GOLDEN SECTIONS AND ARCHIMEDEAN CIRCLES IN AN ARBELOS

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Abstract. We construct some golden ratios in the arbelos and Archimedean circles in this configuration.

Consider an arbelos formed by semi-circles \((O_1), (O_2),\) and \((O)\) of radii \(a, b,\) and \(a + b\). The semi-circles \((O_1)\) and \((O)\) meet at \(A, (O_2)\) and \((O)\) at \(B, (O_1)\) and \((O_2)\) at \(C\). Let \(CD\) be the divided line of the smaller semi-circles.

A segment \(PQ\) is called to be divided in the golden ratio by a point \(R\) if 
\[
\frac{PQ}{PR} = \frac{PR}{RQ}
\]
In this case, the divided ratio is the golden ratio \(\phi := \frac{\sqrt{5}+1}{2}\), which satisfies \(\phi^2 = \phi + 1\).

Theorem 1. Let the segment \(AD\) and semi-circle \((O_1)\) meet again at \(E\), \(BD\) and \((O_2)\) at \(F, AF\) and \((O_2)\) at \(G, BE\) and \((O_1)\) at \(H\). If the rays \(CG\) and \(CH\) meet the semi-circle \((O)\) at \(I\) and \(J\) respectively, \(K\) and \(L\) are the incenter and \(C\)-excenter of triangle \(CIJ\), respectively, then \(K\) divides both the segments \(CD\) and \(LC\) in the golden ratios (see Figure 1).

Proof. Since \(DC\) is the altitude of right triangle \(ABD\) and the quadrilateral \(ACHE\) is concyclic, 
\[
\angle BDC = \angle BAD = \angle CAE = \angle BHC.
\]
It follows that quadrilateral \(BCHD\) is concyclic.

Similarly, the quadrilateral \(ACGD\) is also concyclic.

Since \(AD\) touches the circles \((O_1)\) and \((O_2)\), \(DA \cdot DE = DC^2 = DB \cdot DF\) by the Power-of-a-point theorem and by the Intersecting-chords theorem, the quadrilateral \(ABFE\) is concyclic. Chasing angles, we have 
\[
\angle GCD = \angle GAD = \angle FAE = \angle FBE = \angle DBH = \angle DCH.
\]
This means that \(CD\) is the internal angle bisector of \(\angle ICJ\), and \(K\) belongs to \(CD\).

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Let $J'$ be the reflection of $J$ across $AB$. By symmetry, $J'$ belongs to the circle $(O)$ and $AJ = AJ'$.

It follows that $\angle JIA = \angle AIJ' = \angle AIC$. Hence, $AI$ is the internal angle bisector of $\angle CIJ$ and $K$ belongs to $AI$.

Similarly, $K$ also belongs to $BJ$.

Since $AJ$ and $BJ$ are perpendicular, $AJ$ is the external angle bisector of $\angle CJI$. We deduce that $L$ belongs to $AJ$. Similarly, $L$ also belongs to $BI$.

Then the quadrilaterals $ACIL$ and $BCJL$ are concyclic.

Let the lines $AH$ and $BG$ meet at $M$. Since $CI$ and $CJ$ are the anti-parallel of triangle $ABL$ and they are perpendicular to $BM$ and $AM$ respectively. This means that $M$ is the circumcenter of triangle $ABL$, and $MO$ is perpendicular to $AB$.

