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INFINITE CIRCLE CHAINS IN BETWEEN INTERNALLY TANGENT CIRCLES AND INTEGER SEQUENCES

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Abstract. In this paper we consider the infinite chains of mutually tangent circles that can be inscribed in between two internally tangent circles and we relate them to certain integer sequences.

1. Introduction

Let us consider the infinite chains of mutually tangent circles that can be drawn inside the area between two internally tangent circles (see Figure 1); as we can see, a generic circle of the chain is tangent to the preceding and succeeding ones and to the outer and inner circles $\mathrm{C_a}$ and $\mathrm{C_b}$ respectively. Let us consider a generic chain and let us define by *major circle* the largest circle belonging to the chain. By looking at Figure 1, the major circle can be considered origin of a first circle chain (named *up chain*) pointing upward and of a second circle chain (named *down chain*) pointing downward; both the chains tend to the origin of the cartesian reference system shown in Figure 1. In [1] and [2] formulas for the centres coordinates and radius of a generic circle belonging the chains can be found. For convenience, we report here such formulas in a different but, suitable form for the following.

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$$x_n = \frac{ab(a+b)}{\left[\left(\frac{\sqrt{ab}}{a-b}\sqrt{\frac{a-b-r_0}{r_0}}+n\right)(a-b)\right]^2+ab} \tag{1}$$

$$y_n = \frac{2ab(a-b)\left[\left(\frac{\sqrt{ab}}{a-b}\sqrt{\frac{a-b-r_0}{r_0}}+n\right)\right]}{\left[\left(\frac{\sqrt{ab}}{a-b}\sqrt{\frac{a-b-r_0}{r_0}}+n\right)(a-b)\right]^2+ab} \tag{2}$$

$$r_n = \frac{ab(a-b)}{\left[\left(\frac{\sqrt{ab}}{a-b}\sqrt{\frac{a-b-r_0}{r_0}}+n\right)(a-b)\right]^2+ab} \tag{3}$$

In the above formulas, a and b are the radius of the outer and inner circles

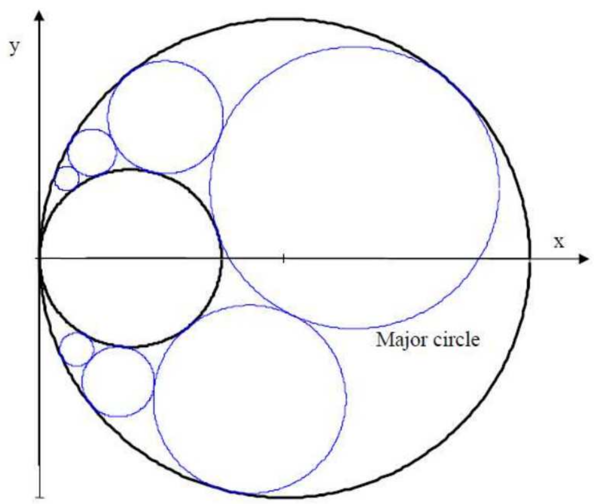

FIGURE 1. Example of circle chains

respectively, while r_0 is the major circle radius. Moreover, for $n > 0$ one gets the up chain while, for $n < 0$, the down chain.
The parameters a, b and r_0 are subject to the following constraints:

$$a > b \tag{4}$$

$$\frac{4ab(a-b)}{(a+b)^2} \leq r_0 \leq a-b \tag{5}$$

In particular, from equation (5), we can notice that when:

$$r_0 = \frac{4ab(a-b)}{(a+b)^2} \tag{6}$$

one obtains two circle chains with *bi-central symmetry* as shown in Figure 2.

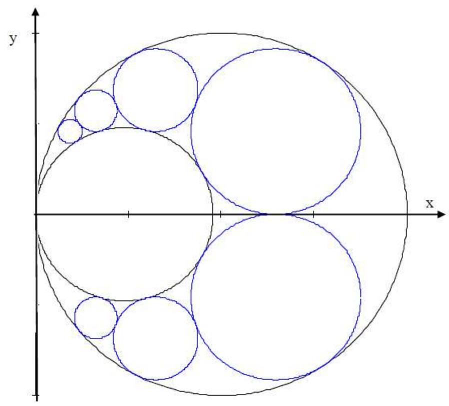

FIGURE 2. Circle chains with bi-central symmetry

Note that, in this particular case, we have two major circles; one for the up-chain and one for the down chain.

