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SOLVING ALHAZEN'S PROBLEM BY ORIGAMI

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Abstract. In this paper we give a procedure of the geometric construction for Alhazen's problem by using origami (paperfolding).

1. Introduction

The famous Alhazen's problem (cf. [1] and [4]) concerns the path of light reflected by a circular mirror. Given two fixed points A and B in the same side to the circle C, to find a point P on the edge of the circular mirror such that the path of light from A after one reflection at P will pass from B (see Figure 1). In other words, locate P on the circle C so that the $\angle APB$ is bisected by the diameter through P. Algebraically, Alhazen's problem corresponds to solving a quartic equation. Therefore, this problem can not be solved by the classical geometric construction by using ruler and compass.

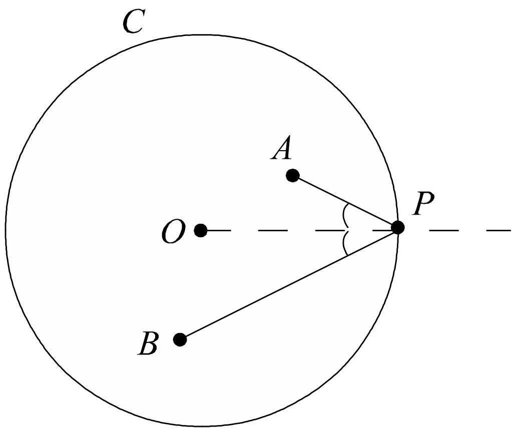

Figure 1. Alhazen's problem

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On the other hand, it is known that general cubic and quartic equations can be solved by Origami (cf. [3, Theorem 10.14] etc.). Therefore Alhazen's problem can be solved by origami, theoretically. This fact was pointed out in [1], and moreover an actual origami construction has been given in [6] in only special case such that A lies on C. In this paper, we give a procedure of origami construction for Alhazen's problem in general cases.

2. Huygens' solution and the dual conic

Huygens solved Alhazen's problem reducing to the intersection of the circle with an equilateral hyperbola in 1672. In this section, we rewrite his solution in terms of algebra. We regard the given points $a = A,\ b = B,\ z = P$ as complex numbers and the unit circle C centered at the origin O. Then we want $z \in \mathbb{C}$ so that $\arg\left(\frac{a-z}{0-z}\right) = \arg\left(\frac{0-z}{b-z}\right)$ or equivalently $\frac{a-z}{z}\left(\frac{z}{b-z}\right)^{-1}$ is a real number. Therefore we obtain the following relation by using $z\bar{z} = 1$:

$$(ab)\bar{z}^2 - (\overline{ab})z^2 = (a+b)\bar{z} - (\overline{a+b})z.$$

We set $p = \mathrm{Im}(ab),\ q = \mathrm{Re}(ab),\ r = \mathrm{Re}(a+b),\ s = \mathrm{Im}(a+b)$ and $z = x+iy$. Then we can rewrite the above equation in terms of real coordinates as follows:

$$p(x^2 - y^2) - 2qxy = sx - ry.$$

By a suitable rotation of the plane about the origin, we may assume that the positive x-axis bisects $\angle AOB$, thus $p = 0$. Then, the solution of Alhazen's problem occurs at one of the intersections of the equilateral hyperbola $Q : 2qxy + sx - ry = 0$ and the circle $C : x^2 + y^2 = 1$ (see Figure 2). We notice that the problem is obvious if $qrs = 0$. Therefore we may assume that $qrs \neq 0$ below.

Here we consider the correlation with the unit circle C. For each point $U(\alpha, \beta)$ except the origin, we denote its polar line $\bar{U} : \alpha x + \beta y - 1 = 0$. Conversely, we denote by $\bar{L}$ the pole of the line L which does not pass through the origin. We notice that a point U lies on the unit circle C if and only if its polar line $\bar{U}$ is tangent to C at the point U. The following lemma is the duality of conics associated with the correlation (cf. [5, Proposition 3.4]).