Since $KC$ touches $(O_1)$,

$$\angle MAO = \angle HAC = \angle HCK = \angle GCD = \angle GAD = \angle FAD.$$ 

It follows that right triangles $MAO$ and $FAD$ are similar,

$$\frac{MO}{AO} = \frac{FD}{AD} \implies MO = AO \cdot \frac{DF}{DA}.$$
We have $AO = \frac{AB}{2} = a + b$, $AD = \sqrt{AB \cdot AC} = 2\sqrt{a(a+b)}$, $DC = \sqrt{CA \cdot CB} = 2\sqrt{ab}$, and

$$DF = \frac{DC^2}{DB} = \frac{CA \cdot CB}{\sqrt{BA \cdot BC}} = \frac{4ab}{2\sqrt{b(a+b)}} = \frac{2a\sqrt{b(a+b)}}{a+b}.$$ 

It follows that

$$MO = AO \cdot \frac{DF}{DA} = \sqrt{ab} = \frac{DC}{2}.$$ 

Let $N$ be the orthogonal projection of $M$ onto $CD$. Then $N$ is the midpoint of $CD$ and $CN = \sqrt{ab}$.

By the Pythagorean theorem,

$$NL^2 = ML^2 - MN^2 = (a+b)^2 + ab - (a-b)^2 = 5ab$$

$$\implies LC = LN + CN = (\sqrt{5} + 1)\sqrt{ab}.$$ 

Let $K'$ be the reflection of $K$ across $AB$. By symmetry, $CK = CK'$ and

$$\angle AK' B = \angle AKB = 180^\circ - \angle ALB.$$ 

It follows that quadrilateral $ALBK'$ is concyclic. By the Intersecting-chords theorem,

$$CK \cdot CL = CK' \cdot CL = CA \cdot CB$$

$$\implies CK = \frac{CA \cdot CB}{CL} = \frac{2a \cdot 2b}{(\sqrt{5} + 1)\sqrt{ab}} = (\sqrt{5} - 1)\sqrt{ab}.$$ 

Hence,

$$\frac{KC}{KD} = \frac{KC}{CD - KC} = \frac{\sqrt{5} - 1}{3 - \sqrt{5}} = \varphi,$$ 

and

$$\frac{KL}{KC} = \frac{LC - CK}{CK} = \frac{2}{\sqrt{5} - 1} = \varphi.$$ 

These prove that $K$ divides both $CD$ and $LC$ in the golden ratios. \hfill \Box

**Remark 2.** It is easy to see that $CK = LD$ and $D$ divides both segments $LK$ and $CL$ in the golden ratios.

The famous Archimedean twin circles associated in the arbelos have equal radii $t := \frac{ab}{a+b}$ (see [2] and [3]). Circles with radius $t$ are called Archimedean and they are congruent to the Archimedean twin circles.

**Theorem 3.** If the perpendicular bisector of $CD$ meets $CG$ and $CH$ at $P$ and $Q$ respectively, then the circle with diameter $PQ$ is Archimedean (see Figure 2).
Proof. Since the right triangles $CPN$ and $AFD$ are similar, $CN = \sqrt{ab}$, $AD = 2\sqrt{a(a+b)}$ and $DF = \frac{2a\sqrt{a(a+b)}}{a+b}$, we deduce that $\frac{PN}{CN} = \frac{FD}{AD} \Rightarrow \frac{PN}{CN} = \frac{FD}{AD} = \frac{ab}{a+b} = t$.

Similarly, $QN = t$.

It follows that the circle with diameter $PQ$ is Archimedean. \hfill \Box

Remark 4. It is easy to see that if the line perpendicular to $CD$ at $D$ meets $CG$ and $CH$ at $R$ and $S$, respectively, then the circles with diameters $DR$ and $DS$ are Archimedean.

For two points $P$ and $Q$ in the plane, the circle with center $P$ passing through $Q$ is denoted by $P(Q)$.

Theorem 5. The circle $D(O)$ meets the perpendicular bisector of $AB$ again at $U$. The semi-circle $(O_1)$ and the segment $AU$ meet at $V$, $(O_2)$ and $BU$ at $W$. If the common external tangent lines of two circles $A(V)$ and $B(W)$ meet $CD$ at $D_1$ and $D_2$ such that $D, D_1, D_2$ are collinear in that order, then $D$ divides $D_2D_1$ in the golden ratio.

