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GOLDEN SECTIONS AND ARCHIMEDEAN CIRCLES IN AN ARBELOS

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Abstract. We construct some golden ratios in the arbelos and Archimedean circles in this configuration.

Consider an arbelos formed by semi-circles (O_1), (O_2), and (O) of radii a, b, and $a+b$. The semi-circles (O_1) and (O) meet at A, (O_2) and (O) at B, (O_1) and (O_2) at C. Let CD be the divided line of the smaller semi-circles.

A segment PQ is called to be divided in the golden ratio by a point R if $\frac{PQ}{PR} = \frac{PR}{RQ}$. In this case, the divided ratio is the golden ratio $\varphi := \frac{\sqrt{5}+1}{2}$, which satisfies $\varphi^2 = \varphi + 1$.

Theorem 1. *Let the segment AD and semi-circle (O_1) meet again at E, BD and (O_2) at F, AF and (O_2) at G, BE and (O_1) at H. If the rays CG and CH meet the semi-circle (O) at I and J respectively, K and L are the incenter and C-excenter of triangle CIJ, respectively, then K divides both the segments CD and LC in the golden ratios (see Figure 1).*

Proof. Since DC is the altitude of right triangle ABD and the quadrilateral $ACHE$ is concyclic,

$$\angle BDC = \angle BAD = \angle CAE = \angle BHC.$$

It follows that quadrilateral $BCHD$ is concyclic.

Similarly, the quadrilateral $ACGD$ is also concyclic.

Since AD touches the circles (O_1) and (O_2), $DA.DE = DC^2 = DB.DF$ by the Power-of-a-point theorem and by the Intersecting-chords theorem, the quadrilateral $ABFE$ is concyclic. Chasing angles, we have

$$\angle GCD = \angle GAD = \angle FAE = \angle FBE = \angle DBH = \angle DCH.$$

This means that CD is the internal angle bisector of $\angle ICJ$, and K belongs to CD.

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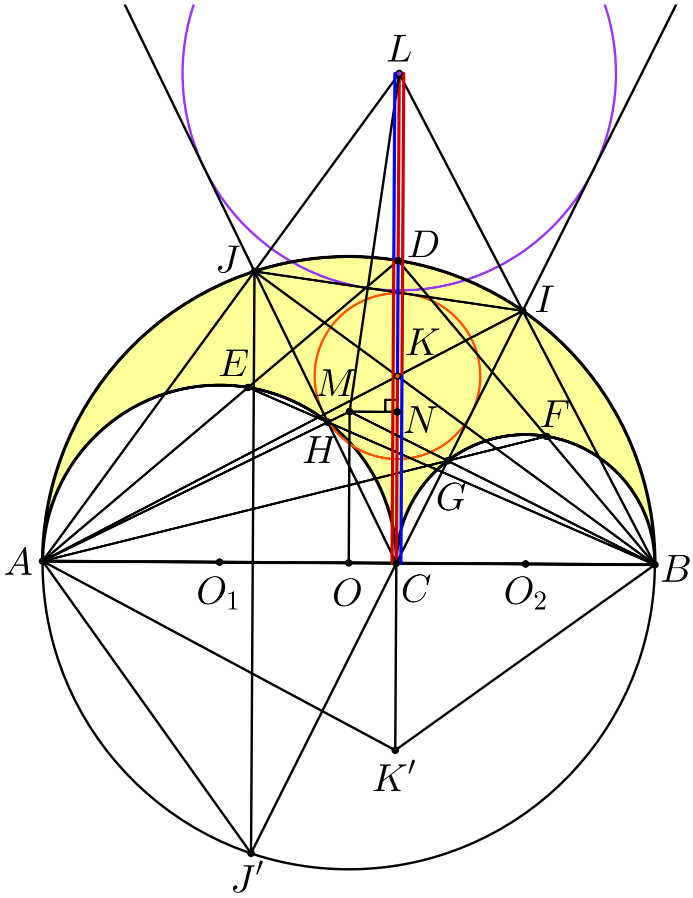

Figure 1.

