Y_n^2 - r_n^2\right) = 0 \tag{6}$$

that yields:

$$Y_n = \frac{1 + 4a^2 r_n^2}{4a} \tag{7}$$

On the other hand, the tangency condition between two consecutive circles of the chain requests that:

$$|Y_{n+1} - Y_n| = r_{n+1} + r_n \tag{8}$$

By substituting equation (7) into equation (8) and after some algebraic steps, one finally obtains:

$$r_{n+1} - r_n = \frac{1}{|a|} \tag{9}$$

Being a a constant, equation (9) means that the radius of the circles belonging to the chain increases according to the arithmetic progression of common difference D given by:

$$D = \frac{1}{|a|} \tag{10}$$

so that one has:

$$r_n = r_0 + \frac{n}{|a|} \qquad n = 0, 1 \ldots \tag{11}$$

It is interesting to remark that the common difference D of the arithmetic progression is equal to the length of the so called *latus rectum* of the parabola i.e. the chord through the focus and parallel to the directrix.
Furthermore, we observe that the characteristics of the circle chain depend not only on the parameter a but also on the coordinates $(0, Y_0)$ choosen for the centre of the first circle labeled by the index 0 or, equivalently, on the value of r_0. Infact, from (7) one has that Y_0 and r_0 are biunivocally related by:

$$Y_0 = \frac{1 + 4a^2 r_0^2}{4a} \tag{12}$$

Nevertheless, by rememembering that it must be $|Y_0| \geq r_0$, one has that Y_0 is subjected to the constraint given by:

$$|Y_0| \geq \frac{1}{2|a|} \tag{13}$$

Formula (13) can be obtained from (12) by imposing $|Y_0| = r_0$; the geometrical meaning is that, in this case, the smallest circle of the chain is tangent to the parabola in its vertex. See Fig.2.

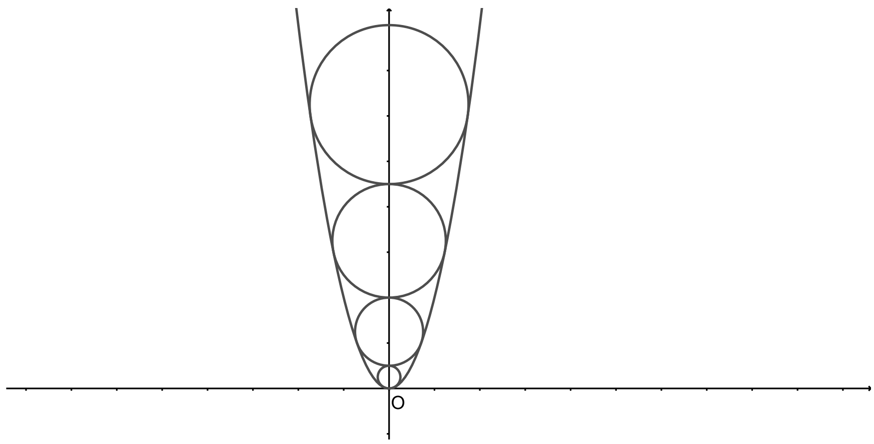

FIGURE 2. Example of circle chain with the smallest circle tangent to the parabola in its vertex.

It is also useful to add that, by means of (7), (11), (12) it possible to write:

$$Y_n = Y_0 + \frac{a}{|a|} 2nr_0 + \frac{n^2}{a} \qquad n = 0, 1 \ldots \tag{14}$$

3. CONDITIONS FOR RELATING THE CIRCLE CHAINS TO INTEGER SEQUENCES

By looking at (11) and (14), we shall consider the sequences having elements represented by r_n and by $|Y_n|$. It may be interesting to find under which conditions the sequences $\{r_n\}$ and $\{|Y_n|\}$ are entirely composed by integer numbers. To this aim we introduce and demonstrate the following theorem:

Theorem 3.1. *Necessary and sufficient condition in order to be both* $\{r_n\}$ *and* $\{|Y_n|\}$ *integer sequences is that the triplet* $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ *is composed by integer numbers and that:*

$$a = \frac{1}{2\left(Y_0 - sign(Y_0)\sqrt{Y_0^2 - r_0^2}\right)} \tag{15}$$

The expression for a in (15) involving the function $sign(Y_0)$ takes into account that Y_0 can be positive or negative and that the condition given by (13) has to be fullfilled.
Moreover, it is useful to notice that if $|Y_0| \neq r_0$, the integer triplet $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ is also Pythagorean.