Proof. Since the right triangles $CAV$ and $CBW$ are similar, $\frac{CA}{CB} = \frac{AV}{BW}$.

This means that $C$ is the internal homothetic center of two circles $A(V)$, $B(W)$, and $CV$, $CW$ are the common internal tangent lines of two circles $A(V)$ and $B(W)$.

Let the line $BU$ and $(O)$ meet again at $B_1$, and let the lines $CV$ and $CW$ meet two common external tangent lines of two circles $A(V)$ and $B(W)$ at $X, Y, Z,$ and $T$ as show in the Figure 3.
Since $XA$ and $XB$ are the internal and external angle bisectors of $\angle CXY$, we get $\angle AXB = 90^\circ$. It follows that $X$ belongs to circle $(O)$. Similarly, the points $Y, Z, T$ also belong to circle $(O)$.

Note that $CY$ and $AB_1$ are both perpendicular to $BB_1$, they are parallel, and since $AX = AT$ by the symmetry under the axis $AB$, $YA$ bisects angle $XTY$. Chasing angles, we have $\angle AB_1X = \angle AYX = \angle AYT = \angle B_1AY$. It follows that $XB_1$ and $AY$ are parallel.

Since $AXB_1Y$ is the isosceles trapezoid with two bases $AY$ and $XB_1$, $AB_1 = XY$.

Let $H_1$ be the orthogonal projection of $D$ onto $OU$. Since $H_1$ is the midpoint of $OU$, we get

$$CD = OH_1 = \frac{OU}{2}.$$ 

Let us denote by $\alpha := \angle ACV$, then $\alpha = \angle BCW = \angle BAB_1 = \angle BCZ$. Since the right triangles $BAB_1$ and $BVO$ are similar,

$$\frac{AB_1}{BB_1} = \frac{UO}{BO} = \frac{2CD}{BO} = \frac{4CD}{AB} \Longrightarrow AB_1 = 4DC \frac{BB_1}{BA} = 4DC \sin \alpha.$$
By the symmetry, $XT$ and $YZ$ are both perpendicular to $AB$. Let the line $CD$ and circle $(O)$ meet again at $D_3$, $AB$ and $XT$ at $M_1$, $AB$ and $YZ$ at $M_2$. By the Intersecting-chords theorem and symmetry,

$$AB_1^2 = 16CD^2 \cdot \sin^2 \alpha = 16CD \cdot CD_3 \cdot \sin \alpha \cdot \sin \alpha$$

$$= 16CX \cdot CZ \cdot \frac{M_1X \cdot M_2Z}{CX \cdot CZ} = 16M_1X \cdot M_2Z = 4XT \cdot YZ.$$

It follows that

$$XY^2 = 4XT \cdot YZ. \quad (1)$$

Note that $D_1D_2$, $XT$ and $YZ$ are pairwise parallel. By the Thales’ theorem,

$$\frac{D_1X}{D_1Y} = \frac{CX}{CZ} = \frac{XT}{YZ} \Rightarrow \frac{D_1X}{XY} = \frac{XT}{XT + YZ}, \quad (2)$$

and similarly,

$$\frac{D_1Y}{XY} = \frac{YZ}{XT + YZ}, \quad (3)$$

Comparing (1), (2) with (3), we obtain

$$D_1X \cdot D_1Y = \frac{4XT^2 \cdot YZ^2}{(XT + YZ)^2}. \quad (4)$$

Again, by the Thales’ theorem,

$$\frac{CD_1}{XT} + \frac{CD_1}{YZ} = \frac{XD_1}{XY} + \frac{YD_1}{YX} = \frac{XD_1 + D_1Y}{XY} = 1 \Rightarrow CD_1 = \frac{XT \cdot YZ}{XT + YZ},$$

and similarly,

$$CD_2 = \frac{XT \cdot YZ}{XT + YZ}. \quad (5)$$

It follows that

$$D_1D_2 = CD_1 + CD_2 = \frac{2XT \cdot YZ}{XT + YZ}. \quad (6)$$

And by the Intersecting-chords theorem and symmetry,

$$D_1X \cdot D_1Y = D_1D_1D_3 = D_1D_2D_2D. \quad (7)$$

From (4), (5) and (6), we deduce that $D_1D_2^2 = DD_1 \cdot DD_2$. This proves that $D_1$ divides $D_2$ in the golden ratio. $\square$