Let J' is the reflection of J across AB. By symmetry, J' belongs to the circle (O) and $AJ = AJ'$.
It follows that $\angle JIA = \angle AIJ' = \angle AIC$. Hence, AI is the internal angle bisector of $\angle CIJ$ and K belongs to AI.

Similarly, K also belongs to BJ.

Since AJ and BJ are perpendicular, AJ is the external angle bisector of $\angle CJI$. We deduce that L belongs to AJ. Similarly, L also belongs to BI. Then the quadrilaterals $ACIL$ and $BCJL$ are concyclic.

Let the lines AH and BG meet at M. Since CI and CJ are the anti-parallel of triangle ABL and they are perpendicular to BM and AM respectively. This means that M is the circumcenter of triangle ABL, and MO is perpendicular to AB.

Since KC touches (O_1),

$$\angle MAO = \angle HAC = \angle HCK = \angle GCD = \angle GAD = \angle FAD.$$

It follows that right triangles MAO and FAD are similar,

$$\frac{MO}{AO} = \frac{FD}{AD} \Longrightarrow MO = AO\frac{DF}{DA}.$$

We have $AO = \frac{AB}{2} = a+b$, $AD = \sqrt{AB.AC} = 2\sqrt{a(a+b)}$, $DC = \sqrt{CA.CB} = 2\sqrt{ab}$, and

$$DF = \frac{DC^2}{DB} = \frac{CA.CB}{\sqrt{BA.BC}} = \frac{4ab}{2\sqrt{b(a+b)}} = \frac{2a\sqrt{b(a+b)}}{a+b}.$$

It follows that

$$MO = AO\frac{DF}{DA} = \sqrt{ab} = \frac{DC}{2}.$$

Let N be the orthogonal projection of M onto CD. Then N is the midpoint of CD and $CN = \sqrt{ab}$.

By the Pythagorean theorem,

$$ML^2 = MA^2 = AO^2 + MO^2 = (a+b)^2 + ab,$$

and since $CNMO$ is the rectangle,

$$MN = CO = |AC - AO| = |2a - (a+b)| = |a-b|\,.$$

It follows that

$$NL^2 = ML^2 - MN^2 = (a+b)^2 + ab - (a-b)^2 = 5ab$$
$$\Longrightarrow LC = LN + CN = (\sqrt{5}+1)\sqrt{ab}.$$

Let K' be the reflection of K across AB. By symmetry, $CK = CK'$ and

$$\angle AK'B = \angle AKB = 180^\circ - \angle ALB.$$

It follows that quadrilateral $ALBK'$ is concyclic. By the Intersecting-chords theorem,

$$CK.CL = CK'.CL = CA.CB$$
$$\Longrightarrow CK = \frac{CA.CB}{CL} = \frac{2a.2b}{(\sqrt{5}+1)\sqrt{ab}} = (\sqrt{5}-1)\sqrt{ab}.$$

Hence,

$$\frac{KC}{KD} = \frac{KC}{CD - KC} = \frac{\sqrt{5}-1}{3-\sqrt{5}} = \varphi,$$

and

$$\frac{KL}{KC} = \frac{LC - CK}{CK} = \frac{2}{\sqrt{5}-1} = \varphi.$$

These prove that K divides both CD and LC in the golden ratios. □

Remark 2. It is easy to see that $CK = LD$ and D divides both segments LK and CL in the golden ratios.

The famous Archimedean twin circles associated in the arbelos have equal radii $t := \frac{ab}{a+b}$ (see [2] and [3]). Circles with radius t are called Archimedean and they are congruent to the Archimedean twin circles.

