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CHARACTERIZATIONS OF CYCLIC QUADRILATERALS

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Abstract. Cyclic quadrilaterals have many famous properties, that is, necessary conditions. However, what is not so well-known is that most of their properties are also sufficient conditions for such quadrilaterals to exist. In this paper we prove 19 characterizations of convex cyclic quadrilaterals.

1. Introduction

If a quadrilateral is inscribed into a circle so that all four vertices lie on the circle, it is most often referred to as a *cyclic quadrilateral* and the vertices are said to be concyclic. This circle is called the circumcircle and its center, which is equidistant from the vertices, is called the circumcenter. Other names that have been used for this class of quadrilaterals are inscribed quadrilateral, concyclic quadrilateral and chordal quadrilateral.

In textbooks covering Euclidean geometry and on mathematical problem solving websites, necessary conditions of cyclic quadrilaterals are often given in theorems and exercises. Then it is assumed that the quadrilateral is cyclic and a relation between angles or lines or distances shall be proved. In contrast to what many textbook authors and problem creators seem to realize or care about, it is common also for the converse theorem to be true. This means that if the relation we just referred to holds in a convex quadrilateral, then it can be concluded that it *is* a cyclic quadrilateral.

In this paper we shall prove 19 such necessary and sufficient conditions. Many of these are not so well known, and several are probably previously unknown characterizations. We have references for all of the direct theorems except for three; it is the converses for about half of the conditions that we consider to be new. In fact, it's very difficult to find any textbook, paper, or website covering more than just three necessary *and* sufficient conditions for cyclic quadrilaterals. Thus collecting a large number of characterizations of cyclic quadrilaterals with proofs is the primary goal for this paper. ——

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2. Five basic characterizations

There are a handful of well-known characterizations of cyclic quadrilaterals that can be found in many books and websites dealing with Euclidean geometry (although seldom all in the same place). We begin by reviewing them here and shall use a couple of them in some of the proofs. *The angle between a side and a diagonal in a convex quadrilateral is equal to the angle between the opposite side and the other diagonal if and only if it is a cyclic quadrilateral.* That is, for example $\angle BAC = \angle BDC$ in quadrilateral $ABCD$. The direct (the "if") part of the theorem was Proposition III.21 in Euclid's Elements and a proof of the converse was given by the French mathematician Jacques Hadamard in the late 1890s according to [8].

The following necessary and sufficient condition establishes the location of the circumcenter: *In a convex quadrilateral, the four perpendicular bisectors to the sides are concurrent if and only if it is cyclic.* Their common point is the circumcenter.

The most famous characterization of cyclic quadrilaterals and in problem solving also the most frequently used is that *any two opposite angles of a convex quadrilateral are supplementary if and only if it is a cyclic quadrilateral.* It means that $ABCD$ is cyclic if and only if

$$\angle A + \angle C = \pi = \angle B + \angle D. \tag{1}$$

This is often referred to as the "cyclic quadrilateral theorem". The direct part of it was Proposition III.22 in Euclid's Elements. The converse is almost always proved using an indirect proof, but an interesting different approach can be found in [16, p. 131]. As a simple corollary we have that *an exterior angle to a convex quadrilateral is equal to the opposite interior angle if and only if it is a cyclic quadrilateral.*

An important theorem in circle geometry is the intersecting chords theorem. It states that *the four vertices A, B, C and D of a convex quadrilateral satisfy the equation*

$$AP \cdot PB = DP \cdot PC$$

if and only if it is a cyclic quadrilateral, where P is the intersection of AB and DC or their extensions. If the chords intersect outside the circle, the four points are vertices in a cyclic quadrilateral $ABCD$. But if the point of intersection lies inside the circle, then the convex cyclic quadrilateral is $ACBD$. The direct part of the case with an internal intersection was Proposition III.35 in Euclid's Elements.

We encourage the reader to write up the proofs of these five characterizations if they are not previously known before reading on. In order to make this collection of theorems self contained, we will give the proofs in an appendix at the end.

3. Characterizations concerning angles or parallel lines

For the first two necessary and sufficient conditions we will prove we have not found any reference at all (neither for the direct nor the converse theorem). But since they are simple angle relations, at least the direct parts have surely been published in some geometry textbook before.

Theorem 3.1. *In a convex quadrilateral $ABCD$,*

$$\angle CAD + \angle DCA = \angle B$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. $ABCD$ is cyclic if and only if $\angle A + \angle C = \pi$, which is equivalent to

$$\begin{aligned} & \angle CAD + \angle CAB + \angle DCA + \angle ACB = \pi \\ \Leftrightarrow \quad & \angle CAD + \angle DCA = \pi - (\angle CAB + \angle ACB) \\ \Leftrightarrow \quad & \angle CAD + \angle DCA = \angle B. \end{aligned}$$

In the last step we used the sum of angles in triangle ABC (see Figure 1). □

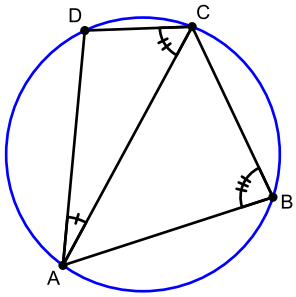

FIGURE 1. $ABCD$ is cyclic iff $\angle CAD + \angle DCA = \angle B$

The second theorem is a simple corollary to the first of the five basic characterization we reviewed in the previous section.

