STRUCTURAL INVARIANCE OF RIGHT-ANGLE TRIANGLE UNDER ROTATION-SIMILARITY TRANSFORMATION

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Abstract. We consider relations in the predecessor-successor chain emerged at rotation-similarity transformation of right-angle triangle with an arbitrary leg ratio in 2-dimensional Euclidean space. Following this, we generalize structural ratios obtained earlier. We find conditions favoring to the structural invariance for above triangle in the predecessor-successor pair. Finally, we discuss image of above transformation and compare probability of folding/unfolding development for random transformation conditions.

1. Introduction

In [2], the family of right-angle triangles $\Delta AB_nC_n$ in 2-dimensional Euclidean space with leg ratio $AC/BC = 2^p$, where $p = \pm 1$, was considered (Figure 1).

It was shown that if to mark off the distance from the vertex $B_n$ along $AB_n$ as $B_nC_{n+1} = B_nC_n = B_nF_n$ then for $n \geq 1$, the following ratios are valid irrelevant to the sign of $\Delta S_n$

\begin{align*}
(1) & \quad \left( \frac{S_n^F}{2S_n^C} \right)^p = \left( \frac{2S_n^T}{S_n^C} \right)^p = \left( \frac{AF_n}{A_n} \right)^p = \left( \frac{AC_n}{AC_{n+1}} \right)^p = \phi^p, \\
(2) & \quad \eta = \frac{S_C}{S_{n+1}^F} = 1 \\
(3) & \quad AC_n = AC(n) \exp[g(\phi)b\theta]
\end{align*}

where $S_n$ is area of $\Delta AB_nC_n$, $S_n^F = S_n^C + 2S_n^T$ is area of $\Delta AF_nC_n$, $S_C$ is area of $\Delta AC_{n+1}C_n$, $S_n^T$ is area of equal triangles $\Delta C_nB_n$ and $C_nF_nB_n$, and $\phi$ is the golden ratio [1].

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So, ratio (2) or equivalent $S_{n+1}^C/S_n^F = 1$ declares invariance of the referenced area ratio under above transformation. In (3), $\Delta \theta_n = \angle C_nAB_n = \arctan(1/2)$, $g(\phi) = \phi - 2$ for folding logarithmic spiral ($\Delta S_n < 0$) and $g(\phi) = \phi - 1$ for unfolding one ($\Delta S_n > 0$), and $b = 2\pi/\arctan(1/2)$.

In this article, we generalize (1-3) for arbitrary rational $k = AC_n/B_nC_n$ and consider some additional features for family $\Delta AB_nC_n$ related to its random structuring.

Write down some relations from [1], we will be using further,

\begin{align}
(4) \quad f_k &= \frac{\sqrt{k^2 + 1} + 1}{k} \\
(5) \quad S_n^C &= \frac{(B_nC_n)^2}{2} \left( k - \sqrt{1 - \frac{1}{k^2 + 1}} \right) \\
(6) \quad S_n^T &= \frac{(B_nC_n)^2}{2} \sqrt{1 - \frac{1}{k^2 + 1}} \\
(7) \quad S_n^C &= \frac{(B_nC_n)^2}{2} \left( k + \sqrt{1 - \frac{1}{k^2 + 1}} \right)
\end{align}

where $f_k$ is a similarity ratio.

2. Generalization theorem

Formulate theorem generalizing (1-3) for any $k$ and prove its statements one by one.

**Theorem 2.1.** If in arbitrary right-angle $\Delta AB_nC_n$ with fixed leg ratio $AC_n/B_nC_n = k$, where $k$ is any rational number, to mark off from the vertex $B_n$ along hypotenuse $AB_n$ the line segments $B_nC_{n+1}$ and $B_nF_n$ with
the length equal to length of $B_nC_n$, then irrelevant to the sign of $\Delta S_n$, for any integer $n \geq 1$, where $\gamma = (4/k^2)^p$, $p = \pm 1$

$$
\left(\frac{AF_n}{AC_n}\right)^p = \left(\frac{AC_n}{AC_{n+1}}\right)^p,
$$

(8)

$$
\left(\frac{2S^T_n}{S^C_n}\right)^p = \gamma(k) \left(\frac{S^F_n}{2S^T_n}\right)^p;
$$