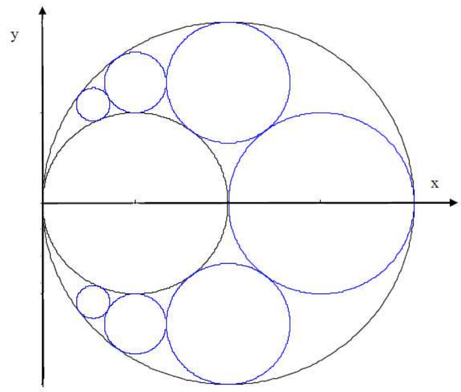

FIGURE 3. Circle chains with central symmetry

On the contrary, if:

$$r_0 = a - b \tag{7}$$

one obtains a circle chain with *central symmetry* as shown in Figure 3.
For any other value of r_0, the up and down chains are not symmetrical as in Figure 1.

2. Sequences related to the circle chains

Let us define the ratio between the major circle radius and the generic n-th circle radius for the up and down chain i.e.:

$$\tau_n^{up} = \frac{\left[\left(\frac{\sqrt{ab}}{a-b}\sqrt{\frac{a-b-r_0}{r_0}}+n\right)(a-b)\right]^2+ab}{ab(a-b)}r_0 \tag{8}$$

$$\tau_n^{down} = \frac{\left[\left(\frac{\sqrt{ab}}{a-b}\sqrt{\frac{a-b-r_0}{r_0}}-n\right)(a-b)\right]^2+ab}{ab(a-b)}r_0 \tag{9}$$

Thus, formulas (8) and (9), introduce two sequences $\{\tau_n^{up}\}$ and $\{\tau_n^{down}\}$ which, in general, are composed by real numbers.
In this paper, we want to investigate if some conditions exist in order to be both $\{\tau_n^{up}\}$ and $\{\tau_n^{down}\}$ integer sequences. In other words, which are the conditions (provided they exist) in order that the radius of any generic circle of the chain is a submultiple of the major circle radius? This is the purpose of the present paper.

3. Conditions for being $\{\tau_n^{up}\}$ and $\{\tau_n^{down}\}$ integer

First of all, it is useful to introduce the following variables:

$$\lambda = \frac{a}{b} \tag{10}$$

$$\mu = \frac{r_0}{b} \tag{11}$$

By taking into account of (4) and (5), one has the following constraints for λ and μ:

$$\lambda > 1 \tag{12}$$

$$\frac{4\lambda(\lambda-1)}{(\lambda+1)^2} \leq \mu \leq \lambda - 1 \tag{13}$$

Therefore, equations (8) and (9) can be written in function of λ and μ as:

$$\tau_n^{up} = \frac{\left[\left(\frac{\sqrt{\lambda}}{\lambda-1}\sqrt{\frac{\lambda-1-\mu}{\mu}}+n\right)(\lambda-1)\right]^2\mu+\lambda\mu}{\lambda(\lambda-1)} \tag{14}$$

$$\tau_n^{down} = \frac{\left[\left(\frac{\sqrt{\lambda}}{\lambda-1}\sqrt{\frac{\lambda-1-\mu}{\mu}}-n\right)(\lambda-1)\right]^2\mu+\lambda\mu}{\lambda(\lambda-1)} \tag{15}$$

The values of λ and μ yielding integer sequences for both $\{\tau_n^{up}\}$ and $\{\tau_n^{down}\}$

are solution of the following system:

$$(16)\qquad \begin{cases} \dfrac{\Big[\Big(\dfrac{\sqrt{\lambda}}{\lambda-1}\sqrt{\dfrac{\lambda-1-\mu}{\mu}}+n\Big)(\lambda-1)\Big]^2\mu+\lambda\mu}{\lambda(\lambda-1)}=M(n) \\ \dfrac{\Big[\Big(\dfrac{\sqrt{\lambda}}{\lambda-1}\sqrt{\dfrac{\lambda-1-\mu}{\mu}}-n\Big)(\lambda-1)\Big]^2\mu+\lambda\mu}{\lambda(\lambda-1)}=N(n) \end{cases}$$

being $M(n)$ and $N(n)$ two unknown integer functions of n.
It is important to remark that the solution λ and μ we are looking for, must not depend on n.
Through some algebraic passages, system (16) can be transformed into:

$$(17)\qquad \begin{cases} \lambda+2n\sqrt{\lambda\mu}\sqrt{\lambda-1-\mu}+n^2(\lambda-1)\mu=\lambda M(n) \\ \lambda-2n\sqrt{\lambda\mu}\sqrt{\lambda-1-\mu}+n^2(\lambda-1)\mu=\lambda N(n) \end{cases}$$

By adding and subtracting the two equations in (17) one gets the equivalent system:

$$(18)\qquad \begin{cases} 2\lambda+2n^2(\lambda-1)\mu=\lambda[M(n)+N(n)] \\ 4n\sqrt{\mu\lambda-\mu-\mu^2}=\sqrt{\lambda}[M(n)-N(n)] \end{cases}$$