Theorem 3.2. *In a convex quadrilateral $ABCD$,*

$$\angle A - \angle B = \angle BAC - \angle ABD$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. Since $\angle A = \angle BAC + \angle DAC$ and $\angle B = \angle ABD + \angle DBC$ (see Figure 2), we have that $ABCD$ is cyclic if and only if

$$\angle DAC = \angle DBC \quad \Leftrightarrow \quad \angle A - \angle BAC = \angle B - \angle ABD$$

and the relation follows. □

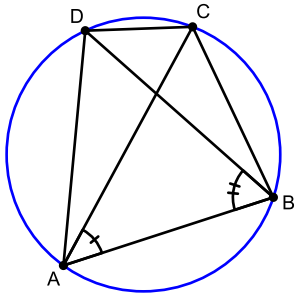

FIGURE 2. $ABCD$ is cyclic iff $\angle A - \angle B = \angle BAC - \angle ABD$

The direct part of the next theorem is almost the same as Problem 574 on the wonderful website [7] by Antonio Gutierrez.

Theorem 3.3. *Let the extensions of opposite sides AB and CD in a convex quadrilateral $ABCD$ intersect at E and the diagonals intersect at P. Then the angle bisectors to the angles AED and APD are parallel if and only if $ABCD$ is a cyclic quadrilateral.*

Proof. Using notations as in Figure 3, we have

$$u = \pi - \angle A - \frac{\pi - \angle A - \angle D}{2} = \frac{\pi - \angle A + \angle D}{2}$$

and, by the exterior angle theorem,

$$v = \frac{\pi - \angle DAC - \angle BDA}{2} + \angle BDA = \frac{\pi - \angle DAC + \angle BDA}{2}.$$

The angle bisectors to the angles AED and APD are parallel if and only if

$$u = v \quad \Leftrightarrow \quad -\angle A + \angle D = -\angle DAC + \angle BDA$$

which is a variant of the characterization in Theorem 3.2. □

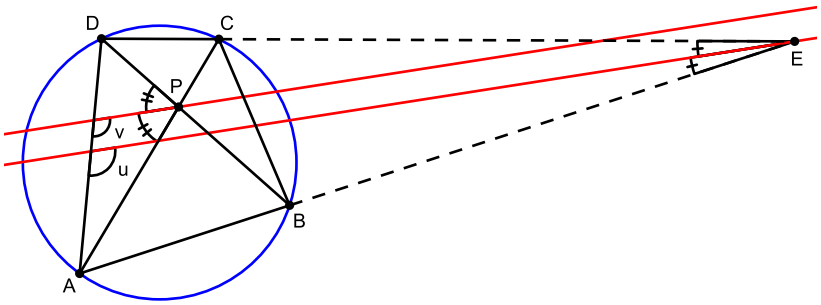

FIGURE 3. $ABCD$ is cyclic iff the angle bisectors at E and P are parallel

The direct part of the following theorem was a problem in [3, p. 134].

Theorem 3.4. *In a convex quadrilateral $ABCD$, let the perpendicular to AB at A meet CD in A', and the perpendicular to CD at C meet AB in C'. The line $A'C'$ is parallel to BD if and only if $ABCD$ is cyclic.*

Proof. The line $A'C'$ is parallel to diagonal BD if and only if $\angle CDB = \angle CA'C'$ (see Figure 4). Quadrilateral $AC'CA'$ is cyclic by construction since it has two opposite right angles. Thus $\angle CAC' = \angle CA'C'$. Quadrilateral $ABCD$ is cyclic if and only if $\angle CDB = \angle CAB = \angle CAC' = \angle CA'C'$. □

The next characterization can be proved via an extensive angle chase. The converse in the theorem was Problem 2 at the first day of the China Western Mathematical Olympiad in 2004 [4, pp. 186–187]. It was proposed by Liu Shixiong.

Theorem 3.5. *In convex quadrilateral $ABCD$, let I_1 and I_2 be the incenters in triangles ABC and BCD respectively, and let the line I_1I_2 intersect the sides AB and CD at I_1' and I_2' respectively. If the extensions of these two sides intersect at E, then $EI_1' = EI_2'$ if and only if $ABCD$ is a cyclic quadrilateral.*

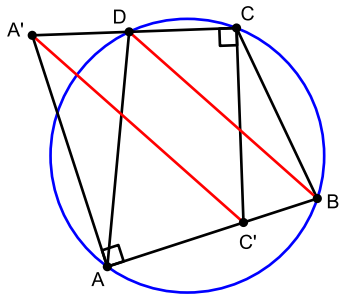

FIGURE 4. $ABCD$ is cyclic iff $A'C' \parallel BD$

Proof. Quadrilateral $ABCD$ is cyclic if and only if (see Figure 5)

$$\begin{aligned}
& \angle BAC = \angle BDC \\
\Leftrightarrow\ & \angle ABC + \angle ACB = \angle DBC + \angle DCB \\
\Leftrightarrow\ & \angle I_1BC + \angle I_1CB = \angle I_2BC + \angle I_2CB \\
\Leftrightarrow\ & \angle BI_1C = \angle BI_2C \\
\Leftrightarrow\ & BI_1I_2C \text{ is a cyclic quadrilateral} \\
\Leftrightarrow\ & \angle I_2I_1B + \angle I_2CB = \angle I_1I_2C + \angle I_1BC \\
\Leftrightarrow\ & \angle I_2I_1B - \angle I_1BA = \angle I_1I_2C - \angle I_2CD \\
\Leftrightarrow\ & \angle EI_1'I_2' = \angle EI_2'I_1' \\
\Leftrightarrow\ & EI_1' = EI_2'
\end{aligned}$$

where we have used the angle sum of a triangle, that the incenter lies on the internal angle bisectors, the exterior angle theorem, and in the last step the isosceles triangle theorem and its converse. □

