INTERNATIONAL JOURNAL OF GEOMETRY
Vol.7 (2018), No. 2, 72 - 86

FROM BÉZIER CURVES AND T-BÉZIER CURVES TO SUPERQUADRICS

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Abstract. In this paper are presented some constructions of superquadrics using the classical Bézier curves and also a new type of curves, called T-Bézier curves. The T-Bézier curves have been discovered recently and are also called trigonometric Bézier curves.

1. Introduction

In paper [2] are presented the classical Bézier curves which are defined in the following way:

$$P(t)=\sum_{k=0}^{n} B_{k,n}(t)\cdot P_k \tag{1}$$

where $B_{k,n}(t)$ are the classical Bernstein polynomials:

$$B_{k,n}(t)=\sum_{k=0}^{n} C_n^k t^k(1-t)^{n-k} \tag{2}$$

and P_k $(k=\overline{0,n})$ are the control points of the curve. The quadratic Bézier curve is obtained, when we choose $n=2$ in the above relation (1).

The quadratic Bézier curves have the following representation:

$$P(t)=C_2^0t^2\cdot P_0+C_2^1t(1-t)P_1+C_2^2(1-t)^2P_2=t^2\cdot P_0+2t(1-t)P_1+(1-t)^2P_2$$

where P_0, P_1, P_2 are the control points.

The cubic Bézier curve (obtained for $n=3$) have the following representation:

$$P(t)=C_3^0t^3\cdot P_0+C_3^1t^2(1-t)P_1+C_3^2t(1-t)^2P_2+C_3^3(1-t)^3P_3=$$
$$=t^3\cdot P_0+3t^2(1-t)P_1+3t(1-t)^2P_2+(1-t)^3\cdot P_3$$

where P_0, P_1, P_2, P_3 are the control points of the curve.

Date: Received: 1.09.2017. In revised form: 13.06.2018. Accepted: 10.07.2018.
2010 *Mathematics Subject Classification.* 53A04, 53A05.
Key words and phrases. Bézier curves, T-Bézier curves, superquadrics.

In paper [1] are presented the T-Bézier curves, also called trigonometric Bézier curves.

In paper [1], the 2T-Bézier curves (obtained for $n = 2$) are defined in the following way:

$$p_2(t) = B_0(t) \cdot P_0 + B_1(t) \cdot P_1 + B_2(t) \cdot P_2, \quad 0 \leq t \leq \frac{\pi}{2} \tag{3}$$

where $B_0(t) = 1 - \sin t$, $B_1(t) = -1 + \sin t + \cos t$, $B_2(t) = 1 - \cos t$.

The equation (3) may be rewritten (see [1]):

$$p_2(t) = (1 - \sin t)P_0 + (-1 + \sin t + \cos t)P_1 + (1 - \cos t)P_2 \tag{4}$$

with $P_i (i = 0, 1, 2)$ the control points of the curve and $t \in [0, \frac{\pi}{2}]$. For more details please see paper [1].

The 3T-Bézier curve (obtained also in paper [1], for $n = 3$) is:

$$p_3(t) = (1,5 - 2\sin t - 0,5\cos 2t)P_0 + (-1 + \sin t + \cos 2t)P_1 +$$

$$+(-1 + 2\cos t - \cos 2t)P_2 + (1,5 - 2\cos t + 0,5\cos 2t)P_3 \tag{5}$$

where $t \in [0, \frac{\pi}{2}]$ and $P_i (i = 0, 1, 2, 3)$ are the control points of the curve.

More generally, the T-Bézier curves can be written in the following way (see [1]):

$$p(t) = \sum_{k=0}^{n} p_{k,n}(t) \cdot P_k \tag{6}$$

where $p_{k,n}(t)$ are the trigonometric polynomials and P_k $(k = 0, 1, ..., n)$ are the control points of the curve.

The classical rational Bézier curves are defined using the classical Bézier curves in the following way (see [2]):

$$R(t) = \frac{\sum\limits_{k=0}^{n} B_{k,n}(t) \cdot w_k \cdot P_k}{\sum\limits_{k=0}^{n} B_{k,n}(t) \cdot w_k} \tag{7}$$

where $B_{k,n}(t)$ are the classical Bernstein polynomials, w_k are the weights and P_k $(k = 0, 1, ..., n)$ are the control points of the curve.

The T-Bézier rational curves can be defined in the following way:

$$R(t) = \frac{\sum\limits_{k-0}^{n} p_{k,n}(t) \cdot w_k \cdot P_k}{\sum\limits_{k=0}^{n} p_{k,n}(t) \cdot w_k} \tag{8}$$

where $p_{k,n}(t)$ are the trigonometric polynomials and P_k are the control points.

A superquadric is an surface with the following implicit equation:

$$\left[\left(\frac{x}{a_1}\right)^{\frac{2}{\varepsilon_2}} + \left(\frac{y}{a_2}\right)^{\frac{2}{\varepsilon_2}}\right]^{\frac{\varepsilon_1}{\varepsilon_2}} + \left(\frac{z}{a_3}\right)^{\frac{2}{\varepsilon_1}} = 1 \tag{9}$$

(9)

where a_1, a_2, a_3 are constants.

