INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 7 (2018), No. 2, 66 - 71

STRUCTURAL INVARIANCE OF RIGHT-ANGLE TRIANGLE UNDER ROTATION-SIMILARITY TRANSFORMATION

ANDREI MOLDAVANOV

Abstract. We consider relations in the predecessor-successor chain emerged at rotation-similarity transformation of right-angle triangle with an arbitrary leg ratio in 2-dimensional Euclidean space. Following this, we generalize structural ratios obtained earlier. We find conditions favoring to the structural invariance for above triangle in the predecessor-successor pair. Finally, we discuss image of above transformation and compare probability of folding/unfolding development for random transformation conditions.

1. Introduction

In [2], the family of right-angle triangles ΔAB_nC_n in 2-dimensional Euclidean space with leg ratio $AC/BC = 2^p$, where $p = \pm 1$, was considered (Figure 1).

It was shown that if to mark off the distance from the vertex B_n along AB_n as $B_nC_{n+1} = B_nC_n = B_nF_n$ then for n ≥ 1, the following ratios are valid irrelevant to the sign of ΔS_n

(1) $$\left(\frac{S_n^F}{2S_n^T}\right)^p = \left(\frac{2S_n^T}{S_n^C}\right)^p = \left(\frac{AF_n}{A_n}\right)^p = \left(\frac{AC_n}{AC_{n+1}}\right)^p = \phi^p,$$

(2) $$\eta = \frac{S_n^C}{S_{n+1}^F} = 1$$

(3) $$AC_n = AC(n)exp[g(\phi)b\theta]$$

where S_n is area of ΔAB_nC_n, $S_n^F = S_n^C + 2S_n^T$ is area of ΔAF_nC_n, S_n^C is area of $\Delta AC_{n+1}C_n$, S_n^T is area of equal triangles $\Delta C_nC_{n+1}B_n$ and $C_nF_nB_n$, and ϕ is the golden ratio [1].

Keywords and phrases: Area invariance, Planar Transformation, Right-angle triangle, Random Development
(2010)Mathematics Subject Classification: 51A99, 51M25
Received: 03.01.2018. In revised form: 27.09.2018. Accepted: 03.10.2018.

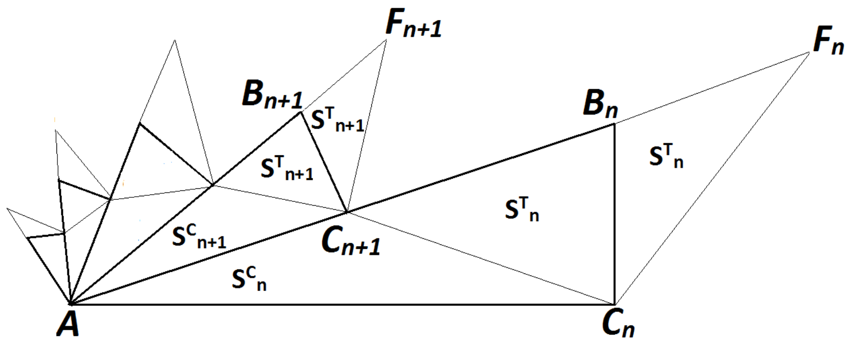

FIGURE 1. Family of right-angle triangles at planar rotation-similarity transformation.

So, ratio (2) or equivalent $S^C_{n+1}/S^F_n = 1$ declares invariance of the referenced area ratio under above transformation. In (3), $\Delta\theta_n = \angle C_nAB_n = arctan(1/2)$, $g(\phi) = \phi - 2$ for folding logarithmic spiral ($\Delta S_n < 0$) and $g(\phi) = \phi - 1$ for unfolding one ($\Delta S_n > 0$), and $b = 2\pi/arctan(1/2)$.

In this article, we generalize (1-3) for arbitrary rational $k = AC_n/B_nC_n$ and consider some additional features for family ΔAB_nC_n related to its random structuring.
Write down some relations from [1], we will be using further,

$$f_k = \frac{\sqrt{k^2+1}+1}{k} \tag{4}$$

$$S^C_n = \frac{(B_nC_n)^2}{2}\left(k - \sqrt{1 - \frac{1}{k^2+1}}\right) \tag{5}$$

$$S^T_n = \frac{(B_nC_n)^2}{2}\sqrt{1 - \frac{1}{k^2+1}} \tag{6}$$

$$S^C_n = \frac{(B_nC_n)^2}{2}\left(k + \sqrt{1 - \frac{1}{k^2+1}}\right) \tag{7}$$

where f_k is a similarity ratio.

2. Generalization theorem

Formulate theorem generalizing (1-3) for any k and prove its statements one by one.