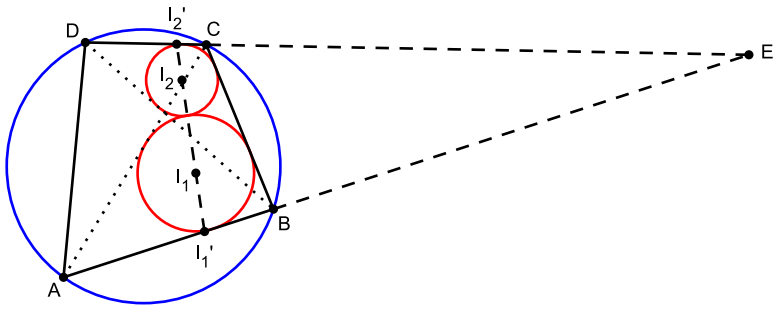

FIGURE 5. $ABCD$ is cyclic iff $EI_1' = EI_2'$

4. A characterization concerning concurrency

A line through the midpoint of a side in a quadrilateral and perpendicular to the opposite side is called a *maltitude*. The proof of the direct part of the next theorem is more or less the same as in [10, p. 36]. We will use the well-known property that the midpoints of the sides in a quadrilateral are the vertices of a parallelogram (called the Varignon parallelogram). Its

diagonals are called the bimedians of the quadrilateral. To prove this characterization of cyclic quadrilaterals was Problem 5 on the third round of the 2001 Brazilian Mathematical Olympiad [1]. Another proof than ours was given in [12, p. 19].

Theorem 4.1. *The four maltitudes of a convex quadrilateral are concurrent if and only if it is a cyclic quadrilateral.*[1]

Proof. ($\Rightarrow$) If the quadrilateral is cyclic, we shall prove that the maltitudes are concurrent. We label the midpoints of the sides as M_a, M_b, M_c and M_d. Let the bimedians M_aM_c and M_bM_d intersect at G, the circumcircle have center O, and let Q be a point on the extension of OG such that $OG = GQ$ (see Figure 6). Then $M_aG = GM_c$ since G is the intersection of the diagonals in a parallelogram. We also have $\angle OGM_c = \angle QGM_a$, so triangles OGM_c and QGM_a are congruent (SAS). Thus $\angle GM_cO = \angle GM_aQ$ and so the lines OM_c and M_aQ are parallel. But OM_c is perpendicular to the side CD, hence so is the extension of M_aQ. In the same way the extensions of M_bQ, M_cQ and M_dQ are perpendicular to DA, AB and BC respectively, so all four maltitudes are concurrent at Q.

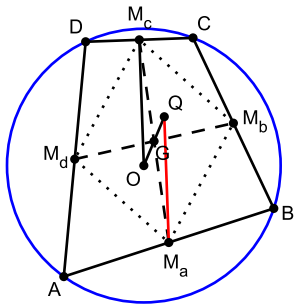

FIGURE 6. The extension of M_aQ is a maltitude to side CD

($\Leftarrow$) If the maltitudes are concurrent in a point, say Q, we shall prove that the quadrilateral is cyclic. Let the bimedians intersect at G. Extend QG to a point O' such that $QG = GO'$. Triangles $O'GM_c$ and QGM_a are congruent (SAS), so $O'M_c$ is perpendicular to side CD since it is parallel to a maltitude. Thus triangles $O'CM_c$ and $O'DM_c$ are congruent (SAS), so $O'C = O'D$. In the same way we have that $O'A = O'B$, $O'B = O'C$ and $O'D = O'A$. Hence O' is equidistant to the vertices, which proves that $ABCD$ is a cyclic quadrilateral with circumcenter O'. □

5. Characterizations concerning similarity

The first theorem on similarity is an extension of a problem by David Monk in [13, p. 6].

[1]Their point of intersection is called the anticenter of the cyclic quadrilateral since it is the reflexion of the circumcenter in the centroid (the point where the bimedians intersect).

Theorem 5.1. *Let $ABCD$ be a convex quadrilateral where the side AB is not parallel to CD and the diagonals intersect at P. Let the parallel to AB through P meet AD at S and BC at T, and the parallel to CD through P meet AD at U and BC at V. Then*

$$SP \cdot PT = UP \cdot PV$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. ($\Rightarrow$) If $ABCD$ is a cyclic quadrilateral, then the exterior angle at B is $\beta = \angle D$ (see Figure 7). But $\angle D = \angle SUP$ and $\beta = \angle VTP$, so $\angle SUP = \angle VTP$. Thus the triangles SPU and VPT are similar (AA). Hence

$$\frac{SP}{PV} = \frac{UP}{PT}$$

and the equality follows.