(9)

the full area $S^F_{n+1}(S^F_n)$ of the first immediate successor (predecessor) in the chain of similar triangles is exactly equal to the area $S^C_n(S^C_{n+1})$ of the core triangle in its immediate predecessor (successor), i.e.

$$
\eta = 1.
$$

(10)

**Proof.** (a) Relation (5.a) immediately follows from definition of $AC_n = kB_nC_n$, $AF_n = AB_n + B_nC_n = B_nC_n(\sqrt{k^2 + 1} + 1)$ and $AC_{n+1} = AB_n - B_nC_n(\sqrt{k^2 + 1} - 1)$.

Now, calculate $\gamma(k)$ using (4 - 7) as

$$
\gamma(k) = \left(\frac{2S^T_n}{S^C_n}\right)^p : \left(\frac{S^F_n}{2S^T_n}\right)^p = \left(\frac{4f^2}{2k_f + k^2}\right)^p = \left(\frac{4}{k^2}\right)^p;
$$

(11)

Now, prove (10). For specificity, consider case $p = 1$, the case $p = -1$ is proved by simple reversing of all the ratios.

Using (4),

$$
S^F_n = \frac{S^F_n}{f^2} = S^F_n \left(\frac{\sqrt{k^2 + 1} + 1}{k}\right)^{-2}
$$

(12)

then apply (7) which yields

$$
S^F_{n+1} = \frac{(B_nC_n)^2}{2} \left(k - \sqrt{1 - \frac{1}{k^2 + 1}}\right) = S^C_n
$$

(13)

that proves (10).

3. Distortion theorem

So far, we believed that $B_nC_n = B_nC_{n+1}$ and proved that at that condition, relation (10) is identity. Now, let the length $\alpha$ of $BC_{n+1}$ be random, i.e. generally $BC_{n+1} \neq B_nC_n$ and evaluate the measure of distortion $\beta$ between $S^F_{n+1}$ and $S^C_n$ ($S^F_n$ and $S^C_{n+1}$). From that, random $\alpha \in [-\sqrt{k^2 + 1}, 1]$ with origin in the vortex $B_n$ and the full range $R = AF = \sqrt{k^2 + 1} + 1$ with the subrange $R^- = k$ for the random event $\Delta S_n \leq 0$ and $R^+ = \sqrt{k^2 + 1} + 1 - k$ for the random event $\Delta S_n > 0$. Assuming uniform probability density distribution in $R$, calculate appropriate probability density function

$$
g^-(k) = \frac{R^-}{R} = \frac{k}{\sqrt{k^2 + 1} + 1}
$$

(14)
and
\begin{equation}
    g^+(k) = \frac{R^+}{R} = \frac{\sqrt{k^2 + 1} + 1}{\sqrt{k^2 + 1} + 1}. \tag{15}
\end{equation}

**Theorem 3.1.** If to supplement Theorem 2.1 by requirement of random $BC_{n+1} = \alpha$ (random selection of the point $C_{n+1}$) at condition of uniform probability density distribution, where $\alpha$ is random number in the range $[-\sqrt{k^2 + 1}, 1]$ with origin in the vortex $B_n$ then:

(a) distortion between area $S_{n+1}^F(S_n^F)$ of successor (predecessor) and $S_n^C(S_{n+1}^C)$ of predecessor (successor)

\begin{equation}
    \beta(\alpha, k) = 1 - \eta \tag{16}
\end{equation}
achieves minimum $\beta = 0$ at $\alpha = \pm 1$ at any $k$;

(b) at $AC_{n+1} = k$, the image of rotation-similarity transformation degenerates to circle with $\beta \neq 0$ at any $k$:

\begin{equation}
    AC_n = AC(n) \exp[g(\phi) b0] \tag{17}
\end{equation}

where $AC_1$ is assumed to be constant, $b = 2\pi/\arctan(1/k)$;

(c) probability of random event ($\Delta S_n < 0$) exceeds the one ($\Delta S_n \geq 0$),

\begin{equation}
    \frac{g^-(k)}{g^+(k)} \geq 1 \tag{18}
\end{equation}
at $f_k \leq 2$ ($k \geq 4/3$).