From the first equation of system (18), one readily obtains:

$$(19)\qquad \mu=\frac{[M(n)+N(n)-2]\lambda}{2n^2(\lambda-1)}$$

Due to the fact that μ must not depend on n and being $M(n)+N(n)$ an integer, we have that it must be $M(n)+N(n)=Jn^2+2$ with $J\in\mathbb{N}^+$; therefore:

$$(20)\qquad \mu=\frac{J\lambda}{2(\lambda-1)}$$

By substituting (20) into the second equation of system (18), one has:

$$(21)\qquad \frac{2n}{\lambda-1}\sqrt{J[2\lambda^2-(J+4)\lambda+2]}=M(n)-N(n)$$

Due to the fact that λ must not depend on n and being $M(n)-N(n)\in\mathbb{Z}$, we have that $M(n)-N(n)=Ln$ with $L\in\mathbb{Z}$; therefore:

$$(22)\qquad \frac{2}{\lambda-1}\sqrt{J[2\lambda^2-(J+4)\lambda+2]}=L$$

By solving for λ, one finally gets:

$$(23)\qquad \begin{cases} \lambda=\dfrac{[2J^2+8J-L^2]+2J\sqrt{J^2+8J-L^2}}{8J-L^2} \\ \mu=\frac{1}{2}\dfrac{[2J^2+8J-L^2]+2J\sqrt{J^2+8J-L^2}}{2J+2\sqrt{J^2+8J-L^2}} \end{cases}$$

Note that J and L must satisfy the following conditions:

$$J^2 + 8J - L^2 \geq 0 \tag{24}$$

$$8J - L^2 \neq 0 \tag{25}$$

It is necessary also to remark that being: $M(n) + N(n) = J^2 + 2$ and $M(n) - N(n) = nL$, one readily obtains:

$$M(n) = \frac{Jn^2 + Ln + 2}{2} \tag{26}$$

$$N(n) = \frac{Jn^2 - Ln + 2}{2} \tag{27}$$

By looking at equations (26) and (27), we have that, in order to be both $M(n)$ and $N(n)$ integers, we have the further condition:

$$J + L = \pm 2k \qquad k = 1, 2 \dots \tag{28}$$

Furthermore, we remark that if $L = 0$ one obtains circle chains with central symmetry (see Figure 3) while if $J = \pm L$ one obtains circle chains with bi-central symmetry (see Figure 2). In all the other cases the chains are not symmetrical. Finally, by looking at equations (26) and (27), we can notice that if one changes L into $-L$, up and down chains are interchanged.
In practice, we can conclude that to each pair (J, L) it corresponds an up and down circle chain together the associated two sequences. Therefore, we can state the following property:

Let us consider the set of pairs (J, L) formed by integer numbers so that they satisfy conditions (24) and (25) and their sum is and even number; then, to each one of those pairs it corresponds a circle chain having the characteristic that the ratio between the major circle radius and the generic n-th circle radius is an integer.

4. Sequences classified according to OEIS

In this paragraph we show examples of several circle chains that can be composed according to the choice for the parameters λ and μ given by formulas (23) in function of J and L (provided the conditions expressed by (12), (13), (24), (25) and (28) are satisfied) and generating integer sequences which are classified in the On Line Encyclopedia of Integer Sequences (OEIS) [3]. In Table 1, we report the result of our analysis for a certain number of pairs (J, L).

J	L	SYMMETRY	OEIS CLASSIFICATION
1	1	bi-central	A000124
2	0	central	A002522
2	2	bi-central	A002016
3	1	none	up A104249, down A143689
3	3	bi-central	A005448
4	0	central	A058331
4	2	none	up A084849, down A130883
4	4	bi-central	A001844
5	1	none	up A116668, down A140066
5	3	none	up A134238, down A192136
5	5	bi-central	A005891
6	0	central	A056107
6	2	none	up A056108, down A056106
6	4	none	up A056109, down A056105
6	6	bi-central	A003215
7	1	none	up not classified, down A140063
7	3	none	up not classified, down A140065
7	5	none	up A100752, down not classified
7	7	bi-central	A069099
8	0	central	A053755
8	2	none	up A054567, down A054556
8	4	none	up A054569, down A054554
8	6	none	up A033951, down A054552
9	1	none	up A006137, down A276819
9	3	none	up A038764, down A080855
9	5	none	up A064225, down A140064
9	7	none	up A081267, down A117625
10	0	central	A212656
10	2	none	up not classified, down A172043
10	4	none	up not classified, down A145995
10	8	none	up not classified, down A190816
12	4	none	up A136392, down not classified
12	6	none	up A085473, down not classified
12	8	none	up A080859, down not classified

Table 1. Some sequences classified in OEIS corresponding to circle chains

References

[1] Tahir, H., *Infinite chains of tangent circles*, Aust. Math. Soc. Gazette, **26(3)(1999)** 128–133.
[2] Tahir, H., *On points of contact in infinite chains*, Aust. Math. Soc. Gazette, **33(4)(2006)** 273–281.
[3] Sloane, N.J.A., *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org

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