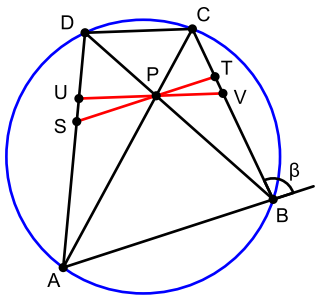

FIGURE 7. $ABCD$ is cyclic iff $SP \cdot PT = UP \cdot PV$

($\Leftarrow$) For the converse, since

$$SP \cdot PT = UP \cdot PV \quad \Rightarrow \quad \frac{SP}{PV} = \frac{UP}{PT}$$

and $\angle SPU = \angle TPV$, the triangles SPU and TPV are similar. Thus $\angle SUP = \angle VTP$. If the exterior angle at B is β, then $\beta = \angle VTP$ since AB and ST are parallel. We also have $\angle SUP = \angle D$, so

$$\beta = \angle VTP = \angle SUP = \angle D.$$

Hence $ABCD$ is a cyclic quadrilateral. □

The direct part of the following theorem was Problem 334 at [7].

Theorem 5.2. *In a convex quadrilateral $ABCD$ where the diagonals intersect at P, let A' and C' be the projections of A and C respectively on diagonal BD, and let B' and D' be the projections of B and D respectively on diagonal AC. Then*

$$A'P \cdot PC' = B'P \cdot PD'$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. We first study a convex quadrilateral $ABCD$. We have (see Figure 8)

$$\sin\angle ADB = \frac{AA'}{DA}, \qquad \sin\angle ACB = \frac{BB'}{BC},$$
$$\sin\angle CAD = \frac{DD'}{DA}, \qquad \sin\angle CBD = \frac{CC'}{BC}. \tag{2}$$

The pair of triangles $AA'P$ and $BB'P$, and also $DD'P$ and $CC'P$, are similar (AA), so

$$\frac{BB'}{AA'} = \frac{B'P}{A'P}, \qquad \frac{CC'}{DD'} = \frac{PC'}{PD'}. \tag{3}$$

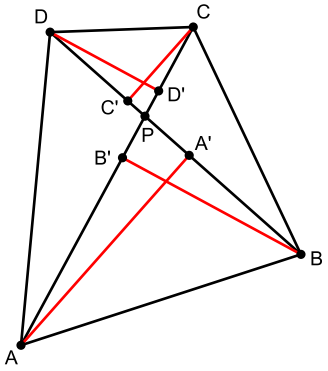

FIGURE 8. Projections on the diagonals

($\Rightarrow$) Now assume $ABCD$ is cyclic. Then $\angle ADB = \angle ACB$ and $\angle CAD = \angle CBD$, so from equations (2) and (3) we get

$$\frac{AA'}{DA} = \frac{BB'}{BC} \quad\Rightarrow\quad \frac{BC}{DA} = \frac{BB'}{AA'} = \frac{B'P}{A'P}, \tag{4}$$
$$\frac{DD'}{DA} = \frac{CC'}{BC} \quad\Rightarrow\quad \frac{BC}{DA} = \frac{CC'}{DD'} = \frac{PC'}{PD'}. \tag{5}$$

Hence

$$\frac{B'P}{A'P} = \frac{PC'}{PD'} \quad\Rightarrow\quad A'P\cdot PC' = B'P\cdot PD'.$$

($\Leftarrow$) Conversely, if $A'P\cdot PC' = B'P\cdot PD'$, we will prove that $ABCD$ is cyclic. Using (3), we get

$$\frac{B'P}{A'P} = \frac{PC'}{PD'} \quad\Rightarrow\quad \frac{BB'}{AA'} = \frac{CC'}{DD'} \quad\Rightarrow\quad AA'\cdot CC' = BB'\cdot DD'.$$

Using (2), this implies

$$DA\sin\angle ADB\cdot BC\sin\angle CBD = BC\sin\angle ACB\cdot DA\sin\angle CAD. \tag{6}$$

From the sum of angles in triangles ADP and BCP, we have $\angle ADB = \pi-(\angle P+\angle CAD)$ and $\angle ACB = \pi-(\angle P+\angle CBD)$ where $\angle P = \angle APD = \angle BPC$. Thus equation (6) can be rewritten as

$$\frac{\sin(P+\angle CAD)}{\sin\angle CAD} = \frac{\sin(P+\angle CBD)}{\sin\angle CBD}.$$

Applying the addition formula for sine leads to

$$\frac{\sin P\cos\angle CAD+\cos P\sin\angle CAD}{\sin\angle CAD}=\frac{\sin P\cos\angle CBD+\cos P\sin\angle CBD}{\sin\angle CBD}$$

which simplifies as

$$\sin P\cot\angle CAD+\cos P=\sin P\cot\angle CBD+\cos P$$

and we get $\cot\angle CAD=\cot\angle CBD$. Hence $\angle CAD=\angle CBD$, so $ABCD$ is a cyclic quadrilateral. □

6. Characterizations concerning extensions of the sides

Where we in this section formulate theorems about the intersections of the extensions of opposite sides, we assume the quadrilateral is not a trapezoid,[2] so these points exist. First we have a simple angle relation which is a corollary to the cyclic quadrilateral theorem.

Theorem 6.1. *Let the extensions of the sides of a convex quadrilateral $ABCD$ intersect at E and F. If the vertices are labeled such that triangle CEF only has one point in common with $ABCD$, then*

$$\angle CEF+\angle CFE=\angle A$$

if and only if $ABCD$ is cyclic.