Theorem 3. *If the perpendicular bisector of CD meets CG and CH at P and Q respectively, then the circle with diameter PQ is Archimedean (see Figure 2).*

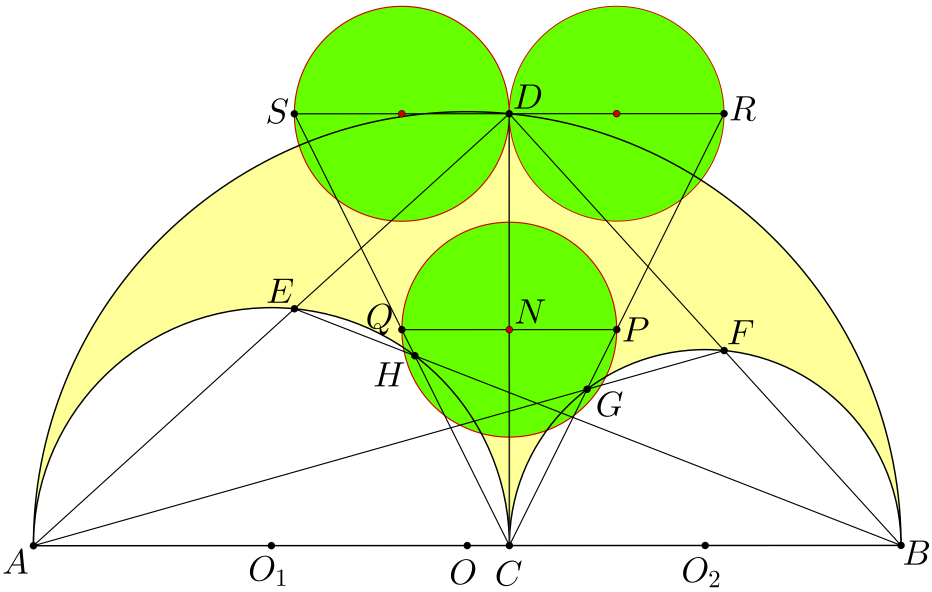

Figure 2.

Proof. Since the right triangles CPN and AFD are similar, $CN = \sqrt{ab}$, $AD = 2\sqrt{a(a+b)}$ and $DF = \frac{2a\sqrt{b(a+b)}}{a+b}$, we deduce that

$$\frac{PN}{CN} = \frac{FD}{AD} \Longrightarrow PN = CN\frac{FD}{AD} = \frac{ab}{a+b} = t.$$

Similarly, $QN = t$.

It follows that the circle with diameter PQ is Archimedean. □

Remark 4. It is easy to see that if the line perpendicular to CD at D meets CG and CH at R and S, respectively, then the circles with diameters DR and DS are Archimedean.

For two points P and Q in the plane, the circle with center P passing through Q is denoted by $P(Q)$.

Theorem 5. *The circle $D(O)$ meets the perpendicular bisector of AB again at U. The semi-circle (O_1) and the segment AU meet at V, (O_2) and BU at W. If the common external tangent lines of two circles $A(V)$ and $B(W)$ meet CD at D_1 and D_2 such that D, D_1, D_2 are collinear in that order, then D divides D_2D_1 in the golden ratio.*

Proof. Since the right triangles CAV and CBW are similar,

$$\frac{CA}{CB} = \frac{AV}{BW}.$$

This means that C is the internal homothetic center of two circles $A(V)$, $B(W)$, and CV, CW are the common internal tangent lines of two circles $A(V)$ and $B(W)$.

Let the line BU and (O) meet again at B_1, and let the lines CV and CW meet two common external tangent lines of two circles $A(V)$ and $B(W)$ at X, Y, Z, and T as show in the Figure 3.