In this paper we will choose $a_1 = a_2 = a_3 = 1$; so we will work with the following reduced equation of the superquadrics:

$$\left[(x)^{\frac{2}{\varepsilon_2}} + (y)^{\frac{2}{\varepsilon_2}}\right]^{\frac{\varepsilon_1}{\varepsilon_2}} + z^{\frac{2}{\varepsilon_1}} = 1 \tag{10}$$

Remark 1.1. *The classical quadratic and cubic rational Bézier curves have the following form:*

$$R_2(t) = \frac{C_2^0 t^2 \cdot w_0 \cdot P_0 + C_2^1 t(1-t)w_1P_1 + C_2^2(1-t)^2 \cdot w_2P_2}{C_2^0 t^2 \cdot w_0 + C_2^1 t(1-t)w_1 + C_2^2(1-t)^2 w_2} =$$

$$\frac{t^2 w_0 P_0 + 2t(1-t)w_1P_1 + (1-t)^2 w_2 P_2}{t^2 \cdot w_0 + 2t(1-t)w_1 + (1-t)^2 w_2} \tag{11}$$

and respectivelly:

$$R_3(t) = \frac{C_3^0 t^3 w_0 P_0 + C_3^1 t^2(1-t)w_1P_1 + C_3^2 t(1-t)^2 w_2P_2 + C_3^3(1-t)^3 w_3P_3}{C_3^0 t^3 w_0 + C_3^1 t^2(1-t)w_1 + C_3^2 t(1-t)^2 w_2 + C_3^3(1-t)^3 w_3} =$$

$$\frac{t^3 w_0 P_0 + 3t^2(1-t)w_1P_1 + 3t(1-t)^2 w_2P_2 + (1-t)^3 w_3P_3}{t^3 \cdot w_0 + 3t^2(1-t)w_1 + 3t(1-t)^2 w_2 + (1-t)^3 w_3} \tag{12}$$

Remark 1.2. *The quadratic and respectivelly the cubic T-Bézier rational curves can be obtained using (4) and (5) in the following way:*

$$r_2(t) = \frac{(1-\sin t)w_0P_0 + (-1+\sin t+\cos t)w_1P_1 + (1-\cos t)w_2P_2}{(1-\sin t)w_0 + (-1+\sin t+\cos t)w_1 + (1-\cos t)w_2} \tag{13}$$

for 2T-Bézier rational curve, and

$$r_3(t) = \frac{b_0(t)w_0P_0 + b_1(t)w_1P_1 + b_2(t)w_2P_2 + b_3(t)w_3P_3}{b_0(t)w_0 + b_1(t)w_1 + b_2(t)w_2 + b_3(t)w_3} \tag{14}$$

with $b_0(t) = 1,5 - 2\sin t - 0,5\cos 2t$, $b_1(t) = -1 + 2\sin t + \cos 2t$, $b_2(t) = -1 + 2\cos t - \cos 2t$, $b_3(t) = 1,5 - 2\cos t + 0,5\cos 2t$ for the 3T-Bézier rational curve. Encouraged by the results from paper [3], we decide to use the above mentionated rational Bézier curves to obtain superquadrics.

2. Main Results

The main goal of this paper is to obtain some types of superquadrics using the classical rational Bézier curves and also the T-rational Bézier curves and replacing them in the exponential part of the implicit equation of one superquadric.This new type of superquadrics will be called T-superquadrics. First, let us consider the following equation of the superquadric:

$$\left[(x)^{\frac{2}{\varepsilon_2}} + (y)^{\frac{2}{\varepsilon_2}}\right]^{\frac{\varepsilon_1}{\varepsilon_2}} + (z)^{\frac{2}{\varepsilon_1}} = 1$$

For the exponential part of this superquadric, ε_1 and ε_2, let us consider the following:

$$\varepsilon_1 = \frac{\sum\limits_{k=0}^{n} p_{k,n}(t) \cdot w_k \cdot f(P_k)}{\sum\limits_{k=0}^{n} p_{k,n}(t) \cdot w_k} \tag{15}$$

where w_k are the weights of the T-rational Bézier curves defined in (8), $p_{k,n}(t)$ are the trigonometric polynomials; P_k are the control points and let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_i, y_i) = \sqrt{(x_i)^2 + (y_i)^2}$, $(x_i, y_i) \in \mathbb{R}^2$.

$$\varepsilon_2 = \frac{\sum\limits_{k=0}^{n} B_{k,n}(t) \cdot w_k \cdot f(P_k)}{\sum\limits_{k=0}^{n} B_{k,n}(t) \cdot w_k} \tag{16}$$

where w_k are the weights of the classical rational Bézier curves defined in (7), $B_{k,n}(t)$ are the classical Bernstein polynominals and P_k are the control points. Again we will consider $f : \mathbb{R}^2 \to \mathbb{R}$,$f(P_k) = \sqrt{(x_i)^2 + (y_i)^2}$, $(x_i, y_i) \in \mathbb{R}^2$.

