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# ON THE JERABEK HYPERBOLA

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**Abstract.** In this article we study the Jerabek hyperbola of a triangle, representing it as a geometric locus of intersections of Steiner lines and trilinear polars of points on the circumcircle of the triangle of reference. In addition we exhibit and study some basic properties of a related and naturally defined projectivity, which maps the circumcircle onto the Jerabek hyperbola.

## 1 The Jerabek hyperbola

The Jerabek hyperbola of the triangle $ABC$ is a rectangular hyperbola generated by the centers of perspectivity $P$ of the triangle $ABC$ and the triangles $A'B'C'$, which are homo-

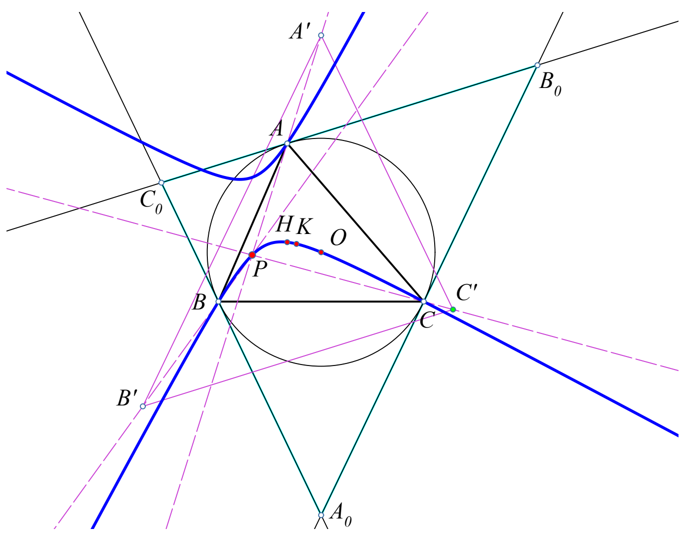


Figure 1: Jerabek's rectangular hyperbola of the triangle $ABC$

thetic to the tangential triangle $A_0B_0C_0$ of $ABC$, w.r. to the circumcenter of $ABC$ ([4, p. 448]) (See Figure 1).

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This hyperbola is the isogonal conjugate of the Euler line of the triangle $ABC$ ([16, p.18]) and passes trhough the vertices, the orthocenter $H$, the circumcenter $O$, the symmedian point $K$ and also through other remarkable triangle centers ([11]) of $ABC$. The hyperbola pertains to the *Poncelet pencil* ([2], [12]) of rectangular hyperbolas of the triangle, consisting of all hyperbolas which pass through the vertices of the triangle and its orthocenter. These hyperbolas are characterized also as the isogonal conjugates of all the diameter-lines of the circumcircle. Their centers are the orthopoles of these diameter-lines and lie on the Euler circle of the triangle ([9]). For the notions of *isogonal* and *isogonal conjugate* and the use of *trilinear coordinates*, which handles them conveniently, a quick reference can be found in the articles [10], [8] and a more detailed reference, extended with applications, in the book [14, vol.II]. The whole subject belongs to the geometry of the triangle, expositions of which can be found in [16], [15], [13] and [7].

## 2 Alternative generation

Next theorem demonstrates the possibility to generate the Jerabek hyperbola of the triangle $ABC$ in a different way, using the *trilinear polars* and the *Steiner lines* of points $P$ on its circumcircle $\kappa$ (See Figure 2). The trilinear polars $t_P$ of points $P$ on the circumcircle $\kappa$ pass