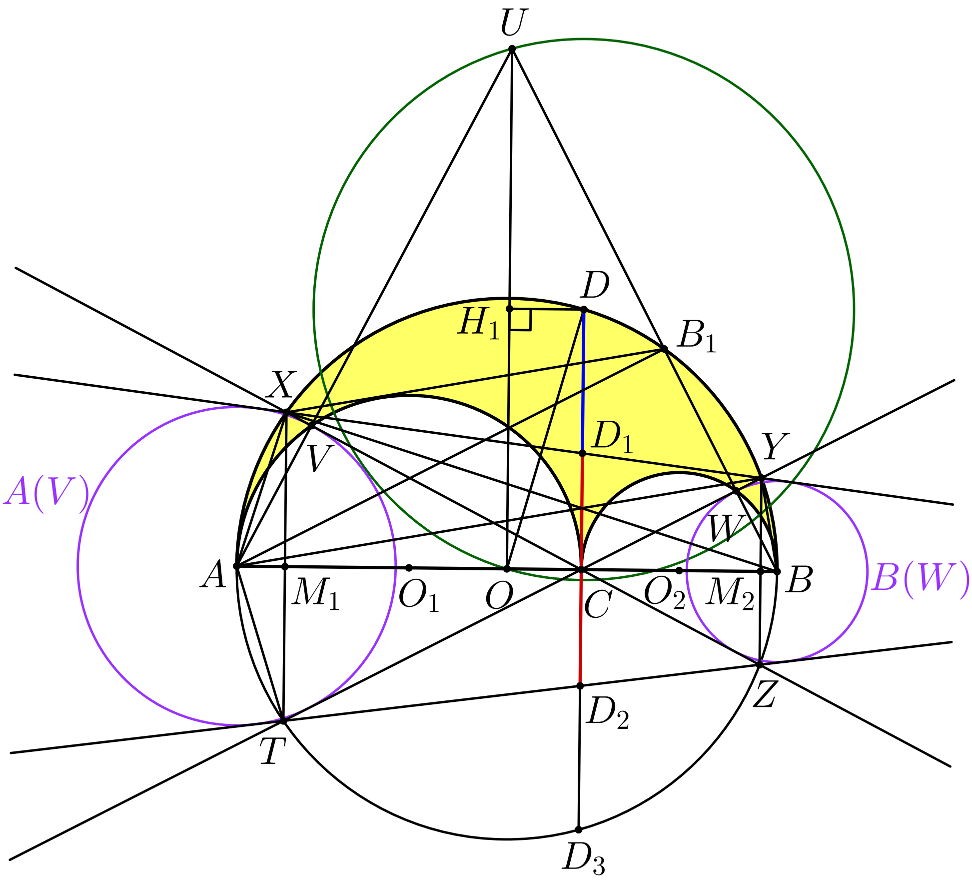

Figure 3.

Since XA and XB are the internal and external angle bisectors of $\angle CXY$, we get $\angle AXB = 90°$. It follows that X belongs to circle (O). Similarly, the points Y, Z, T also belong to circle (O).

Note that CY and AB_1 are both perpendicular to BB_1, they are parallel, and since $AX = AT$ by the symmetry under the axis AB, YA bisects angle XTY. Chasing angles, we have $\angle AB_1X = \angle AYX = \angle AYT = \angle B_1AY$. It follows that XB_1 and AY are parallel.

Since AXB_1Y is the isosceles trapezoid with two bases AY and XB_1, $AB_1 = XY$.

Let H_1 be the orthogonal projection of D onto OU. Since H_1 is the midpoint of OU, we get

$$CD = OH_1 = \frac{OU}{2}.$$

Let us denote by $\alpha := \angle ACV$, then $\alpha = \angle BCW = \angle BAB_1 = \angle BCZ$. Since the right triangles BAB_1 and BUO are similar,

$$\frac{AB_1}{BB_1} = \frac{UO}{BO} = \frac{2CD}{BO} = \frac{4CD}{AB} \Longrightarrow AB_1 = 4DC\frac{BB_1}{BA} = 4DC.\sin\alpha.$$

By the symmetry, XT and YZ are both perpendicular to AB. Let the line CD and circle (O) meet again at D_3, AB and XT at M_1, AB and YZ at M_2. By the Intersecting-chords theorem and symmetry,

$$AB_1^2 = 16CD^2.\sin^2\alpha = 16CD.CD_3.\sin\alpha.\sin\alpha$$
$$= 16CX.CZ\frac{M_1X}{CX}\frac{M_2Z}{CZ} = 16M_1X.M_2Z = 4XT.YZ.$$