**Theorem 6.** The external tangent line of two semi-circles $(O_1)$ and $(O_2)$ meets the semi-circle $(O)$ at $P_1$ and $P_2$ such that $A$, $P_1$, $D$, $P_2$ and $B$ lie on the semi-circle $(O)$ in that order. The line passing through $P_1$ perpendicular to $CP_1$ meets $(O)$ at $P_1$ and $Q_1$. Let $C_1$ be the circumcenter of triangle $CDQ_1$. Circle $C_1P_2$ meets $CD$ at $E_1$ and $F_1$ such that $E_1$, $D$, $C$ and $F_1$ lie on $CD$ in that order. Then $D$ divides both the segments $CE_1$ and $F_1C$ in the golden ratios.

**Proof.** (see Figure 4). Segment $DA$ meets the semi-circle $(O_1)$ at $E$, and segment $DB$ meets the semi-circle $(O_2)$ at $F$; let $M_0$ be the mid-point of $CD$.

We easily see that $CEDF$ is a rectangular. Hence $M_0$ is the mid-point of $EF$. Furthermore $\angle CEF = \angle CDB = \angle CAE$. It follows that $EF$ is
tangent with \((O_1)\). Similarly, \(EF\) is also tangent with \((O_2)\). Hence \(EF\) is the external common tangent line of two semi-circles \((O_1)\) and \((O_2)\). Hence \(EF\) and \(PQ\) are coincident.

Since two semi-circles \((O)\) and \((O_1)\) are inner tangent at \(A\), the homothety \(H_{\frac{a}{a+b}}\), center \(A\), ratio \(\frac{a}{a+b}\) transforms \((O)\) into \((O_1)\). Under the homothety \(H_{\frac{a}{a+b}}\), points \(O\), \(D\) go into points \(O_1\), \(E\). Hence \(OD\) and \(O_1E\) are parallel. Since \(EF\) is tangent with \((O_1)\), \(O_1E\) is perpendicular to \(EF\). It follows that \(OD\) is perpendicular to \(P_1P_2\). This thing proves that \(OD\) is the perpendicular bisector of \(P_1P_2\). It follows \(DP_1 = DP_2\).

We have \(\angle DP_1P_2 = \angle DP_2P_1 = \angle DAP_1\). This thing proves that \(DP_1\) is tangent with the circumcircle of triangle \(AEP_1\). Applying the Power-of-a-point theorem, we have \(DP_1^2 = DA \cdot DE = DC^2\). It follows \(DP_1 = DC\). Since \(DP_1 = DP_2\), \(DC = DP_1 = DP_2\), it follows that

\[
\text{(7)} \quad D \text{ is the circumcenter of triangle } C_1 P_1 P_2.
\]