Definition 2.1. *The implicit equation of one T-superquadric* $[x^{\eta_2} + y^{\eta_2}]^{\frac{\eta_1}{\eta_2}} + z^{\eta_1} = 1$ *where*

$$\eta_2 = \frac{2\sum\limits_{k=0}^{n} B_{k,n}(t) w_k}{\sum\limits_{k=0}^{n} B_{k,n}(t) \cdot w_k \cdot f(P_k)}$$

is presented in (16) and

$$\eta_1 = \frac{2\sum\limits_{k=0}^{n} P_{k,n}(t)w_k}{\sum\limits_{k=0}^{n} P_{k,n}(t)\cdot w_k \cdot f(P_k)}$$

is presented in (15) and we choose the same weights for the both curves.

Theorem 2.1. *The 2T-superquadric (obtained for $n=2$) have the following equation:*

$$\left[x^{\frac{2}{\eta_2}} + y^{\frac{2}{\eta_2}}\right]^{\frac{\eta_1}{\eta_2}} + z^{\frac{2}{\eta_1}} = 1 \tag{17}$$

where

$$\frac{2}{\eta_2} = \frac{2\left(t^2\omega_0 + 2t\left(1-t\right)\omega_1 + \left(1-t\right)^2\cdot\omega_2\right)}{t^2\omega_0\cdot f\left(P_0\right) + 2t\left(1-t\right)\omega_1\cdot f\left(P_1\right) + (1-t)^2\omega_2\cdot f\left(P_2\right)}$$

and

$$\frac{2}{\eta_1} = \frac{2[(1-\sin t)\,\omega_0 + (-1+\sin t+\cos t)\,\omega_1 + (1-\cos t)\omega_2]}{(1-\sin t)\omega_0\cdot f\left(P_0\right) + (-1+\sin t+\cos t)\,\omega_1\cdot f\left(P_1\right) + (1-\cos t)\omega_2\cdot f\left(P_2\right)}$$

Proof.[Proof] Using definition 2.1. and also the expresions of the T-rational Bėzier curves and classical Bėzier rational curves for $n=2$ one obtains:

- for the classical Bėzier rational curve:

$$P\left(t\right) = \frac{C_n^0 t^2\cdot\omega_0 f(P_0) + C_n^1 t(1-t)\omega_1\cdot f\left(P_1\right) + C_n^2(1-t)^2\omega_2\cdot f(P_2)}{C_n^0 t^2\cdot\omega_0 + C_n^1\cdot t(1-t)\omega_1 + C_n^2(1-t)^2\omega_2} \tag{18}$$

where ω_i, $i=0,1,2$, are the weights, $f\left(P_i\right)$ are the values of the factor $f(x_i,y_i) = \sqrt{x_i^2+y_i^2}$;

- for the T-Bėzier rational curve:

$$P_1\left(t\right) = \frac{(1-\sin t)\omega_0\cdot f\left(P_0\right) + (-1+\sin t+\cos t)\omega_1\cdot f\left(P_1\right) + (1-\cos t)\omega_2\cdot f(P_2)}{(1-\sin t)\omega_0 + (-1+\sin t+\cos t)\omega_1 + (1-\cos t)\omega_1} \tag{19}$$

Replacing this two expressions (18) and (19) in (17), we got the assertion of the theorem.

Theorem 2.2. *The 3T-superquadric, has the following implicit equation:*

$$\left[x^{\frac{2}{\xi_2}} + y^{\frac{2}{\xi_2}}\right]^{\frac{\xi_1}{\xi_2}} + z^{\frac{2}{\xi_1}} = 1 \tag{20}$$

where,

$$\frac{2}{\xi_2} = \frac{2\left(t^3\cdot\omega_0 + 3t^2(1-t)\omega_1 + 3t(1-t)^2\omega_2 + (1-t)^3\cdot\omega_3\right)}{t^3\cdot\omega_0\cdot f(P_0) + 3t^2(1-t)\omega_1\cdot f\left(P_1\right) + 3t(1-t)^2\cdot\omega_2\cdot f(P_2) + (1-t)^3\omega\cdot f\left(P_3\right)} \tag{21}$$

and

$$\frac{2}{\xi_1}=\frac{2\left[q_0\omega_0+q_1\omega_1+q_2\omega_2+q_3\omega_3\right]}{q_0\omega_0\cdot f\left(P_0\right)+q_1\omega_1\cdot f(P_1)+q_2\omega_2\cdot f(P_2)+q_3\omega_3\cdot f(P_3)}$$

where

$$q_0=\left(1,5-2\sin t-0,5\cos 2t\right),$$

$$q_1=(-1+2\sin t+\cos 2t),$$

$$q_2=(-1+2\cos t-\cos 2t),$$

$$q_3=(1,5-2\cos t+0,5\cos 2t).$$

Proof. Using definition 2.1 and also the expresion of the T-rational Bèzier curves and classical rational Bèzier curves (12) and (14) and replacing them in (20), one obtains the assertion of the theorem.

Next, we will construct some examples for 2T superquadrics using T-rational Bèzier curves and quadratic classical rational Bèzier curves. In all the bellow examples, we use 3 exact zecimals for the evaluations.