Proof.

The condition is sufficient

Being the triplet $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ integer, from (15), one has that a is the reciprocal of an integer (positive or negative); hence, from (11) and (14), it directly follows that $|Y_n|$ and r_n are integers for each value of n.

The condition is necessary

If $\{r_n\}$ and $\{|Y_n|\}$ are integer sequences then, in particular, the elements r_0 and $|Y_0|$ are integers. By solving (12) with respect to a and choosing among the two solutions the one that fullfills (13), one obtains formula (15). From (11) and from the fact that r_0 is an integer, one deduces that a must be the reciprocal of an integer number (positive or negative). From this, from (15) and from the fact that the elements r_0 and $|Y_0|$ are integers, it follows that also $\sqrt{Y_0^2 - r_0^2}$ must be integer.

□

The sequences $\{r_n\}$ and $\{|Y_n|\}$ have another interesting property expressed by the following theorem:

Theorem 3.2. *If the sequences $\{r_n\}$ and $\{|Y_n|\}$ are integer, then the sequence $\left\{\sqrt{Y_n^2 - r_n^2}\right\}$ is integer too.*

Note that this is equivalent to say that, for each n, the triplet $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ is Pythagorean.

Proof.

Being r_n and $|Y_n|$ integers and being $|Y_n| \geqslant r_n$ for each n, one has that also $Y_n^2 - r_n^2$ is a positive integer.
Moreover $Y_n^2 - r_n^2$ is a perfect square; infact, by remembering (7), one can write:

$$Y_n^2 - r_n^2 = \frac{1 - 8a^2 r_n^2 + 16a^4 r_n^4}{16a^2} = \left(\frac{4a^2 r_n^2 - 1}{4a}\right)^2 \tag{16}$$

so that $\sqrt{Y_n^2 - r_n^2}$ is an integer for each n.
Therefore, it directly follows that the triplet $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ is Pythagorean.

□

It is useful, for the following, to explicitly write the expression for $\sqrt{Y_n^2 - r_n^2}$; infact, after some algebraical steps, that we omit for brevity, one can write:

$$\sqrt{Y_n^2 - r_n^2} = 2n^2\left(|Y_0| - \sqrt{Y_0^2 - r_0^2}\right) + 2nr_0 + \sqrt{Y_0^2 - r_0^2} \tag{17}$$

One might ask if the the Pythagorean triplets $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ are also primitive. The answer is affirmative and the following theorem holds:

Theorem 3.3. *The triplets* $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ *are primitive Pythagorean for each n, if and only if the triplet* $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ *is primitive Pythagorean.*

Proof. First of all it is convenient to remark that, under the hypothesis of Theorem 3.1, r_n and $|Y_n|$ can be written as follows:

$$r_n = r_0 + 2n\left[|Y_0| - \sqrt{Y_0^2 - r_0^2}\right] \tag{18}$$

$$|Y_n| = |Y_0| + 2nr_0 + 2n^2\left[|Y_0| - \sqrt{Y_0^2 - r_0^2}\right] \tag{19}$$

If the triplet $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ is primitive, one has that the GCD (Greatest Common Divisor) among r_0, $\sqrt{Y_0^2 - r_0^2}$ and $|Y_0|$ is 1; thus, one can write respectively:

$$r_0 = \prod_{i=1}^{L} a_i \tag{20}$$

$$\sqrt{Y_0^2 - r_0^2} = \prod_{j=1}^{M} b_j \tag{21}$$

$$|Y_0| = \prod_{k=1}^{N} c_k \tag{22}$$

where the integers a_i are the L factors composing r_0, the integers b_j are the M factors composing $\sqrt{Y_0^2 - r_0^2}$, the integers c_k are the N factors composing $|Y_0|$ and the following relation among them must hold:

$$a_i \neq b_j \neq c_k \qquad \forall i, j, k \in \mathbb{N} \tag{23}$$

Thus, by substituting (20), (21), (22) into (18), (17), (19), one obtains:

$$r_n = \prod_{i=1}^{L} a_i + 2n\prod_{k=1}^{N} c_k - 2n\prod_{j=1}^{M} b_j \tag{24}$$

$$\sqrt{Y_n^2 - r_n^2} = \prod_{j=1}^{M} b_j + 2n \prod_{i=1}^{L} a_i + 2n^2 \prod_{k=1}^{N} c_k - 2n^2 \prod_{j=1}^{M} b_j \tag{25}$$

$$|Y_n| = \prod_{k=1}^{N} c_k + 2n \prod_{i=1}^{L} a_i + 2n^2 \prod_{k=1}^{N} c_k - 2n^2 \prod_{j=1}^{M} b_j \tag{26}$$

By looking at (24), (25), (26)and by taking into account (23), one can see that, for each n, the elements r_n, $\sqrt{Y_n^2 - r_n^2}$, $|Y_n|$ have no common factors thus we can conclude that the triplet $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ is primitive.

If the triplet $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ is non-primitive, one has that:

$$\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right) = \left(K\widehat{r_0}, K\sqrt{\widehat{Y_0}^2 - \widehat{r_0}^2}, K|\widehat{Y_0}|\right) \quad K = 2, 3, \ldots \tag{27}$$

being K a common integer factor and $\left(\widehat{r_0}, \sqrt{\widehat{Y_0}^2 - \widehat{r_0}^2}, |\widehat{Y_0}|\right)$ a primitive triplet.

By substituting (27) into (18), (17) and (19) one has respectively:

$$r_n = K\widehat{r_0} + 2nK\left(|\widehat{Y_0}| - \sqrt{\widehat{Y_0}^2 - \widehat{r_0}^2}\right) \tag{28}$$

$$\sqrt{Y_n^2 - r_n^2} = 2n^2K\left(|\widehat{Y_0}| - \sqrt{\widehat{Y_0}^2 - \widehat{r_0}^2}\right) + 2nK\widehat{r_0} + K\sqrt{\widehat{Y_0}^2 - \widehat{r_0}^2} \tag{29}$$

$$|Y_n| = K|\widehat{Y_0}| + 2nK\widehat{r_0} + 2n^2K\left(|\widehat{Y_0}| - \sqrt{\widehat{Y_0}^2 - \widehat{r_0}^2}\right) \tag{30}$$

Formulas (28)-(30) show that r_n, $\sqrt{Y_n^2 - r_n^2}$ and $|Y_n|$ all have the common factor K so that the triplet $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ is non-primitive.
To conclude, if the triplet $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ is primitive, it generates primitive triplets; likewise if $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ is non-primitive, it generates non-primitive triplets.

□

Let us summarize the main points:

- to each Pythagorean triplet $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$, one has a corresponding value for a, characterizing the parabola, given by formula (15);

- related to the circle chain one has three different integer sequences given by: $\{r_n\}$, $\left\{\sqrt{Y_n^2 - r_n^2}\right\}$ and $\{|Y_n|\}$ with the property that the corresponding elements of the three sequences (i.e. the ones having the same index n) form a Pythagorean triplet;
- if $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$ is a primitive Pythagorean triplet, then, for every n, $\left(r_n, \sqrt{Y_n^2 - r_n^2}, |Y_n|\right)$ is primitive Pythagorean too.

It is useful to remark that the sequence $\left\{\sqrt{Y_n^2 - r_n^2}\right\}$ is integer also when $|Y_0| = r_0$.
It is interesting to notice that formulas (18), (17) and (19) can be used as an algorithm to generate Pythagorean triplets starting from a *seed* Pythagorean triplet represented by $\left(r_0, \sqrt{Y_0^2 - r_0^2}, |Y_0|\right)$.

To conclude the paragraph, we consider a simple example.
If $r_0 = 5$ and $|Y_0| = 13$ so that $\sqrt{Y_0^2 - r_0^2} = 12$, one gets from (15) the value $a = \frac{1}{2}$.
The three integer sequences associated to the circle chain are:

$$\{r_n\} = \quad \{5, 7, 9, 11, 13, 15, 17 \ldots\}$$

$$\{\sqrt{Y_n^2 - r_n^2}\} = \{12, 24, 40, 60, 84, 112, 144 \ldots\}$$

$$\{|Y_n|\} = \quad \{13, 25, 41, 61, 85, 113, 145 \ldots\}$$

One can easily verify that the triplets generated by the primitive triplet (5, 12, 13) i.e.: (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), (15, 112, 113), (17, 144, 145) … are primitive Pythagorean.