Lemma 2.1. *When $qrs \neq 0$, the dual conic of the equilateral hyperbola $Q : 2qxy + sx - ry = 0$ is expressed as $\bar{Q} : r^2x^2 + 2rsxy + s^2y^2 - 4qrx + 4qsy + 4q^2 = 0$. Then a point U (except the origin) lies on Q if and only if its polar line $\bar{U}$ is tangent to $\bar{Q}$.*

Proof. Suppose that a line $L : \alpha x + \beta y - 1 = 0$ is tangent to the hyperbola $Q : 2qxy + sx - ry = 0$. Assume that $\beta \neq 0$ and therefore $y = (-\alpha x + 1)/\beta$. Then a quadratic equation $2qx(-\alpha x + 1)/\beta + sx - r(-\alpha x + 1)/\beta = 0$ for the variable x has a double root, that is, its discriminant $\Delta = (2q + s\beta + r\alpha)^2 - 8qr\alpha = r^2\alpha^2 + 2sr\alpha\beta + s^2\beta^2 - 4qr\alpha + 4qs\beta + 4q^2$ must be zero. This is equivalent to the condition that the pole $\bar{L}(\alpha, \beta)$ lies on $\bar{Q}$.

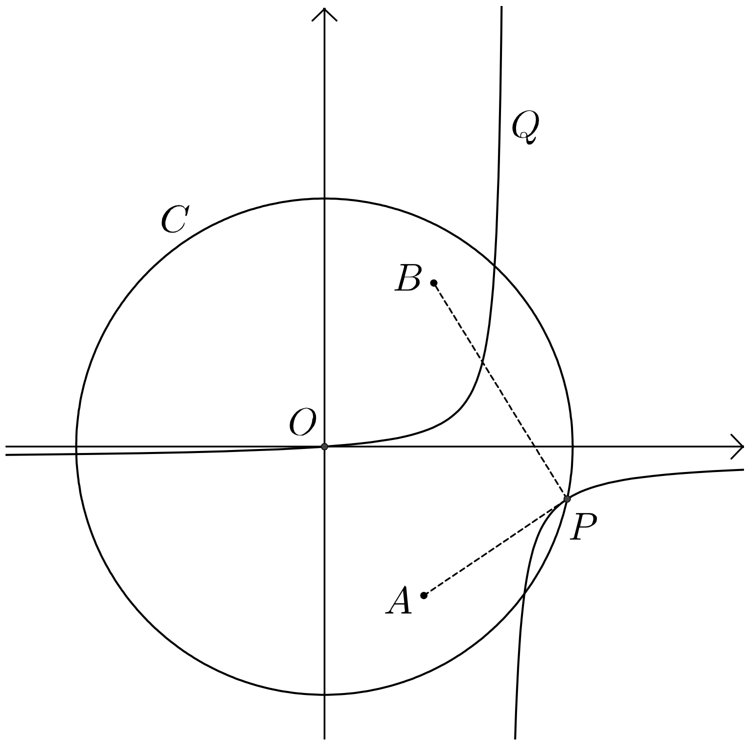

FIGURE 2. Huygen's solution

Conversely, if a line $L : \alpha x + \beta y - 1 = 0$ $(\beta \neq 0)$ is tangent to $\bar{Q}$ then a quadratic equation

$$r^2x^2+2rsx\left(\frac{-\alpha x+1}{\beta}\right)+s^2\left(\frac{-\alpha x+1}{\beta}\right)^2-4qrx+4qs\left(\frac{-\alpha x+1}{\beta}\right)+4q^2=0$$

has a double root, that is, its discriminant $\Delta = qrs(2q\alpha\beta + s\alpha - r\beta)$ is zero. This is equivalent to the condition that the pole $\bar{L}(\alpha, \beta)$ lies on Q.

The case when $\alpha \neq 0$ (involve $\beta = 0$) is similar to the case when $\beta \neq 0$. $\square$

Lemma 2.2. *The conic* $\bar{Q} : r^2x^2 + 2rsxy + s^2y^2 - 4qrx + 4qsy + 4q^2 = 0$ *is a parabola defined by the focus* $F(\frac{2qr}{r^2+s^2}, -\frac{2qs}{r^2+s^2})$ *and the directrix* $D : sx - ry = 0$.