Proof. In triangle CEF we have $\angle CEF+\angle CFE+\angle C=\pi$ (see Figure 9). It is a cyclic quadrilateral if and only if $\angle A+\angle C=\pi$, which is equivalent to $\angle CEF+\angle CFE=\angle A$. □

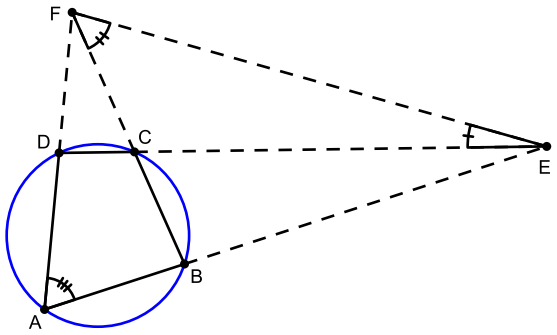

Figure 9. $ABCD$ is cyclic iff $\angle CEF+\angle CFE=\angle A$

The direct part of the next theorem was a problem in [9, p. 255], Problem 573 at [7], Problem 483 (a) in [5], and the first geometry problem ever to appear in the American Mathematical Monthly (in 1894) according to [5]. It seems that the converse was considered for the first time in [5].

Theorem 6.2. *If the extensions of the sides of a convex quadrilateral intersect at E and F, then the angle bisectors at E and F are perpendicular if and only if it is a cyclic quadrilateral.*

[2]And thus not a parallelogram, rhombus, rectangle or square, which are special cases of trapezoids when using inclusive definitions.

Proof. We start with a convex quadrilateral $ABCD$. Let the extension of the sides AB and DC intersect at E, and the extension of the sides BC and AD intersect at F (see Figure 10). From the sum of angles in triangle ADE we get that $\angle E = \pi - \angle A - \angle D$ and in triangle ABF that $\angle F = \pi - \angle A - \angle B$. If ζ is the angle between CD and the angle bisector at F, we have by the exterior angle theorem that $\angle C = \zeta + \frac{1}{2}\angle F$. Let ψ be the angle between the two considered angle bisectors. From the exterior angle theorem we get that $\zeta = \psi + \frac{1}{2}\angle E$, so

$$\begin{aligned} 2\psi &= 2\zeta - \angle E \\ &= 2\angle C - \angle F - \angle E \\ &= 2\angle C - \pi + \angle A + \angle B - \pi + \angle A + \angle D \\ &= (\angle A + \angle B + \angle C + \angle D - 2\pi) + \angle A + \angle C. \end{aligned}$$

The expression in the parenthesis is zero, so in all convex quadrilaterals

$$\psi = \frac{\angle A + \angle C}{2}.$$

Quadrilateral $ABCD$ is cyclic if and only if

$$\angle A + \angle C = \pi \quad \Leftrightarrow \quad \psi = \frac{\pi}{2},$$

that is, if and only if the angle bisectors are perpendicular. □

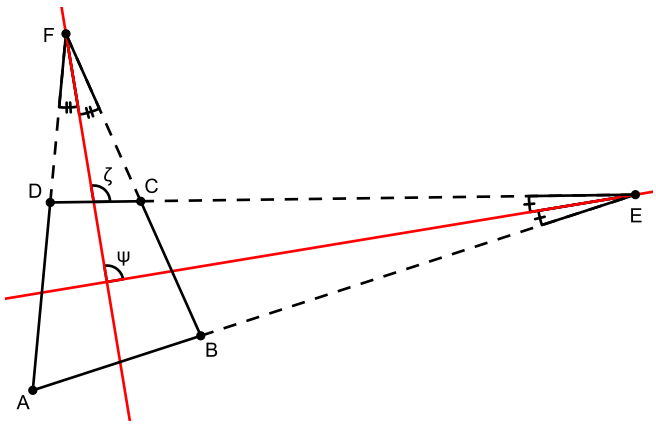

FIGURE 10. $ABCD$ is cyclic iff $\psi = \frac{\pi}{2}$

The direct part of the following characterization was a problem by G. B. M. Zerr in the American Mathematical Monthly in 1898 according to [11, pp. 218–219]. It was also Problem 359 at [7]. The converse is a direct application of Theorem 6.2.

Theorem 6.3. *Let the extensions of the sides of a convex quadrilateral $ABCD$ intersect at E and F, and let the angle bisectors at E and F intersect the sides AB, BC, CD and DA at W, X, Y and Z respectively. The quadrilateral $WXYZ$ is a rhombus if and only if $ABCD$ is cyclic.*

Proof. ($\Rightarrow$) In a cyclic quadrilateral $ABCD$, we have $\angle ECX = \angle EAZ$ (see Figure 11). Since $\angle CEX = \angle AEZ$, the triangles AEZ and CEX are similar. Thus $\angle CXE = \angle AZE$. Since $\angle CXE = \angle BXZ$ we conclude that $\angle FXZ = \angle FZX$, so triangle FXZ is isosceles. Then $FX = FZ$ and $XJ = JZ$, where J is the intersection of the angle bisectors. In the same way $YJ = JW$, so triangles ZJW, ZJY, YJX and XJW are congruent (SAS). Hence $WX = XY = YZ = ZW$ so $WXYZ$ is a rhombus.