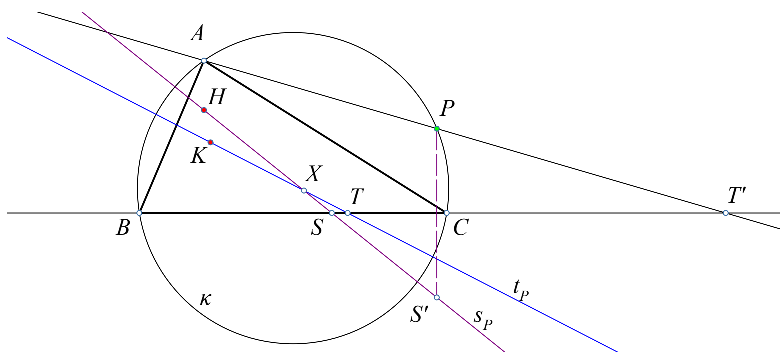


Figure 2: Steiner line $s_P$ and trilinear polar $t_P$ of $P \in \kappa$

through the symmedian point $K$ of the triangle ([4, p.126]) and the harmonic conjugate $T$ of the intersection $T' = (PA, BC)$. The Steiner lines $s_P$ pass through the orthocenter $H$ and the reflected $S'$ of $P$ on $BC$. They are parallel to the corresponding Wallace-Simson line of $P$ and carry the reflected points of $P$ w.r. to the sides of the triangle ([1, p.54]). Thus, as point $P$ wanders on the circumcircle $\kappa$, the correspoding lines $\{s_P, t_P\}$ belong to the pencils $\{H^*, K^*\}$ of lines passing respectively through the points $\{H, K\}$. The theorem shows that the intersection points $\{S = (s_P, BC), T = (t_P, BC)\}$ of these lines with $BC$ are homographically related, i.e. their line coordinates $\{x, x'\}$ satisfy a homographic relation of the form

$$x' = \frac{ax + b}{cx + d}, \qquad (1)$$

hence, by the Chasles-Steiner principle of generation of conics ([5, p.5], [3, p.72], [6, p. 259]), their intersection point $X = (s_P, t_P)$ generates a conic.

**Theorem 2.1.** *The geometric locus of intersections $X = (s_P, t_P)$ of the Steiner line and the trilinear polar w.r. to points $P$ of the circumcircle $\kappa$ of the triangle $ABC$ is the Jerabek hyperbola of the triangle.*

**Proof.** Consider the system of coordinates whose $x$-axis is the side $BC$ and the $y$-axis is the altitude $AO$ (See Figure 3). In this system, the coordinates of the vertices are

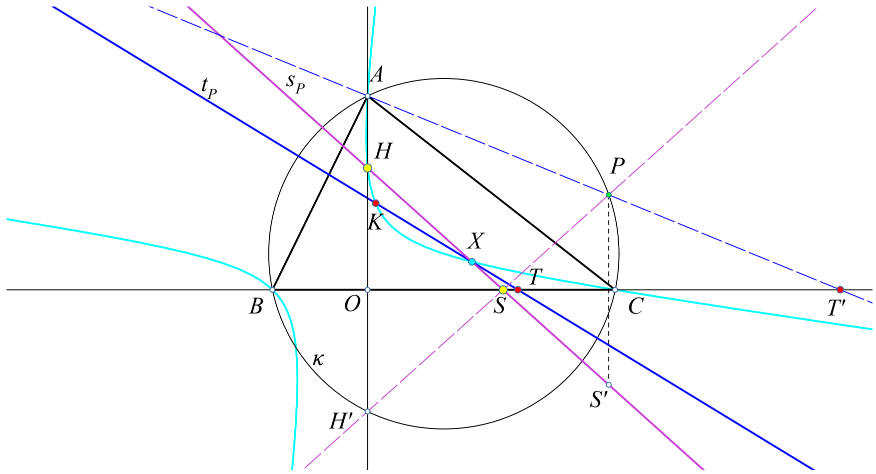


Figure 3: Steiner line $s_P$ and trilinear polar $t_P$ of $P \in \kappa$

$\{A(0, a), B(b, 0), C(c, 0)\}$. The coordinates of the orthocenter $H$ and its reflection $H'$ on $BC$ are then easily seen to be $\{H(0, -bc/a), H'(0, bc/a)\}$ and the equation of the circumcircle

$$a(x^2 + y^2) - a(b + c)x - (bc + a^2)y + abc = 0. \tag{2}$$

Starting with $S(s, 0)$ on $BC$, the point $P$ is found as intersection of the circle with line $H'S$, which is the reflection of line $HS$ on $BC$.

$$P(p_1, p_2) \quad \text{with} \quad p_1 = \frac{s(a^2cs + a^2bs + b^2c^2 - a^2bc)}{a^2s^2 + b^2c^2}, \;\; p_2 = \frac{abc(s - b)(s - c)}{a^2s^2 + b^2c^2}. \tag{3}$$