It follows that

(1) $$XY^2 = 4XT.YZ.$$

Note that D_1D_2, XT and YZ are pairwise parallel. By the Thales' theorem,

(2) $$\frac{D_1X}{D_1Y} = \frac{CX}{CZ} = \frac{XT}{YZ} \Longrightarrow \frac{D_1X}{XY} = \frac{XT}{XT+YZ},$$

and similarly,

(3) $$\frac{D_1Y}{XY} = \frac{YZ}{XT+YZ}.$$

Comparing (1), (2) with (3), we obtain

(4) $$D_1X.D_1Y = \frac{4XT^2.YZ^2}{(XT+YZ)^2}.$$

Again, by the Thales' theorem,

$$\frac{CD_1}{XT} + \frac{CD_1}{YZ} = \frac{XD_1}{XY} + \frac{YD_1}{YX} = \frac{XD_1 + D_1Y}{XY} = 1 \Longrightarrow CD_1 = \frac{XT.YZ}{XT+YZ},$$

and similarly,

$$CD_2 = \frac{XT.YZ}{XT+YZ}.$$

It follows that

(5) $$D_1D_2 = CD_1 + CD_2 = \frac{2XT.YZ}{XT+YZ}.$$

And by the Intersecting-chords theorem and symmetry,

(6) $$D_1X.D_1Y = D_1D.D_1D_3 = D_1D.D_2D.$$

From (4), (5) and (6), we deduce that $D_1D_2^2 = DD_1.DD_2$. This proves that D_1 divides D_2D in the golden ratio. □

Theorem 6. *The external tangent line of two semi-circles (O_1) and (O_2) meets the semi-circle (O) at P_1 and P_2 such that A, P_1, D, P_2 and B lie on the semi-circle (O) in that order. The line passing through P_1 perpendicular to CP_1 meets (O) at P_1 and Q_1. Let C_1 be the circumcenter of triangle CDQ_1. Circle $C_1(P_2)$ meets CD at E_1 and F_1 such that E_1, D, C and F_1 lie on CD in that order. Then D divides both the segments CE_1 and F_1C in the golden ratios.*

Proof. (see Figure 4). Segment DA meets the semi-circle (O_1) at E, and segment DB meets the semi-circle (O_2) at F; let M_0 be the mid-point of CD.

We easily see that $CEDF$ is a rectangular. Hence M_0 is the mid-point of EF. Furthermore $\angle CEF = \angle CDB = \angle CAE$. It follows that EF is