Theorem 2.1. *If in arbitrary right-angle ΔAB_nC_n with fixed leg ratio $AC_n/B_nC_n = k$, where k is any rational number, to mark off from the vertex Bn along hypotenuse AB_n the line segments B_nC_{n+1} and B_nF_n with* *the length equal to length of B_nC_n, then irrelevant to the sign of ΔS_n, for any integer n $n \geq 1$, where $\gamma = (4/k^2)^p$, $p = \pm 1$*

$$\left(\frac{AF_n}{AC_n}\right)^p = \left(\frac{AC_n}{AC_{n+1}}\right)^p, \tag{8}$$

$$\left(\frac{2S_n^T}{S_n^C}\right)^p = \gamma(k)\left(\frac{S_n^F}{2S_n^T}\right)^p; \tag{9}$$

the full area $S_{n+1}^F (S_n^F)$ of the first immediate successor (predecessor) in the chain of similar triangles is exactly equal to the area $S_n^C (S_{n+1}^C)$ of the core triangle in its immediate predecessor (successor), i.e.

$$\eta = 1. \tag{10}$$

Proof. (*a*) *Relation (5.a) immediately follows from definition of $AC_n = kB_nC_n, AF_n = AB_n + B_nC_n = B_nC_n(\sqrt{k^2+1}+1)$ and $AC_{n+1} = AB_n - B_nC_n(\sqrt{k^2+1}-1)$.*
Now, calculate $\gamma(k)$ using (4 − 7) as

$$\gamma(k) = \left(\frac{2S_n^T}{S_n^C}\right)^p : \left(\frac{S_n^F}{2S_n^T}\right)^p = \left(\frac{4f_k^2}{2kf_k + k^2}\right)^p = \left(\frac{4}{k^2}\right)^p; \tag{11}$$

Now, prove (10). For specificity, consider case p = 1, the case p = -1 is proved by simple reversing of all the ratios.
Using (4),

$$S_n^F = \frac{S_n^F}{f_k^2} = S_n^F\left(\frac{\sqrt{k^2+1}+1}{k}\right)^{-2} \tag{12}$$

then apply (7) which yields

$$S_{n+1}^F = \frac{(B_nC_n)^2}{2}\left(k - \sqrt{1 - \frac{1}{k^2+1}}\right) = S_n^C \tag{13}$$

that proves (10).

3. Distortion theorem

So far, we believed that $B_nC_n = B_nC_{n+1}$ and proved that at that condition, relation (10) is identity. Now, let the length α of BC_{n+1} be random, *i.e.* generally $BC_{n+1} \neq B_nC_n$ and evaluate the measure of distortion β between S_{n+1}^F and S_n^C (S_n^F and S_{n+1}^C). From that, random $\alpha \in [-\sqrt{k^2+1}, 1]$ with origin in the vortex B_n and the full range $R = AF = \sqrt{k^2+1}+1$ with the subrange $R^- = k$ for the random event $\Delta S_n \leq 0$ and $R^+ = \sqrt{k^2+1}+1-k$ for the random event $\Delta S_n > 0$. Assuming uniform probability density distribution in R, calculate appropriate probability density function

$$g^-(k) = \frac{R^-}{R} = \frac{k}{\sqrt{k^2+1}+1} \tag{14}$$

and

$$g^{+}(k)=\frac{R^{+}}{R}=\frac{\sqrt{k^2+1}+1-k}{\sqrt{k^2+1}+1}. \tag{15}$$

Theorem 3.1. *If to supplement Theorem 2.1 by requirement of random $BC_{n+1}=\alpha$ (random selection of the point C_{n+1}) at condition of uniform probability density distribution, where α is random number in the range $[-\sqrt{k^2+1},1]$ with origin in the vortex B_n then:*

(a) distortion between area $S^F_{n+1}(S^F_n)$ of successor (predecessor) and $S^C_n(S^C_{n+1})$ of predecessor (successor)

$$\beta(\alpha,k)=1-\eta \tag{16}$$

achieves minimum $\beta=0$ at $\alpha=\pm 1$ at any k;

(b) at $AC_{n+1}=k$, the image of rotation-similarity transformation degenerates to circle with $\beta\neq 0$ at any k:

$$AC_n=AC(n)exp[g(\phi)b\theta] \tag{17}$$

where AC_1 is assumed to be constant, $b=2\pi/arctan(1/k)$;

(c) probability of random event ($\Delta S_n<0$) exceeds the one ($\Delta S_n\geq 0$), i.e.

$$\frac{g^{-}(k)}{g^{+}(k)}\geq 1 \tag{18}$$

at $f_k\leq 2$ ($k\geq 4/3$).