4. Three invariant sequences common to each parabola

Let us introduce now three other sequences related to the circle chains: $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ having their elements defined as follows:

$$\alpha_n = \frac{r_n}{r_0} \qquad n = 0, 1 \ldots$$

$$\beta_n = \frac{|Y_n|}{r_0} \qquad n = 0, 1 \ldots$$

$$\gamma_n = \frac{\sqrt{Y_n^2 - r_n^2}}{r_0} \qquad n = 0, 1 \ldots$$

By remembering formulas (11), (14) and (17), after some algebraical steps that we omit for brevity, one can write:

$$\alpha_n = 1 + \frac{n}{|a|r_0} \qquad n = 0, 1 \ldots \tag{31}$$

$$\gamma_n = \frac{|Y_0|}{r_0} + 2n + \frac{n^2}{|a|r_0} \qquad n = 0, 1 \ldots \tag{32}$$

$$\beta_n = |a|r_0 + 2n + \frac{4n^2 - 1}{4|a|r_0} \qquad n = 0, 1 \ldots \tag{33}$$

By looking at the above formulas, one can notice that each element of the sequences depends on the choice for $|Y_0|$ and r_0 that, in turn, influences the value of the parameter a of the parabola (see formula (15)). We pose now the following question: is it possible to find a choice for $|Y_0|$ and r_0 so that the three sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are integer and depend only on the index n? The answer is in the following theorem:

Theorem 4.1. *If $|Y_0| = r_0$ then the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are integer and depend only on the index n.*

Proof. From (15) and from $|Y_0| = r_0$ one has that $|a|r_0 = \frac{1}{2}$; thus, in this case the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are given by:

$$\alpha_n = 2n + 1 \qquad 0, 1 \ldots \tag{34}$$

$$\gamma_n = 2n^2 + 2n + 1 \qquad 0, 1 \ldots \tag{35}$$

$$\beta_n = 2n^2 + 2n \qquad 0, 1 \ldots \tag{36}$$

As one can see, formulas (34)-(36) generate integer numbers and depend on n only.

□

We immediately recognize that α_n given by (34) represents the odd integers sequence which is classified in the On-Line Encyclopedia of Integer Sequences OEIS [2] as A005408. As far as β_n and γ_n (respectively formulas (35) and (36)) are concerned, they are classified as and A046092 and A001844.
Those three integer sequences that are associated to the family of the circle chains having the smallest circle tangent to the parabola in its vertex (see

Fig.2), do not depend on the parameter a and, consequently, they are not related to a particular parabola. On the contrary, they can be considered as common and invariant sequences to be related to the set of all the parabolas with inscribed circle chain disposed like in Fig.2.

It is worth to mention another property belonging to the sequences represented by (34)-(36); infact, for $n = 1, 2 \ldots$, those formulas generate an infinite set of primitive Pythagorean triplets where the *seed* is given by the simplest one i.e. $(3, 4, 5)$. Indeed, by looking at (34)-(36), we have that, for each n, $\beta_n = \frac{\alpha_n^2 - 1}{2}$ and $\gamma_n = \frac{\alpha_n^2 + 1}{2}$. On the other hand, a well known algorithm, attributed to Pythagoras himself, allows to generate a primitive Pythagorean triplet starting from any any odd integer number n; according to it, the primitive triplet is given by $\left(n, \frac{n^2-1}{2}, \frac{n^2+1}{2}\right)$.

Being α_n an odd integer, we have that the triplet $\left(n, \frac{n^2-1}{2}, \frac{n^2+1}{2}\right)$ is identical to the triplet $(\alpha_n, \beta_n, \gamma_n)$ so deducing that it is primitive.

References

[1] Minton, G., *Problems of Circle Tangency*, The Harvard College Mathematics Review, **1(2)(2007)** 47–53.

[2] Sloane, N.J.A., *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org

VIA CORVI 20
29122 PIACENZA ITALY
E-mail address: vanni_lucca@inwind.it