By the definition of the trilinear polar, point $T$ is the harmonic conjugate w.r. to $(B, C)$ of $T' = (BC, AP)$, whose coordinates are easily seen to be $T'(0, t')$, with $t' = ap_1/(a - p_2)$. The harmonic-conjugate relation gives

$$t = \frac{t'(b + c) - 2bc}{2t' + (b + c)}. \tag{4}$$

Replacing in this the values for $t'$ and $(p_1, p_2)$, we arive after a short calculation, at the expression of $t$ in dependence of $s$

$$t = \frac{(a^2(b^2 + c^2) + 2b^2c^2)s - bc(c + b)(bc + a^2)}{(c + b)(bc + a^2)s - bc(b^2 + c^2 + 2a^2)}. \tag{5}$$

This proves that points $\{S, T\}$ are homographically related, hence $X = (s_P, t_P)$ describes a conic. It is then easy to see that this conic passes through the vertices of the triangle and, by the general properties of the Chasles-Steiner generation method, passes also from the centers of the pencils $\{H^*, K^*\}$ i.e. points $\{H, K\}$. This identifies the conic with the rectangular hyperbola of Jerabek.

**Remark.** Notice that the determinant of the homographic relation (5) is readily seen to be

$$D = -bc(a^2 + b^2)(a^2 + c^2)(c - b)^2,$$

implying that the conic is genuine, except in the case $b = 0$ or $c = 0$, which corresponds to a right triangle $ABC$. These are precisely the cases in which the corresponding Jerabek hyperbola degenerates in the product of two orthogonal lines: the hypotenuse and the altitude to it.

## 3 A projectivity

We will need to pass from the cartesian coordinate system of the preceding section to trilinear coordinates and show the existence of a naturally defined projectivity, mapping the circumcircle onto the Jerabek hyperbola. Now the meaning of symbols changes and $\{a,b,c\}$ denote the side lengths of the triangle $ABC$. In addition we use the symbols $S_A = (b^2+c^2-a^2)/2 = bc\cos(\widehat{A})$ and the analogous $\{S_B, S_C\}$ resulting from $S_A$ by cyclic permutations of the letters. In dealing with trilinear coordinates it is advisable to shorten notation, especially for long expressions, using symbols like $(ab, \dots)$ to denote triples $(ab, bc, ca)$, resulting by cyclic permutations of the letters, and similarly sums $ab+\dots = 0$, meaning $ab+bc+ca = 0$. We use also $(u,\dots) \sim (v,\dots)$ to express that the triples are non zero multiples of each other, defining the same point in trilinears.

Jerabek's hyperbola is described in trilinears $(x,y,z)$ by the equation

$$(b^2-c^2)S_A\frac{a}{x} + \dots = 0 \tag{6}$$

It is readily seen that also

$$(b^2-c^2)S_A + \dots = 0 \tag{7}$$

and from these two equations follows the relation of the triples

$$\left((b^2-c^2)S_A, \dots\right) = \lambda \cdot \left(\frac{b}{y} - \frac{c}{z}, \dots\right). \tag{8}$$

With this preparation, we consider now a point $X(x,y,z)$ of the hyperbola, and the chord-line it defines through the symmedian point $K(a,b,c)$. The equation of this line for running trilinears $(u,v,w)$ is

$$(cy-bz)u + \dots = 0.$$

And the trilinear pole of this line, which is a point on the circumcircle, is

$$P(x',y',z') \quad \text{with} \quad (x',y',z') \sim \left(\frac{1}{cy-bz}, \dots\right). \tag{9}$$

Using the fact that trilinears are defined modulo a multiplicative constant, we show that this triple depends linearly on $(x,y,z)$. In fact, the triple on the right side is a multiple of

$$\left(\frac{1}{x^2y^2z^2}((az-cx)(bx-ay), \dots\right). \tag{10}$$

Working with the first coordinate of this, we notice that

$$\frac{1}{x^2y^2z^2}(az-cx)(bx-ay) = \frac{az-cx}{zx} \cdot \frac{bx-ay}{xy} \cdot \frac{1}{yz} = \left(\frac{a}{x} - \frac{c}{z}\right) \cdot \left(\frac{b}{y} - \frac{a}{x}\right) \cdot \frac{1}{yz}.$$