Example 2.1. *In this example we will choose the following weights $\omega_0=1$, $\omega_1=0,5$, $\omega_2=0,25$ and for the 2T rational Bèzier curves, we will take $t=\frac{\pi}{3}\backsimeq 1,047$ and then $t=\frac{\pi}{4}\backsimeq 0,785$. For the control points, we will take: $P_0\left(1,1\right),\ P_1\left(4,4\right),\ P_2\left(7,1\right)$ and for the function $f\left(x,y\right)=\sqrt{x^2+y^2}$, we will obtain:*

$$f\left(P_0\right)=\sqrt{2}\backsimeq 1,414, f\left(P_1\right)=\sqrt{32}\backsimeq 5,656, f(P_2)=\sqrt{50}\backsimeq 7,071$$

Replacing in (18) and (19) we get:

$$\frac{2}{\eta_1}=\frac{2\left[\left(1-\sin\frac{\pi}{3}\right)\cdot 1+\left(-1+\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)\cdot 0,5+\left(1-\frac{1}{2}\right)\cdot 0,25\right]}{\left(1-\sin\frac{\pi}{3}\right)\cdot 1,414\cdot f\left(P_0\right)+\left(-1+\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)\cdot 0,5\cdot f\left(P_1\right)+\left(1-\frac{1}{2}\right)\cdot 0,25\cdot f\left(P_2\right)}$$

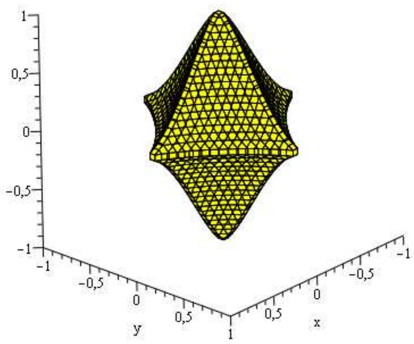
$$=\frac{2\cdot(1-0,866+0,1833+0,125)}{1,414\cdot(1-0,866)+1,035+0,883}=\frac{0,884}{2,107}=0,419$$

$$\frac{2}{\eta_2}=\frac{2\left[(1,047)^2\cdot 1-2\cdot 1,047\cdot 0,047\cdot 0,5+(0,047)^2\cdot 0,25\right]}{(1,047)^2\cdot 1,414-2\cdot 1,047\cdot 0,047\cdot 0,5\cdot 5,656+(0,047)^2\cdot 7,071\cdot 0,25}=\frac{1,047}{1,302}=0,804$$

So, for $t=\frac{\pi}{3}$, we get the following implicit equation for the 2T-superquadric:

$$\left[\left(x^2\right)^{0,402}+\left(y^2\right)^{0,402}\right]^{1,918}+\left(z^2\right)^{0,209}=1.$$

Bellow is the graphical reprezentation of these 2T-superquadric with yellow:



Next, using the same weights and the same control points, we will obtain the 2T superquadric for $t = \frac{\pi}{4} \backsimeq 0,785, f(P_0) \backsimeq 1,414,\ f(P_1) \backsimeq 5,656,$ $f(P_2) \backsimeq 7,071$

$$\frac{2}{\eta_1} = \frac{2\left[\left(1-\sin\frac{\pi}{4}\right)+\left(-1+\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\cdot 0,5+\left(1-\cos\frac{\pi}{4}\right)\cdot 0,25\right]}{\left(1-\sin\frac{\pi}{4}\right)\cdot 1,414+\left(-1+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)\cdot 0,5\cdot 5,656+\left(1-\frac{\sqrt{2}}{2}\right)\cdot 0,25\cdot 7,071} =$$

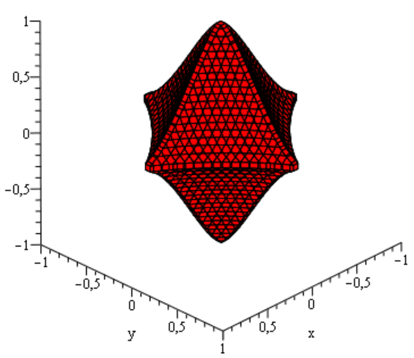
$$\frac{2\left(\frac{3}{4}-\frac{\sqrt{2}}{8}\right)}{0,414+1,170+0,517} = \frac{0,968}{2,101} = 0,46,$$

$$\frac{2}{\eta_2} = \frac{2\left[(0,785)^2+2\cdot 0,785\cdot 0,215\cdot 0,5+(0,215)^2\cdot 0,25\right]}{1,414\cdot(0,785)^2+2\cdot 0,785\cdot 0,215\cdot 0,5\cdot 5,656+(0,215)^2\cdot 0,25\cdot 7,071} = \frac{1,59}{1,906} = 0,834$$

The implicit equation of the 2T-superquadric will be:

$$\left[\left(x^2\right)^{0,417}+\left(y^2\right)^{0,417}\right]^{1,815}+\left(z^2\right)^{0,23}=1$$

The graph is plotted bellow with red



Example 2.2. *Let us consider now, the following weights:* $\omega_0 = 9$, $\omega_1 = 0,2$, $\omega_2 = 0,1$ *and* $t = \frac{\pi}{3} \backsimeq 1,047$