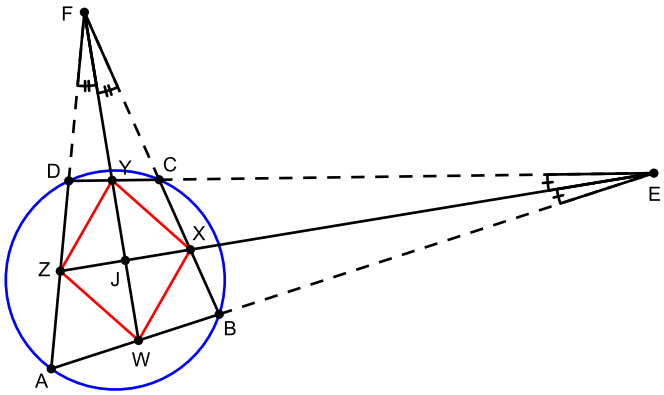

FIGURE 11. $ABCD$ is cyclic iff $WXYZ$ is a rhombus

($\Leftarrow$) If quadrilateral $WXYZ$ is a rhombus, its diagonals are perpendicular, and so are the angle bisectors at E and F. Then $ABCD$ is cyclic according to Theorem 6.2. □

In the proof of the next characterization we need the following lemma. It was discovered by Jakob Steiner in 1827, but a detailed proof was given by Auguste Miquel according to [14, p. 96].

Lemma 6.1. *Let the extensions of the sides AB and CD intersect at E, and the extensions of the sides BC and AD intersect at F in a convex quadrilateral $ABCD$. Then all four circumcircles to triangles BCE, ABF, ADE and ABF are concurrent in a point called the* Miquel point *of the complete quadrilateral $ABECFD$.*

Proof. Let the circumcircles of triangles BCE and ABF intersect at J (see Figure 12). In the cyclic quadrilaterals $ABJF$ and $CBEJ$ we have that

$$\angle DEJ = \angle CEJ = \angle CBJ = \angle FBJ = \angle FAJ = \angle DAJ$$

which proves that $DAEJ$ is also a cyclic quadrilateral. Thus the circumcircle of triangle ADE also goes through J. In the same way it is proved that the circumcircle of ABF goes through J as well. □

In [8, p. 109] it was given as a part of Exercise 106 to prove the following characterization of cyclic quadrilaterals (formulated differently).

Theorem 6.4. *Let the extensions of opposite sides in a convex quadrilateral $ABCD$ intersect at E and F. The Miquel point of the complete quadrilateral $ABECFD$ lies on the line EF if and only if $ABCD$ is a cyclic quadrilateral.*

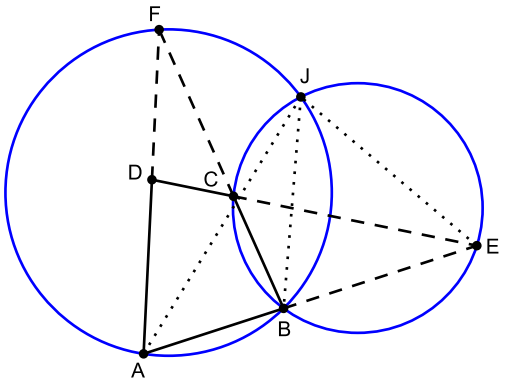

FIGURE 12. The cyclic quadrilaterals $ABJF$ and $CBEJ$

Proof. Due to Lemma 6.1, it is enough to prove that the circumcircles of two arbitrary of the four triangles BCE, ABF, ADE and ABF intersect at EF if and only if the quadrilateral $ABCD$ is cyclic. To this end, let the circumcircle of BCE intersect EF at K (see Figure 13). Quadrilateral $EBCK$ is cyclic, so $\angle EKB = \angle ECB$. Also, $\angle BKF = \pi - \angle EKB$ so

$$\angle BKF = \pi - \angle ECB = \angle C. \tag{7}$$

In the same way, let the circumcircle of ABF intersect EF at L, so $FABL$ is cyclic and $\angle FLB + \angle FAB = \pi$. From the exterior angle theorem,

$$\angle BKF = \angle FLB + \angle KBL = \pi - \angle FAB + \angle KBL. \tag{8}$$

Combining (7) and (8) yields

$$\angle KBL = \angle A + \angle C - \pi.$$

Hence

$$\angle A + \angle C = \pi \quad \Leftrightarrow \quad \angle KBL = 0 \quad \Leftrightarrow \quad K = L$$

so quadrilateral $ABCD$ is cyclic if and only if the two circumcircles intersect in $K = L$ on EF. □

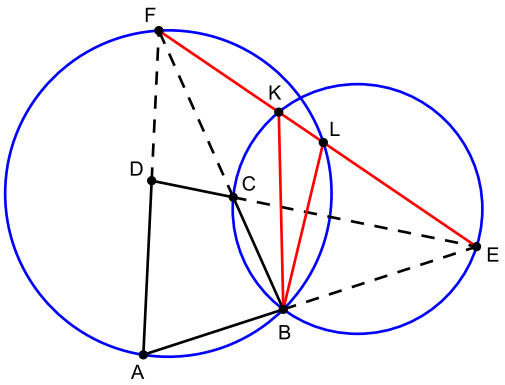

FIGURE 13. $ABCD$ is cyclic iff $K = L$

The formula in the next characterization appears in a problem in [6, p. 26]. Some time after we proved this theorem, we found out that the converse

was Problem 13 at the mathematics competition Baltic Way in 2006 [15, p. 19]. We will give our proof, which differs from the official solution to that problem.

Theorem 6.5. *Let the extensions of the sides AB and CD in a convex quadrilateral $ABCD$ intersect at E, and let the extensions of the sides BC and AD intersect at F. Then*

$$EF^2 = AE \cdot EB + BF \cdot FC$$

if and only if $ABCD$ is a cyclic quadrilateral.