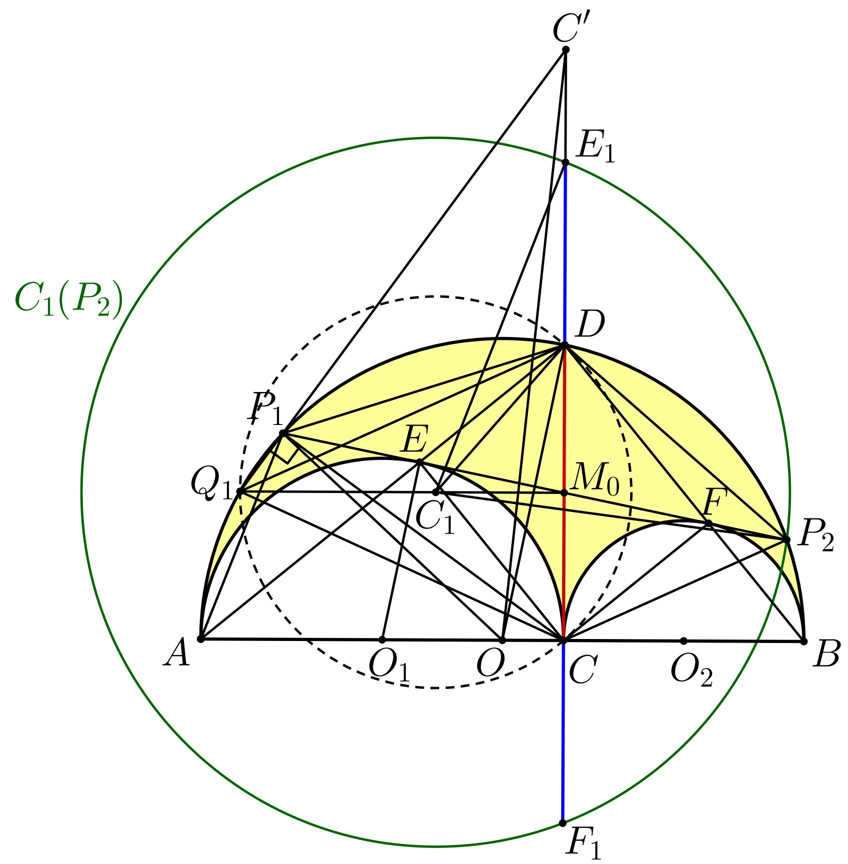

Figure 4.

tangent with (O_1). Similarly, EF is also tangent with (O_2). Hence EF is the external common tangent line of two semi-circles (O_1) and (O_2). Hence EF and PQ are coincident.

Since two semi-circles (O) and (O_1) are inner tangent at A, the homothety $\mathcal{H}_A^{\frac{a}{a+b}}$ center A, ratio $\frac{a}{a+b}$ transforms (O) into (O_1). Under the homothety $\mathcal{H}_A^{\frac{a}{a+b}}$, points O, D go into points O_1, E. Hence OD and O_1E are parallel. Since EF is tangent with (O_1), O_1E is perpendicular to EF. It follows that OD is perpendicular to P_1P_2. This thing proves that OD is the perpendicular bisector of P_1P_2. It follows $DP_1 = DP_2$.

We have $\angle DP_1P_2 = \angle DP_2P_1 = \angle DAP_1$. This thing proves that DP_1 is tangent with the circumcircle of triangle AEP_1. Applying the Power-of-a-point theorem, we have $DP_1^2 = DA.DE = DC^2$. It follows $DP_1 = DC$. Since $DP_1 = DP_2$, $DC = DP_1 = DP_2$, it follows that

(7) $\qquad D$ is the circumcenter of triangle CP_1P_2.

Let C' be the point symmetric to C across D. Then $DC = DP_1 = DC'$. It follows $\angle CP_1C' = 90°$. Hence three points C', P_1 and Q_1 are collinear. Applying the Power-of-a-point theorem with note that $CC' = 2CD = 4\sqrt{ab}$

and $CM_0 = \frac{1}{2}CD = \sqrt{ab}$, and $C'M_0 = C'C - CM_0 = 3\sqrt{ab}$, we have

$$C'P_1.C'Q_1 = C'O^2 - DO^2 = C'C^2 - DC^2 = 12ab = C'M_0.C'C.$$

Applying the Intersecting-chords theorem, we have quadrilateral $CM_0P_1Q_1$ being concyclic.
Hence $\angle CM_0Q_1 = \angle CP_1Q_1 = 90°$. Hence Q_1M_0 is perpendicular to CD at the mid-point M_0 of CD so triangle CDQ_1 is isosceles at Q_1. Thus, C_1 belongs to Q_1M_0. Hence E_1 and F_1 are symmetric about point M_0 and $CF_1 = DE_1$.

On the other hand, since triangle CDQ_1 is isosceles at Q_1 and D is the circumcenter of triangle CP_1P_2, $\angle CDP_2 = 2\angle CP_1P_2 = 2\angle CQ_1M_0 = \angle CQ_1D$. Hence DP_2 is tangent with the circumcircle of triangle CDQ_1. Thus, DC_1 is perpendicular to DP_2.