Proof. Suppose that a point $X(x, y)$ satisfies the condition that XF coincides with the distance between X and D. Then we obtain the equation:

$$\left(x - \frac{2qr}{r^2+s^2}\right)^2 + \left(y + \frac{2qs}{r^2+s^2}\right)^2 = \frac{(sx-ry)^2}{s^2+r^2}.$$

The above equation can be changed into $r^2x^2 + 2rsxy + s^2y^2 - 4qrx + 4qsy + 4q^2 = 0$, thus the locus of X coincides with the conic $\bar{Q}$. $\square$

3. Origami construction for Alhazen's problem

For given points A, B and the unit circle C, we use notations P, Q, F and D according as the previous section. That is, the point P is an intersection of the unit circle C and an equilateral hyperbola Q associated with A, B. Then the polar line $\bar{P}$ is a common tangential line of C and the dual parabola $\bar{Q}$ from Lemma 2.1. It is known that a tangential line of the parabola $\bar{Q}$ is a perpendicular bisector of its focus F and a point which lies on the directrix D (cf. [3, Theorem 10.2]). Therefore we obtain the line $\bar{P}$ as a crease by the following folding move.

Theorem 3.1. *Let Q be a conic, and C is a unit circle centered at origin. Suppose that the dual conic $\bar{Q}$ is a parabola with a focus F and a directrix D. We fold so that F is reflected onto D and its crease is tangent to C. Then, the contact point P of the crease $\bar{P}$ and C coincides with the intersection of C and Q (see Figure 3).*

Remark 3.1. *The folding move in Figure 3 use the circle, and does not contain in Huzita's axioms O1 $\sim$ O7 (see* [2, Chapter 19]*). In* [5] *we studied an extension of origami constructions by using circles, and this folding move is named the axiom O18.*

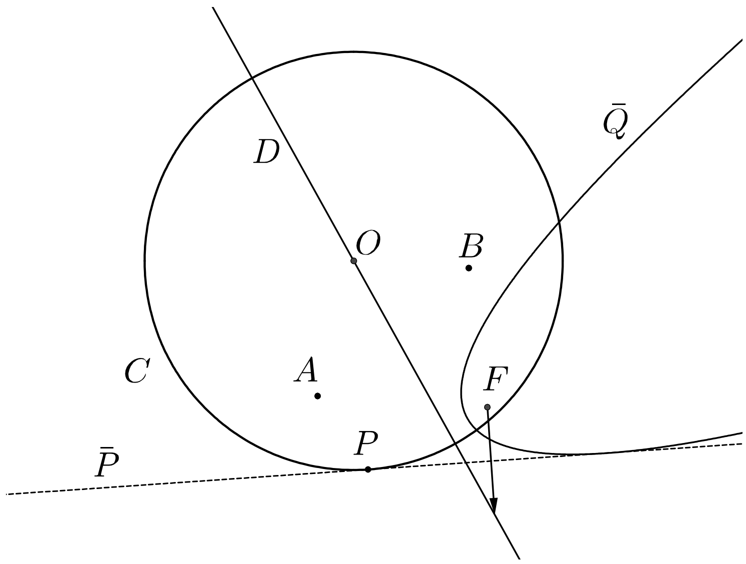

Figure 3. Construction of P

We show a complete procedure of origami construction for Alhazen's problem as follows. Suppose that two points $a = A$ and $b = B$ on the complex plane $\mathbb{C}$ are given in interior of the unit circle C centered at the origin O. When two points A, B are exterior to the circle, the procedure is the same.

1) The line OA is folded and we denote an intersection of the ray OA and C by E.

2) The line AB is folded.

3) The perpendicular bisector of A and B is folded and we denote the middle point of AB by M. Note that $OM = \sqrt{s^2 + r^2}/2 = |a + b|/2$.

4) The line OM is folded and we denote the crease and the intersection of the ray OM and C by D and E', respectively.

5) The line EB is folded.

6) The perpendicular line of EB through B is folded and we denote the crease by l_1.

7) The perpendicular line of l_1 through A is folded and we denote the crease by l_2.