Proof. Let the circumcircles of triangles BCE and ABF intersect EF at K and L respectively (see Figure 13). Then

$$EF^2 = EF(EL \pm LK + KF) = EF \cdot EL \pm EF \cdot LK + EF \cdot KF$$

where the choice of signs depends on where the circles intersect EF. Using the intersecting chords theorem, we have that $FE \cdot EL = AE \cdot EB$ in cyclic quadrilateral $ABLF$ and $EF \cdot FK = BF \cdot FC$ in cyclic quadrilateral $BCKE$. Hence

$$EF^2 = AE \cdot EB \pm EF \cdot LK + BF \cdot FC$$

so

$$EF^2 = AE \cdot EB + BF \cdot FC \quad \Leftrightarrow \quad LK = 0$$

since $EF \neq 0$. We have $LK = 0$ if and only if the quadrilateral $ABCD$ is cyclic according to Theorem 6.4. □

The direct part of the last theorem was Problem 4 at the competition Tournament of the Towns in the fall of 2015 for senior A-level [2].

Theorem 6.6. *Let the extensions of opposite sides in a convex quadrilateral intersect at E and F, and the midpoints of the diagonals be M and N. Then*

$$\angle EMF + \angle ENF = \pi$$

if and only if it is a cyclic quadrilateral.

Proof. ($\Rightarrow$) In a cyclic quadrilateral $ABCD$, the pairs of triangles ECA and EBD, and also FAC and FBD are similar (AA), so $\angle EMA = \angle END$ and $\angle FMC = \angle FND$ (see Figure 14). Hence

$$\pi - \angle EMC = \angle ENF + \angle FND \quad \Rightarrow \quad \pi = \angle ENF + \angle EMC + \angle FMC$$

and we get $\pi = \angle ENF + \angle EMF$.

($\Leftarrow$) We do an indirect proof of the converse. If the vertex A in a convex quadrilateral $ABCD$ lies inside the circumcircle to triangle BCD, then $\angle EMA < \angle END$ and $\angle FMC > \angle FND$. Hence

$$\pi - \angle EMC < \angle ENF + \angle FND < \angle ENF + \angle FMC$$

and we get that $\pi < \angle ENF + \angle EMF$. In the case when the vertex A is outside the circle, this inequality is reversed, completing the proof. □

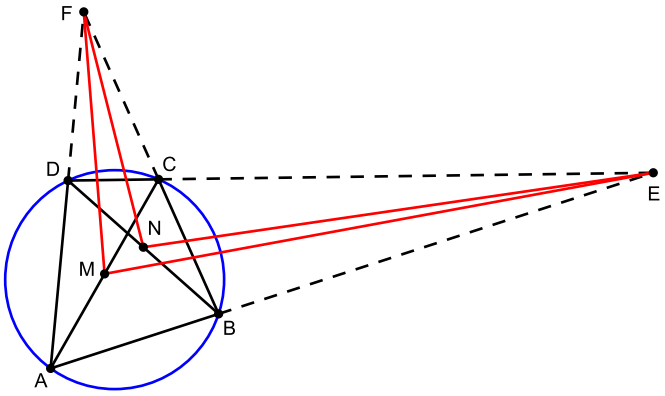

FIGURE 14. $ABCD$ is cyclic iff $\angle EMF + \angle ENF = \pi$

7. Concluding remark

Was this the end? What about Ptolemy's theorem some readers might object. Yes, it was deliberately excluded, but not forgotten. There will be a sequel to this paper covering for instance necessary and sufficient conditions of cyclic quadrilaterals concerning the diagonals.

Appendix A. Proofs of the five basic characterizations

Here we prove the five well-known characterizations of cyclic quadrilaterals that were stated after the introduction.

Theorem A.1. *The angle between a side and a diagonal in a convex quadrilateral is equal to the angle between the opposite side and the other diagonal if and only if it is a cyclic quadrilateral.*

Proof. ($\Rightarrow$) There are several similar cases; we prove that $\angle BAC = \angle BDC$ if and only if $ABCD$ is cyclic. In a cyclic quadrilateral, this angle equality is a direct corollary to the inscribed angle theorem.

($\Leftarrow$) If $\angle BAC = \angle BDC$, we can prove that the quadrilateral is cyclic by an indirect proof. Assume it is not cyclic, but triangle ABC always has a circumcircle. The vertex D lies either outside this circle or inside it. Assume first it is outside. The line BD cuts the circle at a point D' and so $ABCD'$ is cyclic (see Figure 15). Then $\angle BAC = \angle BD'C$ by the direct part of the theorem and therefor $\angle BDC = \angle BD'C$. By the exterior angle theorem, we have $\angle BD'C = \angle BDC + \angle DCD'$. But this is a contradiction to $\angle BDC = \angle BD'C$, so the assumption that D lies outside the circle is incorrect. Likewise we prove that D cannot be inside the circle, hence it's on it and $ABCD$ is cyclic. □

Theorem A.2. *In a convex quadrilateral, the four perpendicular bisectors to the sides are concurrent if and only if it is cyclic.*

Proof. ($\Rightarrow$) In a cyclic quadrilateral $ABCD$ with circumcenter O, triangles AOB, BOC, COD and DOA are all isosceles since $AO = BO = CO = DO$. Thus the perpendicular bisectors to the sides are concurrent in O.