Proof. *(a) Rewrite (5) assuming α to be random as*

$$S^C_n=\frac{(B_nC_n)^2}{2}\left(1-\frac{\alpha}{\sqrt{k^2+1}}\right). \tag{19}$$

Also, write down

$$S^F_{n+1}=\frac{S^F_n}{f_k^2}=\frac{(B_nC_n)^2}{2}\frac{1}{\sqrt{k^2+1}}\frac{k^2+1-\alpha^2}{k^2(\sqrt{k^2+1}+\alpha)}. \tag{20}$$

Insert ($19,20$) to (16), so we have

$$\beta=\frac{S^F_{n+1}-S^C_n}{S^F_{n+1}}=1-\frac{k^2+1-\alpha^2}{k^2}. \tag{21}$$

Function $\beta(\alpha,k)$ has minimum $\beta=0$ at $\alpha=\pm 1(B_nC_n=B_nC_{n+1})$ for any k. Plot $\beta(\alpha,k)$ for different k is shown in Fig.2. So, invariance relation (10) matches (16) at $\alpha=\pm 1$ only. At $\alpha\neq\pm 1$, we should account distortion factor $1-\beta$, i.e.

$$\eta=1-\beta(\alpha,k). \tag{22}$$

(b) If random $AC_{n+1}=k$, $|AC_n|=|AC_{n+1}|$, so (17) obviously describes a circle, where $|z|$ denotes module of number z. In ΔAB_nC_n, random $B_nC_{n+1}=\alpha=AB_n-AC_{n+1}$, i.e.

$$\alpha=B_nC_{n+1}=\sqrt{k^2+1}-k=1 \tag{23}$$

Solution (23) is $k = 0$, which means that α can never be ± 1, so $\beta \neq 0$ for any k. It also directly follows from the well-known triangle inequality declaring that any side (B_nC_n) of a triangle is greater than the difference $(AB_n - AC_n)$ between two other sides.(c) From (14, 15), compose ratio

$$\delta = \frac{g^-(k)}{g^+(k)} = \frac{k}{\sqrt{k^2+1}+1-k} \geq 1 \tag{24}$$

Solution (24) is

$$1 \leq f_k \leq 2 \tag{25}$$

or

$$k \geq \frac{4}{3}. \tag{26}$$

Plots for $\beta(\alpha, k)$ on α for three different k and $\delta(k)$ are in Figure 2 and Figure 3, appropriately.

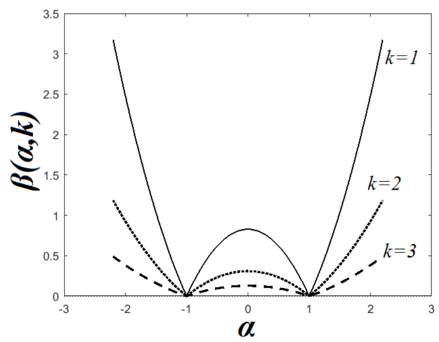

FIGURE 2. Dependence measure of distortion $\beta(\alpha, k)$ on α for three different k.

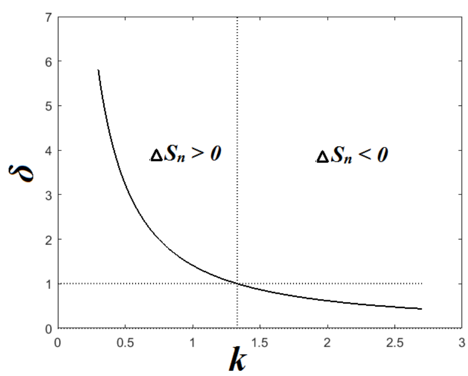

FIGURE 3. Dependence of relative probability for folding/unfolding development at random transformation conditions.

4. Conclusions and outlook

We have shown that the family of right-angle triangles with arbitrary legs ratio k being under planar rotation-similarity transformation, demonstrates consistent internal structuring in the successor/predecessor chain at any transformation step n. In particular, we proved that there is exact equality between the full area of the successor (precursor) triangle and the core area of the built-in core triangle in its precursor (successor), *i.e.* quantity $\eta = 1$ is invariant under above transformation irrelevant to the sign of ΔS_n.
In those cases when $\eta \neq 1$, it is possible to evaluate measure of distortion β and adequately describe variations of the structure in the triangles family occurred under above transformation. In this sense, observe the smooth and continuous dependence of β on proximity of selection point α to the vertexes B_n and C_n.
Finally, note that probability of the triangles family ΔAB_nC_n to evolve in two possible directions, $\Delta S_n > 0$ and $\Delta S_n < 0$, is not the same. Whereas at the fast mode ($f_k > 2$), statistically preferable direction is the one with $\Delta S_n > 0$, at the slow mode ($f_k < 2$), the development with $\Delta S_n < 0$ will prevail will prevail thereby making preferable direction of development depending on k.

5. Acknowledgements

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

References

[1] Bogomolny, A., Golden Ratio in Geometry, http://www.cut-the-knot.org/doyouknow=GoldenRatio.
[2] Moldavanov, A., Classical Right-Angled Triangles and the Golden Ratio, *Forum Geometricorum*, **17(2017),** 433-437, http://forumgeom.fau.edu/FG2017volume17/FG201742index.htm.

2774 SUNNYBRIDGE DRIVE
BURNABY, BC, V5A 3V1, CANADA
E-mail address: trandrei8@gmail.com