For the same control points: $P_0\,(1,1)$, $P_1\,(4,4)$, $P_2\,(7,1)$, one obtains:

$$\frac{2}{\eta_1} = \frac{2\left[\left(1-\frac{\sqrt{3}}{2}\right)\cdot 9 + \left(-1+\frac{\sqrt{3}+1}{2}\right)\cdot 0,2 + \frac{1}{2}\cdot 0,1\right]}{1,414\cdot\left(1-\frac{\sqrt{3}}{2}\right)\cdot + \left(-1+\frac{\sqrt{3}+1}{2}\right)\cdot 0,2\cdot 5,656 + \frac{1}{2}\cdot 0,1\cdot 7,071} = \frac{2,658}{2,472} = 1,075$$

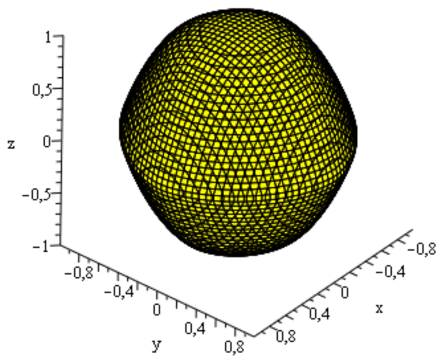
and

$$\frac{2}{\eta_2} = \frac{2\left[(1,047)^2\cdot 9 - 2\cdot 1,047\cdot 0,047\cdot 0,2 + (0,047)^2\cdot 0,1\right]}{1,414\cdot(1,047)^2\cdot 9 - 2\cdot 1,047\cdot 5,656\cdot 0,047\cdot 0,2 + (0,047)^2\cdot 0,1\cdot 7,071} = \frac{19,692}{11,582} = 1,70$$

so

$$\left[\left(x^2\right)^{0,85} + \left(y^2\right)^{0,85}\right]^{1,581} + \left(z^2\right)^{0,537} = 1$$

The graph is plotted bellow:



Now, for the same weights and the same control points, but for $t = \frac{\pi}{4} \backsimeq 0,785$, we obtain:

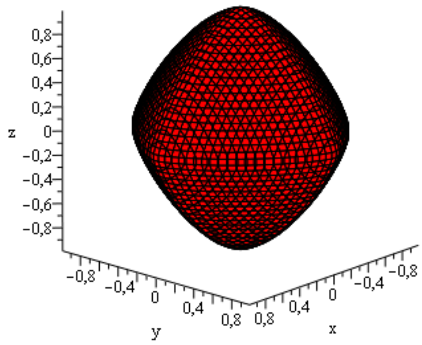
$$\frac{2}{\eta_1} = \frac{2\left[\left(1-\frac{\sqrt{2}}{2}\right)9 + (-1+\sqrt{2})\cdot 0,2 + \left(1-\frac{\sqrt{2}}{2}\right)\cdot 0,1\right]}{1,414\cdot\left(1-\frac{\sqrt{2}}{2}\right)\cdot 9 + \left(-1+\sqrt{2}\right)\cdot 0,2\cdot 5,656 + \left(1-\frac{\sqrt{2}}{2}\right)\cdot 0,1\cdot 7,071} = \frac{5,496}{4,403} = 1,248$$

$$\frac{2}{\eta_2} = \frac{2\left[(0,785)^2\cdot 9 + 2\cdot 0,785\cdot 0,215\cdot 0,2 + (0,215)^2\cdot 0,1\right]}{1,414\cdot(0,785)^2\cdot 9 + 2\cdot 0,785\cdot 0,215\cdot 0,2\cdot 5,656 + (0,215)^2\cdot 7,071\cdot 0,1} = 1,361$$

So, the 2T-superquadric has the following equation:

$$\left[\left(x^2\right)^{0,68}+\left(y^2\right)^{0,68}\right]^{1,09}+\left(z^2\right)^{0,624}=1$$

The graph of this 2T-superquadric is plotted with red:



Example 2.3. *In this example we will choose the following weights:* $\omega_0 = 3$, $\omega_1 = \omega_2 = 0,1$. *Also, we will take two values for* t. *First we take* $t = \frac{\pi}{3}$ *and then* $t = \frac{\pi}{4}$.

The control points are the same: $P_0\,(1,1)$, $P_1(4,4)$, $P_2(7,1)$.

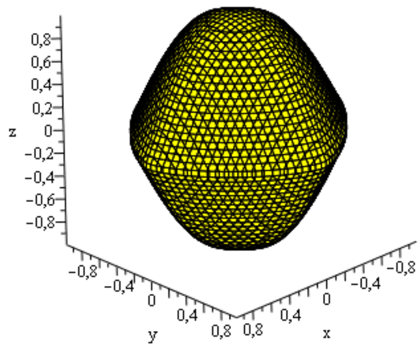
$$\frac{2}{\eta_1}=\frac{2\left[\left(1-\frac{\sqrt{3}}{2}\right)\cdot 3+\left(\frac{\sqrt{3}-1}{2}\right)\cdot 0,1+\frac{1}{2}\cdot 0,1\right]}{1,414\cdot\left(1-\frac{\sqrt{3}}{2}\right)\cdot 3+\left(\frac{\sqrt{3}-1}{2}\right)\cdot 0,1\cdot 5,656+\frac{1}{2}\cdot 0,1\cdot 7,071}=\frac{0,976}{1,128}=0,865$$

$$\frac{2}{\eta_2}=\frac{2\left[(1,047)^2\cdot 3-2\cdot 1,047\cdot 0,047\cdot 0,1+(0,047)^2\cdot 0,1\right]}{1,414\cdot(1,047)^2\cdot 3-2\cdot 1,047\cdot 0,047\cdot 0,1\cdot 5,656+(0,047)^2\cdot 0,1\cdot 7,071}=\frac{6,558}{4,596}=1,426$$