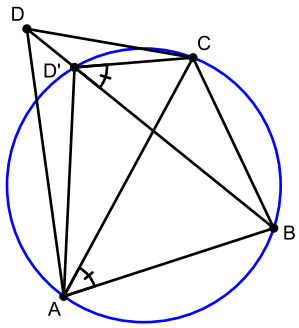

FIGURE 15. $ABCD$ is cyclic iff $\angle BAC = \angle BDC$

($\Leftarrow$) In a convex quadrilateral $ABCD$ where the perpendicular bisectors to the sides are concurrent in a point, say O, we have that triangles AOB, BOC, COD and DOA are all isosceles. Indeed, take for instance AOB and let M_a be the midpoint of AB. Then triangles AM_aO and BM_aO are congruent (SAS), so $AO = BO$ (see Figure 16). In the same way $AO = BO = CO = DO$ which proves that O is the circumcenter in a cyclic quadrilateral $ABCD$. □

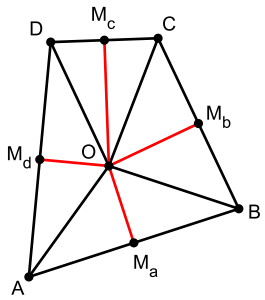

FIGURE 16. The concurrent perpendicular bisectors

Theorem A.3. *Any two opposite angles of a convex quadrilateral are supplementary if and only if it is a cyclic quadrilateral.*

Proof. ($\Rightarrow$) If quadrilateral $ABCD$ is cyclic with circumcenter O and if the angles at the center are $\mathscr{B}$ and $\mathscr{D}$, then by the inscribed angle theorem $\mathscr{B} = 2\angle B$ and $\mathscr{D} = 2\angle D$. Thus $2(\angle B + \angle D) = \mathscr{B} + \mathscr{D} = 2\pi$ so $\angle B + \angle D = \pi$. The other equality $\angle A + \angle C = \pi$ can be derived in the same way, or it follows from the sum of angles in a quadrilateral.

($\Leftarrow$) If $\angle B + \angle D = \pi$ we can prove that the quadrilateral is cyclic by an indirect proof. Assume that it is not cyclic, but triangle ABC has a circumcircle. The vertex D lies either outside this circle or inside it. Assume this time it is inside. Extend CD to D' on the circle (see Figure 17). Then $ABCD'$ is a cyclic quadrilateral and by the direct part of the theorem, we get $\angle B + \angle D' = \pi$. That means $\angle D' = \angle D$, but according to the exterior angle theorem $\angle D = \angle D' + \angle DAD'$, which is a contradiction. In the same

way it can be proved that D is not outside the circle, hence it is on it and $ABCD$ is cyclic. □

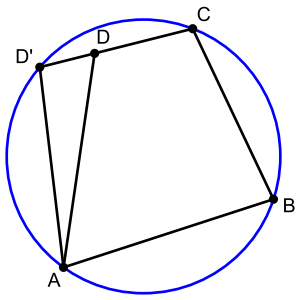

FIGURE 17. $ABCD$ is cyclic iff $\angle B + \angle D = \pi$

Theorem A.4. *An exterior angle to a convex quadrilateral is equal to the opposite interior angle if and only if it is a cyclic quadrilateral.*

Proof. If β is the exterior angle to B, then $\beta + \angle B = \pi$. The convex quadrilateral $ABCD$ is cyclic if and only if

$$\angle B + \angle D = \pi \quad \Leftrightarrow \quad \angle B + \angle D = \beta + \angle B \quad \Leftrightarrow \quad \angle D = \beta$$

and the proof is complete. □

Theorem A.5. *The four vertices A, B, C and D of a convex quadrilateral satisfy the equation*

$$AP \cdot PB = DP \cdot PC$$

if and only if it is a cyclic quadrilateral, where P is the intersection of AB and DC or their extensions.

Proof. ($\Rightarrow$) We consider only the case where the chords intersect outside the quadrilateral; the other case is similar. In a cyclic quadrilateral $ABCD$, $\angle CAB = \angle CDB$ according to Theorem A.1, so triangles CAP and BDP are similar (AA) since the angle at P is common to these triangles (see Figure 18). Then

$$\frac{AP}{DP} = \frac{PC}{PB} \tag{9}$$

and it follows that $AP \cdot PB = DP \cdot PC$.

($\Leftarrow$) If $AP \cdot PB = DP \cdot PC$ in a convex quadrilateral, then (9) holds. Since the angle at P is common to triangles CAP and BDP, they are similar (SAS). Thus $\angle CAB = \angle CDB$, and so $ABCD$ is a cyclic quadrilateral according to the converse in Theorem A.1. □

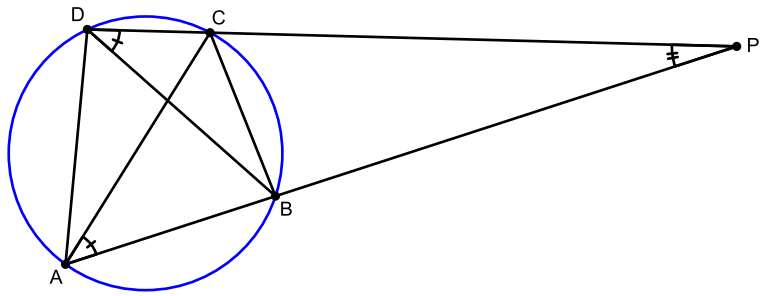

FIGURE 18. $ABCD$ is cyclic iff $AP \cdot PB = DP \cdot PC$

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