8) The line OB is folded and we denote an intersection of OB and l_2 by G (see Figure 4). Note that $OG = q = \mathrm{Re}(ab)$.

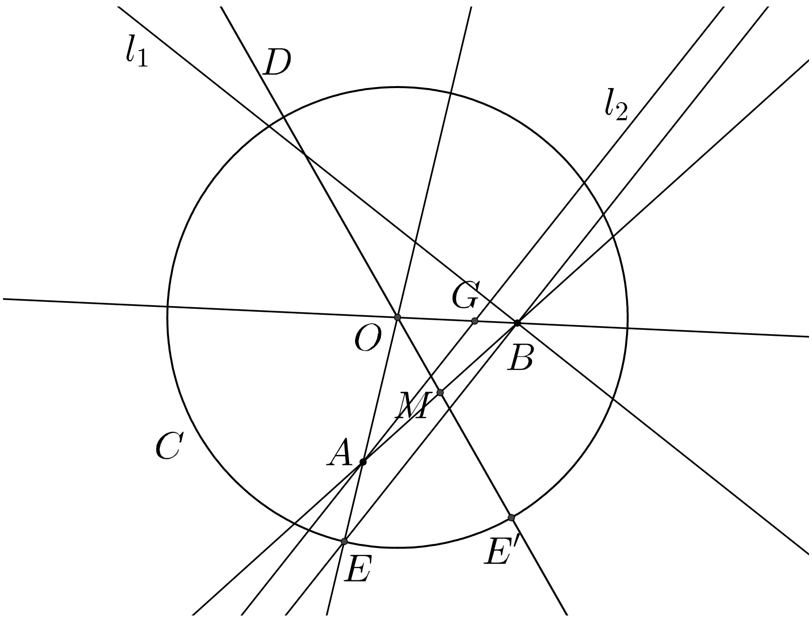

FIGURE 4. $OB \cap l_2 = \{G\}$

9) The line MG is folded.

10) The perpendicular line of MG through M is folded and we denote the crease by l_3.

11) The perpendicular line of l_3 through E' is folded and we denote an intersection of the crease and OB by H. Note that $OH = OG/OM$.

12) The line through O which reflects H onto D is folded and we denote the reflected point of H by K.

13) The line through O which reflects B onto OA is folded and we denote the reflected point of K by F (see Figure 5).

14) The tangential line of C which reflects F onto D is folded. Then we obtain P as its contact point of C and the crease $\bar{P}$ (see Figure 3).

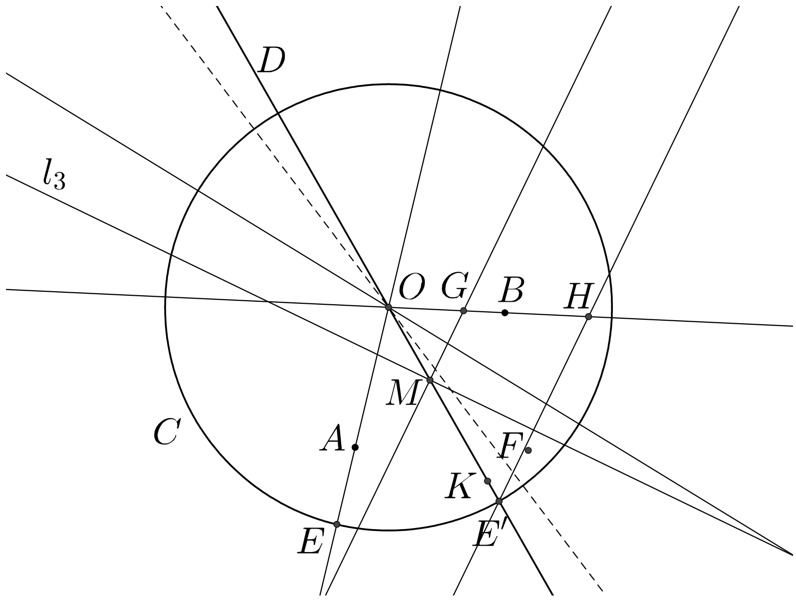

FIGURE 5. The line through O which reflects B onto OA is folded

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