Let \(C'\) be the point symmetric to \(C\) across \(D\). Then \(DC = DP_1 = DC'\). It follows \(\angle CP_1C' = 90^\circ\). Hence three points \(C', P_1\) and \(Q_1\) are collinear. Applying the Power-of-a-point theorem with note that \(CC' = 2CD = 4\sqrt{ab}\)
and $CM_0 = \frac{1}{2} CD = \sqrt{ab}$, and $C'M_0 = C'C - CM_0 = 3\sqrt{ab}$, we have
\[ C'P_1.C'Q_1 = C'O^2 - DO^2 = C'C^2 - DC^2 = 12ab = C'M_0.C'C. \]
Applying the Intersecting-chords theorem, we have quadrilateral $CM_0P_1Q_1$ being concyclic.
Hence $\angle CM_0Q_1 = \angle CP_1Q_1 = 90^\circ$. Hence $Q_1M_0$ is perpendicular to $CD$ at the mid-point $M_0$ of $CD$ so triangle $CDQ_1$ is isosceles at $Q_1$. Thus, $C_1$ belongs to $Q_1M_0$. Hence $E_1$ and $F_1$ are symmetric about point $M_0$ and $CF_1 = DE_1$.
On the other hand, since triangle $CDQ_1$ is isosceles at $Q_1$ and $D$ is the circumcenter of triangle $CP_1P_2$, $\angle CDQ_2 = 2\angle CP_1P_2 = 2\angle CQ_1M_0 = \angle CQ_1D$. Hence $DP_2$ is tangent with the circumcircle of triangle $CDQ_1$.
Thus, $DC_1$ is perpendicular to $DP_2$.
Applying the Pythagorean theorem, we have
\[ CD^2 = P_2D^2 = P_2C_1^2 - C_1D^2 = E_1C_1^2 - (C_1M_0^2 + M_0D^2) \
= (E_1C_1^2 - C_1M_0^2) - M_0D^2 \
= E_1M_0^2 - M_0D^2 \
= (E_1M_0 - M_0D) \cdot (E_1M_0 + M_0D) \
= E_1D \cdot (E_1M_0 + M_0C) \
= E_1D \cdot E_1C \
= E_1D (E_1D + CD). \]
It follows $CD^2 = E_1D^2 + E_1D.CD$. The above equality proves that $\frac{CD}{E_1D} = \frac{\sqrt{5+1}}{2} = \varphi$, is golden ratio.
Furthermore, since $CF_1 = DE_1$, it follows $\frac{DC}{F_1C} = \varphi$ and $\frac{DF_1}{DC} = \frac{\varphi + 1}{\varphi} = \varphi$. This means that point $D$ divides $F_1C$ in the golden ratio. $\square$
Since the figuration of theorem 6, we obtain a following result on the Archimedean circle.
Theorem 7. Line $P_1Q_1$ meets the circumcircle of triangle $CDQ_1$ at $K_1$. Let $L_1$ be the projection from $K_1$ onto $P_1P_2$. Then the circle with diameter $K_1L_1$ is Archimedean (see Figure 5).

Proof. Let $H_0$ be the point of intersection of $DO$ and $P_1P_2$. Since (7), we have $OD$ perpendicular to $P_1P_2$ at $H_0$, quadrilateral $CM_0P_1Q_1$ is concyclic.

Hence $\angle DK_1P_1 = 180^\circ - \angle DCQ_1 = \angle M_0P_1Q_1$. It follows that $DK_1$ and $M_0P_1$ are parallel. This means that $K_1L_1 = DH_0$.

On the other hand, the circle with diameter $DH_0$ is Archimedean (it is the circle $(W_4)$ in [2], also the circle $(A_3)$ in [3]).

Thus, the circle with diameter $K_1L_1$ is Archimedean. □

Theorem 8. The external tangent line of two semi-circles $(O_1)$ and $(O_2)$ meets the semi-circle $(O)$ at $P_1$ and $P_2$, and meets $CD$ at $M_0$. Line $AM_0$ meets $DO_2$ at $A_0$. Let $O_0$ be the point symmetric to $O$ across $D$. Circle $O_0(A_0)$ meets $DP_1$ at $X_1$ and $Y_1$ such that $P_1, X_1, D$ and $Y_1$ lie on $DP_1$ in that order. Then point $X_1$ divides $DP_1$ in the golden ratio, and point $Y_1$ divides $P_1D$ in the golden ratio.

Proof. (see Figure 6). Let $H_0$ be the point of intersection of $OD$ and $P_1P_2$. Then $OD$ is perpendicular to $P_1P_2$ at $H_0$ and $DH_0 = 2t$.