The last expression is, by means of equation 8,

$$\begin{aligned} \left(\frac{a}{x} - \frac{c}{z}\right) \cdot \left(\frac{b}{y} - \frac{a}{x}\right) \cdot \frac{1}{yz} &= \frac{1}{\lambda^2}(c^2-a^2)S_B(a^2-b^2)S_C\frac{1}{yz} \\ &= \frac{x}{\lambda^2 xyz}(c^2-a^2)(a^2-b^2)S_BS_C. \end{aligned}$$

Working analogously with the other coordinates of the triple in 10, we see that this can be expressed as a multiple of

$$\left((c^2-a^2)(a^2-b^2)S_BS_Cx, \dots\right)$$

and by means of equation 9, the function $P = g(X)$ can be expressed by

$$g(x,y,z) \sim \left((c^2-a^2)(a^2-b^2)S_BS_Cx,\dots\right) \quad \sim \quad \left(\frac{(S_A-S_C)(S_A-S_B)}{S_A}x,\dots\right). \quad (11)$$

This shows that the map $P = g(X)$ is projective, hence its inverse $X = f(P)$, which coincides with the correspondence $P \mapsto X$ of the preceding section, is a projective transformation, mapping the circle onto the hyperbola. We formulate this as a theorem.

**Theorem 3.1.** *The map, which to the point $P$ of the circumcircle $\kappa$ of the triangle $ABC$, corresponds the intersection point $X = (s_P, t_P)$ of the Steiner line and the trilinear polar of $P$, is the restriction on $\kappa$ of a projective transformation, mapping the circumcircle $\kappa$ onto the Jerabek hyperbola of $ABC$.*

# 4 X(74)

The triangle center $X(74)$ is the fourth intersection point of the Jerabek hyperbola with the circumcircle $\kappa$ of the triangle of reference $ABC$. It is also the isogonal conjugate of the point at infinity of the Euler line of the triangle ([11]). The next few lemmata show some other aspects of this triangle center, related to the projectivity $f$, established in the previous section.

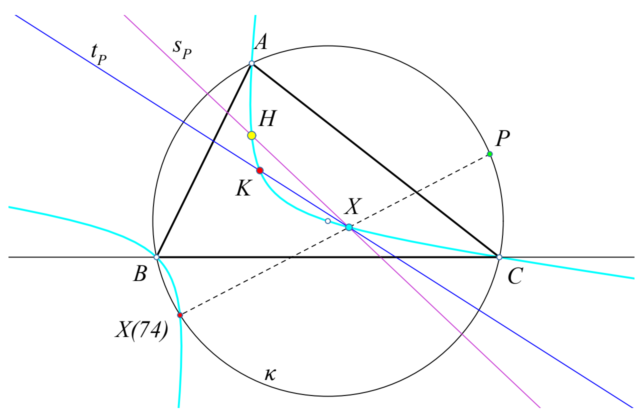


Figure 4: Points $\{P, X, X(74)\}$ are collinear

**Lemma 4.1.** *All lines $PX$, for $P$ on the circumcircle $\kappa$ and $X = f(P)$, pass through $X(74)$.*

**Proof.** The trilinears of $X(74)$ being given by ([11])

$$\left(\frac{a}{S_AS_C+S_AS_B-2S_BS_C},\dots\right) \quad \sim \quad \left(\frac{a}{(b^2-c^2)^2+a^2(c^2+b^2-2a^2)},\dots\right),$$

it suffices to show that, for a point $X(x,y,z)$ of Jerabek's hyperbola, the determinant with the above first row and the other two equal to

$$(x,y,z), \quad \text{and} \quad \left((c^2-a^2)(a^2-b^2)S_BS_Cx,\dots\right),$$

is zero. But this determinant is easily seen to evaluate to a multiple of equation 6, which by assumption now is satisfied by $(x,y,z)$.

**Lemma 4.2.** *The Euler line of the triangle is the line send to infinity by $f$.*

**Proof.** Since the line at infinity in trilinears is $ax + by + cz = 0$, it suffices to show that the two points of it $\{U = (-b, a, 0), V = (-c, 0, a)\}$ map via the inverse $g$ of $f$ onto the Euler line, whose equation is given by

$$S_A(S_B - S_C)ax + \cdots = 0.$$

Using equation 11 we see that

$$g(U) \sim (-b(c^2 - a^2)(a^2 - b^2)S_B S_c, \;\; a(a^2 - b^2)(b^2 - c^2)S_C S_A, \;\; 0),$$

which is readily verified to be on the Euler line. Analogous is the proof for $V$.