**Proof.** (a) Rewrite (5) assuming $\alpha$ to be random as

\begin{equation}
    S_n^C = \frac{(B_nC_n)^2}{2} \left(1 - \frac{\alpha}{\sqrt{k^2 + 1}}\right). \tag{19}
\end{equation}

Also, write down

\begin{equation}
    S_{n+1}^F = \frac{S_n^F}{f_k^2} = \frac{(B_nC_n)^2}{2} \frac{k^2 + 1 - \alpha^2}{\sqrt{k^2 + 1} k^2(\sqrt{k^2 + 1} + \alpha)}. \tag{20}
\end{equation}

Insert (19,20) to (16), so we have

\begin{equation}
    \beta = \frac{S_{n+1}^F - S_n^C}{S_{n+1}^F} = 1 - \frac{k^2 + 1 - \alpha^2}{k^2}. \tag{21}
\end{equation}

Function $\beta(\alpha, k)$ has minimum $\beta = 0$ at $\alpha = \pm 1$ ($B_nC_n = B_nC_{n+1}$) for any $k$. Plot $\beta(\alpha, k)$ for different $k$ is shown in Fig.2. So, invariance relation (10) matches (16) at $\alpha = \pm 1$ only. At $\alpha \neq \pm 1$, we should account distortion factor $1 - \beta$, i.e.

\begin{equation}
    \eta = 1 - \beta(\alpha, k). \tag{22}
\end{equation}

(b) If random $AC_{n+1} = k$, $|AC_n| = |AC_{n+1}|$, so (17) obviously describes a circle, where $|z|$ denotes module of number $z$. In $\Delta AB_nC_n$, random $B_nC_{n+1} = \alpha = AB_n - AC_{n+1}$, i.e.

\begin{equation}
    \alpha = B_nC_{n+1} = \sqrt{k^2 + 1} - k = 1 \tag{23}
\end{equation}
Solution (23) is $k = 0$, which means that $\alpha$ can never be $\pm 1$, so $\beta \neq 0$ for any $k$. It also directly follows from the well-known triangle inequality declaring that any side $(B_nC_n)$ of a triangle is greater than the difference $(AB_n - AC_n)$ between two other sides. (c) From (14, 15), compose ratio

$$\delta = \frac{g^-(k)}{g^+(k)} = \frac{k}{\sqrt{k^2 + 1 + 1} - k} \geq 1$$

Solution (24) is

$$1 \leq f_k \leq 2$$

or

$$k \geq \frac{4}{3}.$$

Plots for $\beta(\alpha, k)$ on $\alpha$ for three different $k$ and $\delta(k)$ are in Figure 2 and Figure 3, appropriately.

![Figure 2](image1)

**Figure 2.** Dependence measure of distortion $\beta(\alpha, k)$ on $\alpha$ for three different $k$.

![Figure 3](image2)

**Figure 3.** Dependence of relative probability for folding/unfolding development at random transformation conditions.
4. Conclusions and outlook

We have shown that the family of right-angle triangles with arbitrary legs ratio $k$ being under planar rotation-similarity transformation, demonstrates consistent internal structuring in the successor/predecessor chain at any transformation step $n$. In particular, we proved that there is exact equality between the full area of the successor (precursor) triangle and the core area of the built-in core triangle in its precursor (successor), i.e. quantity $\eta = 1$ is invariant under above transformation irrelevant to the sign of $\Delta S_n$.

In those cases when $\eta \neq 1$, it is possible to evaluate measure of distortion $\beta$ and adequately describe variations of the structure in the triangles family occurred under above transformation. In this sense, observe the smooth and continuous dependence of $\beta$ on proximity of selection point $\alpha$ to the vertexes $B_n$ and $C_n$.

Finally, note that probability of the triangles family $\Delta AB_nC_n$ to evolve in two possible directions, $\Delta S_n > 0$ and $\Delta S_n < 0$, is not the same. Whereas at the fast mode ($f_k > 2$), statistically preferable direction is the one with $\Delta S_n > 0$, at the slow mode ($f_k < 2$), the development with $\Delta S_n < 0$ will prevail thereby making preferable direction of development depending on $k$.

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References


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