![Figure 6](image-url)

Since $O_2, M_0$ are the midpoints of $CB, CD$, respectively, $O_2M_0$ is parallel to $BD$, from $BD$ is perpendicular to $AD$, $O_2M_0$ is perpendicular to $AD$. 
It follows that $M_0$ is the orthocenter of triangle $ADO_2$. Hence $AA_0$ is perpendicular to $DO_2$ at $A_0$.

Since $CD = 2\sqrt{ab}$ and $CO_2 = b$,

\[(8) \quad O_2A_0 + DA_0 = DO_2 = \sqrt{O_2C^2 + CD^2} = \sqrt{b^2 + 4ab}.
\]

Note that $AO_2 = 2a + b$, $AD = \sqrt{AC \cdot AB} = 2\sqrt{a(a + b)}$, we have

\[O_2A_0^2 - DA_0^2 = O_2A^2 - DA^2 = (2a + b)^2 - 4a(a + b) = b^2\]

\[\implies (O_2A_0 + DA_0)(O_2A_0 - DA_0) = b^2.
\]

Combining with (8), we obtain

\[(9) \quad O_2A_0 - DA_0 = \frac{b^2}{O_2A_0 + DA_0} = \frac{b^2}{\sqrt{b^2 + 4ab}}
\]

Since (8) and (9), it follows

\[(10) \quad DA_0 = \frac{2ab}{\sqrt{b^2 + 4ab}} \text{ và } O_2A_0 = \frac{b^2 + 2ab}{\sqrt{b^2 + 4ab}}.
\]

Let $R_1$ be the projection from $O$ onto $DO_2$. Applying the Thales’ theorem with the note (10), we have $\frac{O_2R_1}{O_2A_0} = \frac{O_2O}{O_2A}$. Since $O_2O = a$,

\[O_2R_1 = O_2A_0 \frac{O_2O}{O_2A} = \frac{b^2 + 2ab}{\sqrt{b^2 + 4ab}} \cdot \frac{a}{2a + b} = \frac{ab}{\sqrt{b^2 + 4ab}}.
\]

Since (10) and (11), it follows $DA_0 = 2O_2R_1$. Let $S_1$ be the point symmetric to $A_0$ across $D$, and let $T_1$ be the point symmetric to $D$ across $O_1$. Then

\[R_1S_1 = R_1D + DS_1 = DA_0 + (DO_2 - O_2R_1) = DA_0 - O_2R_1 + DO_2
\]

\[= O_2R_1 + DO_2 = O_2R_1 + O_2T_1
\]

\[= R_1T_1.
\]

It follows that $OR_1$ is the perpendicular bisector of $S_1T_1$. Hence $OS_1 = OT_1$.

Applying the property of rotation about center, we have $A_0O_0 = OS_1 = OT_1$, and $BT_1$ is parallel and equal to $DC$, so $BT_1$ is perpendicular to $OB$.

Applying the Pythagorean theorem, we have

\[(12) \quad O_0A_0 = OT_1 = \sqrt{T_1B^2 + BO^2} = \sqrt{4ab + (a + b)^2} = \sqrt{a^2 + b^2 + 6ab}.
\]

Since (7) we have $DP_1 = DC = 2\sqrt{ab}$. Hence

\[(13) \quad \cos \angle H_0DP_1 = \frac{DH_0}{DP_1} = \frac{2t}{DP_1} = \frac{2ab}{(a + b)2\sqrt{ab}} = \frac{\sqrt{ab}}{a + b}.
\]