The equation of the 2T-superquadric will be:

$$\left[\left(x^2\right)^{0,713}+\left(y^2\right)^{0,713}\right]^{1,648}+\left(z^2\right)^{0,432}=1.$$

The graph is plotted bellow:



Now, for the same control points and the same weights, we will compute for $t = \frac{\pi}{4} \simeq 0,785$.

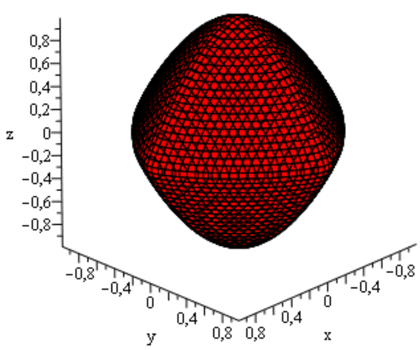
$$\frac{2}{\eta_1} = \frac{2 \cdot \left[\left(1-\frac{\sqrt{2}}{2}\right) \cdot 3 + \left(\sqrt{2}-1\right) \cdot 0,1 + \left(1-\frac{\sqrt{2}}{2}\right) \cdot 0,1\right]}{1,414 \cdot \left(1-\frac{\sqrt{2}}{2}\right) \cdot 3 + \left(\sqrt{2}-1\right) \cdot 0,1 \cdot 5,656 + \left(1-\frac{\sqrt{2}}{2}\right) \cdot 0,1 \cdot 7,071} = \frac{1,898}{1,683} = 1,127$$

$$\frac{2}{\eta_2} = \frac{2 \cdot \left[(0,785)^2 \cdot 3 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,1 + (0,215)^2 \cdot 0,1\right]}{1,414 \cdot (0,785)^2 \cdot 3 + 2 \cdot 0,785 \cdot 0,215 \cdot 0.1 \cdot 5,656 + (0,215)^2 \cdot 0,1 \cdot 7,071} = \frac{3,77}{2,831} = 1,331$$

One obtains:

$$\left[\left(x^2\right)^{0,665} + \left(y^2\right)^{0,665}\right]^{1,181} + \left(z^2\right)^{0,563} = 1$$

The graph is plotted with red:



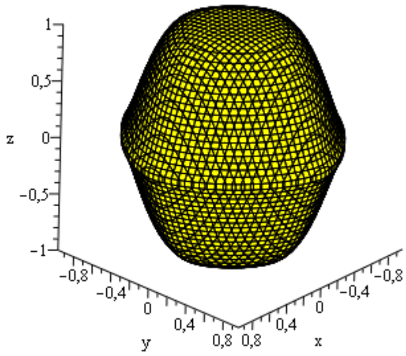
Example 2.4. *At this example and also at the following 2 next examples we will choose the following control points:* $P_0\,(1,1)$, $P_1(1,5;4,5)$, $P_2(4,2)$ *for the T-rational Bézier curves. First, we choose the following weights:* $w_0 = 1$, $w_1 = 0,5$, $w_2 = 0,25$ *and* $t = \frac{\pi}{3}$*. for the control points* P_0, P_1, P_2 *one obtains:* $f(P_0) = \sqrt{2} \simeq 1,414$, $f(P_1) = \sqrt{22,5} \simeq 4,473$

$f(P_2) = \sqrt{20} \simeq 4,472$

$$\frac{2}{\eta_1} = \frac{2 \cdot \left[\left(1-\frac{\sqrt{3}}{2}\right)\cdot 1 + \left(-1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\cdot 0,5 + \left(1-\frac{1}{2}\right)\cdot 0,25\right]}{\left(1-\frac{\sqrt{3}}{2}\right)\cdot 1,414 + \left(-1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\cdot 0,5 \cdot 4,743 + \frac{1}{2}\cdot 0,85 \cdot 4,472} = \frac{0,884}{1,615} = 0,547$$

$$\frac{2}{\eta_2} = \frac{2 \cdot \left[(1,047)^2 - 2 \cdot 1,047 \cdot 0,047 \cdot 0,5 + (0,047)^2 \cdot 0,5\right]}{1,414 \cdot (1,047)^2 - 2 \cdot 1,047 \cdot 0,5 \cdot 4,743 + (0,047)^2 \cdot 0,25 \cdot 4,472} = \frac{2,096}{1,318} = 1,59$$

So, one obtains: $\left[\left(x^2\right)^{0,795} + \left(y^2\right)^{0,795}\right]^{2,906} + \left(z^2\right)^{0,273} = 1$
The graph is plotted bellow:

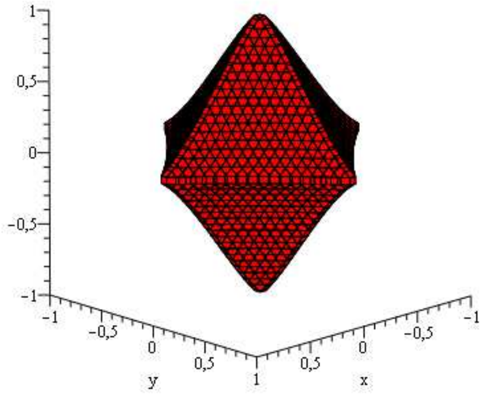


Now, for the same weights and the same control points, but for $t = \frac{\pi}{4}$, one obtains:

$$\frac{2}{\eta_1} = \frac{2 \cdot \left[1 - \frac{\sqrt{2}}{2} + \left(\sqrt{2}-1\right)\cdot\frac{1}{2} + \frac{1}{4} - \frac{\sqrt{2}}{8}\right]}{1,414 \cdot \left(1-\frac{\sqrt{2}}{2}\right) + \left(\sqrt{2}-1\right)\cdot 0,5 \cdot 4,743 + \left(1-\frac{\sqrt{2}}{2}\right)\cdot 0,25 \cdot 4,472} = \frac{1,147}{0,722} = 0,666$$

$$\frac{2}{\eta_2} = \frac{2 \cdot \left[(0.785)^2 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,5 + (0,215)^2 \cdot 0,25\right]}{1,414 \cdot (0,785)^2 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,5 \cdot 4,473 + (0,215)^2 \cdot 0,25 \cdot 4,472} = \frac{1,59}{1,722} = 0,923$$

$\Longrightarrow \left[\left(x^2\right)^{0,461} + \left(y^2\right)^{0,461}\right]^{1,392} + \left(z^2\right)^{0,333} = 1$
The graph is plotted with red:



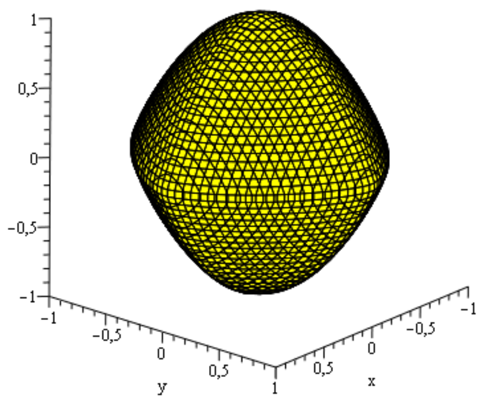
Example 2.5. *Now let us consider the following weights:* $w_0 = 9$, $w_1 = 0,2$, $w_2 = 0,1$ *and the same control points:* $P_0\,(1,1)\,,\ P_1\,(1,5;4,5)\,,\ P_2\,(4,2)$

First we compute the implicit equation of the 2T - superquardric for $t = \frac{\pi}{3}$:

$$\frac{2}{\eta_1} = \frac{2\cdot\left[\left(1-\frac{\sqrt{3}}{2}\right)\cdot 9 + \left(-1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\cdot 0,2 + \frac{1}{2}\cdot 0,1\right]}{\left(1-\frac{\sqrt{3}}{2}\right)\cdot 1,414\cdot 9 + \left(-1+\frac{\sqrt{3}}{2}+\frac{1}{2}\right)\cdot 0,2\cdot 4,769 + \frac{1}{2}\cdot 0,1\cdot 4,472} = \frac{2,658}{2,285} = 1,163$$

$$\frac{2}{\eta_2} = \frac{2\cdot\left[(1,047)^2\cdot 9 - 2\cdot 1,047\cdot 0,047\cdot 0,2 + (0,047)^2\cdot 0,1\right]}{1,414\cdot 1,096 - 2\cdot 1,047\cdot 0,047\cdot 0,2\cdot 4,743 + (0,047)^2\cdot 0,1\cdot 4,472} = \frac{19,69}{13,854} = 1,42$$

$$\Longrightarrow \left[\left(x^2\right)^{0,71} + \left(y^2\right)^{0,71}\right]^{1,22} + \left(z^2\right)^{0,581} = 1$$

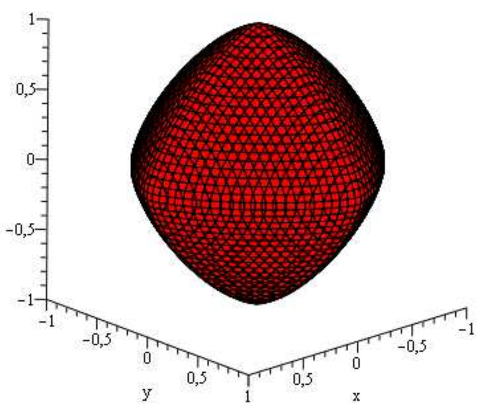


For $t = \frac{\pi}{4}$, one obtains for the same weights, the following results:

$$\frac{2}{\eta_1} = \frac{2 \cdot \left[\left(1 - \frac{\sqrt{2}}{2}\right) \cdot 9 + \left(\sqrt{2} - 1\right) \cdot 0,2 + \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 0,1\right]}{\left(1 - \frac{\sqrt{2}}{2}\right) \cdot 9 \cdot 1,414 + \left(\sqrt{2} - 1\right) \cdot 0,2 \cdot 4,743 + \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 0,1 \cdot 4,472} = \frac{5,896}{4,251} = 1,386$$

$$\frac{2}{\eta_2} = \frac{2\left[(0,785)^2 \cdot 9 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,2 + (0,215)^2 \cdot 0,1\right]}{(0,785)^2 \cdot 9 \cdot 1,414 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,2 \cdot 4,473 + (0,215)^2 \cdot 0,1 \cdot 4,472} = \frac{11,234}{8,182} = 1,373$$