**Lemma 4.3.** *The tangents $\{\delta, \varepsilon\}$ to the circumcircle $\kappa$ at its intersection points $\{D, E\}$ with the Euler line of $ABC$ map under $f$ to the the asymptotes of the Jerabek hyperbola, which are respectively parallel to $\{DX(74), EX(74)\}$.*

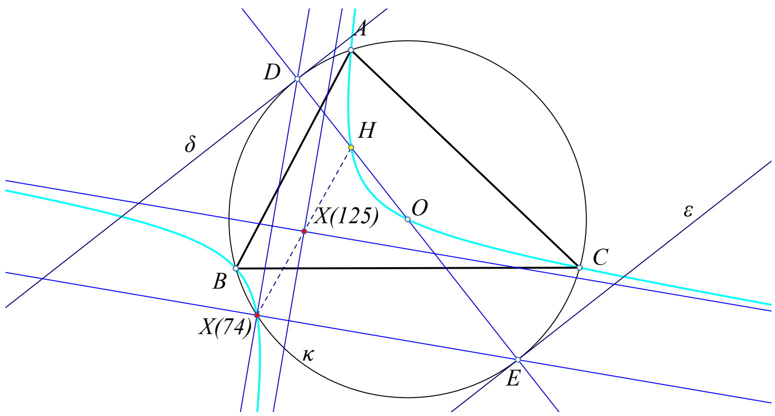


Figure 5: Parallels to the asymptotes of Jerabek's hyperbola

**Proof.** In fact, by lemma 4.2 points $\{D, E\}$ map under $f$ to points at infinity of the hyperbola lying respectively on lines $\{DX(74), EX(74)\}$ (See Figure 5). In addition, the tangents at $\{D, E\}$ of $\kappa$ map to corresponding tangents at the points at infinity, i.e. the asymptotes of the hyperbola.

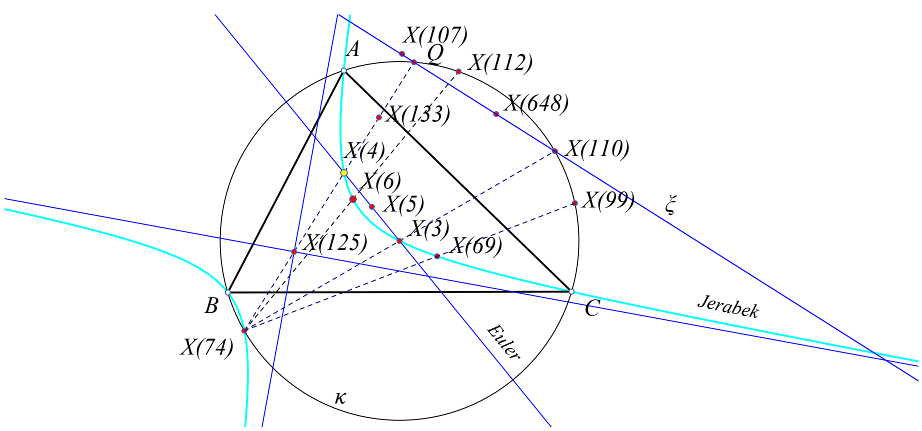


Figure 6: Points corresponding under the projectivity $f$

By lemma 4.1 all triangle centers $P$ lying on the circumcircle define via $f$, points $X = f(P)$ lying on the Jerabek hyperbola, identified by the second intersection of line

$PX(74)$ with the hyperbola. Figure 6 shows a few of them, the verification of which can be deduced from Kimberling's great list [11] of triangle centers with a few additional computations left as an exercise. Line $\xi = X(107)X(110)$ maps via $f$ onto the Euler line, point $X(110)$ mapping onto the circumcenter $O = X(3)$ and its other intersection point with the circumcircle $Q = \left(\frac{1}{a(b^2-c^2)S_A^2}, \dots\right)$ mapping onto the orthocenter $H = X(4)$.

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