Applying the Cosine’s law to triangle $O_0DX_1$, we have

\[O_0X_1^2 = O_0D^2 + X_1D^2 - 2O_0.D.X_1.D \cdot \cos \angle O_0DX_1
\]

\[= O_0D^2 + X_1D^2 + 2O_0.D.X_1.D \cdot \cos \angle H_0DP_1
\]

\[= (a + b)^2 + X_1D^2 + 2(a + b).X_1.D \cdot \cos \angle H_0DP_1.
\]
Note that $O_0X_1 = O_0A_0$. Combining with (12) and (13), we deduce that
\[
a^2 + b^2 + 6ab = (a + b)^2 + X_1D^2 + 2(a + b).X_1D\frac{\sqrt{ab}}{a + b}
\]
\[
= (a + b)^2 + X_1D^2 + 2\sqrt{ab}.X_1D.
\]
It follows $X_1D^2 + 2\sqrt{ab}X_1D - 4ab = 0$. From this, we get
\[
X_1D = (-1 + \sqrt{5})\sqrt{ab}.
\]  
(14)

Since (14), it follows
\[
X_1P_1 = DP_1 - X_1D = 2\sqrt{ab} - (-1 + \sqrt{5})\sqrt{ab} = (3 - \sqrt{5})\sqrt{ab}.
\]  
(15)

Since (14) and (15), it follows
\[
\frac{X_1D}{X_1P_1} = \frac{(-1 + \sqrt{5})\sqrt{ab}}{(3 - \sqrt{5})\sqrt{ab}} = \frac{1 + \sqrt{5}}{2} = \varphi.
\]

This means that $X_1$ divides $DP_1$ in the golden ratio.

On the other hand, applying the Power-of-a-point theorem with the note that $O_0D = OD = a + b$ and (12), (14), we have
\[
DX_1DY_1 = O_0A_0^2 - DO_0^2 = 4ab \implies Y_1D = \frac{4ab}{DX_1} = (1 + \sqrt{5})\sqrt{ab}.
\]

It follows
\[
\frac{Y_1D}{DP_1} = \frac{(1 + \sqrt{5})\sqrt{ab}}{2\sqrt{ab}} = \varphi \quad \text{and} \quad \frac{Y_1P_1}{Y_1D} = \frac{\varphi + 1}{\varphi} = \varphi.
\]

Hence point $Y_1$ divides segment $P_1D$ in the golden ratio. □

From the configuration of theorem 8, we obtain a pair of Archimedean circles as follows

**Theorem 9.** The circle $O_0(A_0)$ meets the semi-circle $(O)$ at $U_1$ and $U_2$. Then the circles that are tangent with $P_1P_2$ and their centers are $U_1$ and $U_2$, respectively are Archimedean (see figure 7).

**Proof.** Let $V_1$ be the projection from $U_1$ onto $OO_0$.

We have
\[
O_0V_1 + OV_1 = OO_0 = 2OD = 2(a + b).
\]  
(16)

Note that $U_1O_0 = O_0A_0 = \sqrt{a^2 + b^2 + 6ab}$ (since (12)), $DH_0 = 2t$ and $U_1O = a + b$. We have
\[
V_1O_0^2 - V_1O^2 = U_1O_0^2 - U_1O^2 = a^2 + b^2 + 6ab - (a + b)^2 = 4ab.
\]

It follows $OO_0(O_0V_1 - OV_1) = 4ab$. From this, we get
\[
O_0V_1 - OV_1 = \frac{4ab}{OO_0} = \frac{4ab}{2(a + b)} = \frac{2ab}{a + b} = 2t.
\]  
(17)

Since (16) and (17), it follows
\[
O_0V_1 = a + b + t \quad \text{and} \quad OV_1 = a + b - t.
\]

Hence
\[
V_1H_0 = |OV_1 - H_0O| = |a + b - t - (a + b)| = t.
\]
This proves that the distance from $U_1$ to $P_1P_2$ is equal to $t$.

Similarly, the distance from $U_2$ to $P_1P_2$ is also equal to $t$.

It follows that the circles are tangent with $P_1P_2$ and their centers are $U_1$ and $U_2$, respectively such that their radii are equal to $t$, so they are Archimedean. The theorem is proved. □

References


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