$\Longrightarrow \left[\left(x^2\right)^{0,686} + \left(y^2\right)^{0,686}\right]^{0,99} + \left(z^2\right)^{0,693} = 1$
The graph is plotted with red:



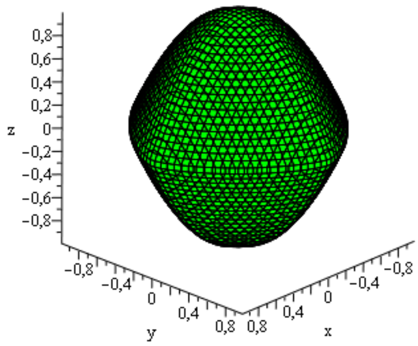
Example 2.6. *Now, we choose the following weights:* $w_0 = 3$, $w_1 = 0,1$, $w_2 = 0,1$ *and also the same control points* $P_0\,(1;1)$. $P_1\,(1,5;4,5)$, *and* $P_2\,(4;2)$.

We compute for $t = \frac{\pi}{3}$ and one obtains:

$$\frac{2}{\eta_1} = \frac{\left[\left(1 - \frac{\sqrt{3}}{2}\right) \cdot 3 + \left(\frac{\sqrt{3}-1}{2}\right) \cdot 0,1 + \frac{1}{2} \cdot 0,1\right]}{\left(1 - \frac{\sqrt{3}}{2}\right) \cdot 3 \cdot 1,414 + \left(\frac{\sqrt{3}-1}{2}\right) \cdot 0,1 \cdot 4,743 + \frac{1}{2} \cdot 4,472 \cdot 0,1} = \frac{0,976}{0,964} = 1,012$$

$$\frac{2}{\eta_2} = \frac{2 \cdot \left[(1,047)^2 \cdot 3 - 2 \cdot 1,47 \cdot 0,047 \cdot 0,1 + (0,047)^2 \cdot 0,1\right]}{(1,047)^2 \cdot 3 \cdot 1,414 - 2 \cdot 1,47 \cdot 0,047 \cdot 4,743 \cdot 0,1 + (0,047)^2 \cdot 4,472 \cdot 0,1} = \frac{6,558}{4,603} = 1,424$$

$\Longrightarrow \left[\left(x^2\right)^{0,712} + \left(y^2\right)^{0,712}\right]^{1,407} + \left(z^2\right)^{0,506} = 1$

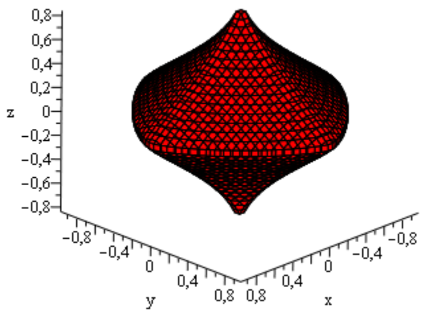


Now, for $t = \frac{\pi}{4} \simeq 0,785$, one obtains:

$$\frac{2}{\eta_2} = \frac{2 \cdot \left[(0,785)^2 \cdot 3 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,1 + (0,215)^2 \cdot 0,1\right]}{(0,785)^2 \cdot 3 \cdot 1,414 + 2 \cdot 0,785 \cdot 0,215 \cdot 0,1 \cdot 4,743 + (0,215)^2 \cdot 0,1 \cdot 4,472} = \frac{3,77}{2,794} = 1,349$$

$$\frac{2}{\eta_1} = \frac{2 \cdot \left[\left(1 - \frac{\sqrt{2}}{2}\right) \cdot 3 + \left(\sqrt{2} - 1\right) \cdot 0,1 + \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 0,1\right]}{\left(1 - \frac{\sqrt{2}}{2}\right) \cdot 3 \cdot 1,414 + \left(\sqrt{2} - 1\right) \cdot 0,1 \cdot 4,743 + \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 0,1 \cdot 4,472} = \frac{1,898}{1,57} = 1,208$$

$$\Longrightarrow \left[\left(x^2\right)^{0,674} + \left(y^2\right)^{0,674}\right]^{0,389} + \left(z^2\right)^{0,604} = 1$$



Conclusion 2.1. *In the previous examples we obtain $2T$ - superquadrics using the Bézier classical rational curves and $2T$ - Bézier rational curves. This surfaces could be studied from the Gauss curvature point of view. Some*

of them have the Gaussian curvature pozitive and some of them negative and could represent models in some type of geometries.

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