Applying the Pythagorean theorem, we have

$$\begin{aligned}
CD^2 = P_2D^2 = P_2C_1^2 - C_1D^2 &= E_1C_1^2 - \left(C_1M_0^2 + M_0D^2\right)\\
&= \left(E_1C_1^2 - C_1M_0^2\right) - M_0D^2\\
&= E_1M_0^2 - M_0D^2\\
&= (E_1M_0 - M_0D)\,.\,(E_1M_0 + M_0D)\\
&= E_1D.\,(E_1M_0 + M_0C)\\
&= E_1D.E_1C\\
&= E_1D.\,(E_1D + CD)\,.
\end{aligned}$$

It follows $CD^2 = E_1D^2 + E_1D.CD$. The above equality proves that $\frac{CD}{E_1D} = \frac{\sqrt{5}+1}{2} = \varphi$, is golden ratio.

Furthermore, since $CF_1 = DE_1$, it follows $\frac{DC}{F_1C} = \varphi$ and $\frac{DF_1}{DC} = \frac{\varphi+1}{\varphi} = \varphi$. This means that point D divides F_1C in the golden ratio. □

Since the figuration of theorem 6, we obtain a following result on the Archimedean circle.

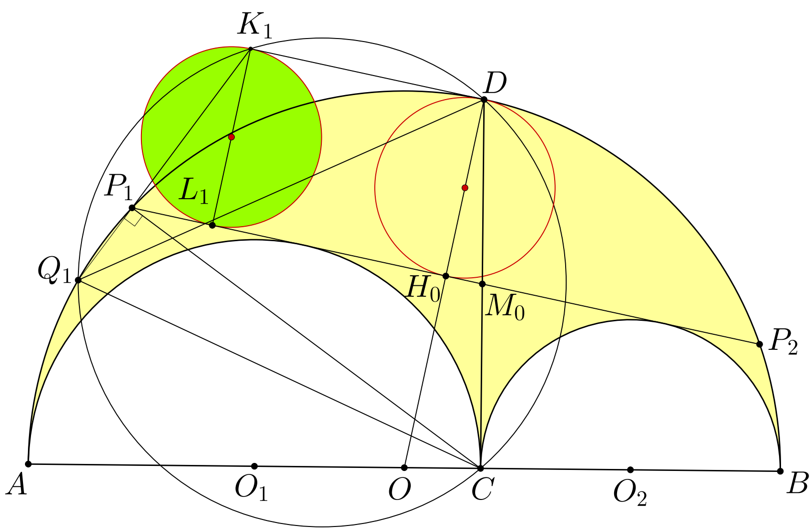

Figure 5.

Theorem 7. *Line P_1Q_1 meets the circumcircle of triangle CDQ_1 at K_1. Let L_1 be the projection from K_1 onto P_1P_2. Then the circle with diameter K_1L_1 is Archimedean (see Figure 5).*

Proof. Let H_0 be the point of intersection of DO and P_1P_2. Since (7), we have OD perpendicular to P_1P_2 at H_0, quadrilateral $CM_0P_1Q_1$ is concyclic.

Hence $\angle DK_1P_1 = 180° - \angle DCQ_1 = \angle M_0P_1Q_1$. It follows that DK_1 and M_0P_1 are parallel. This means that $K_1L_1 = DH_0$.

On the other hand, the circle with diameter DH_0 is Archimedean (it is the circle (W_4) in [2], also the circle (A_3) in [3]).

Thus, the circle with diameter K_1L_1 is Archimedean. □

Theorem 8. *The external tangent line of two semi-circles (O_1) and (O_2) meets the semi-circle (O) at P_1 and P_2, and meets CD at M_0. Line AM_0 meets DO_2 at A_0. Let O_0 be the point symmetric to O across D. Circle $O_0(A_0)$ meets DP_1 at X_1 and Y_1 such that P_1, X_1, D and Y_1 lie on DP_1 in that order. Then point X_1 divides DP_1 in the golden ratio, and point Y_1 divides P_1D in the golden the ratio.*

Proof. (see Figure 6). Let H_0 be the point of intersection of OD and P_1P_2. Then OD is perpendicular to P_1P_2 at H_0 and $DH_0 = 2t$.

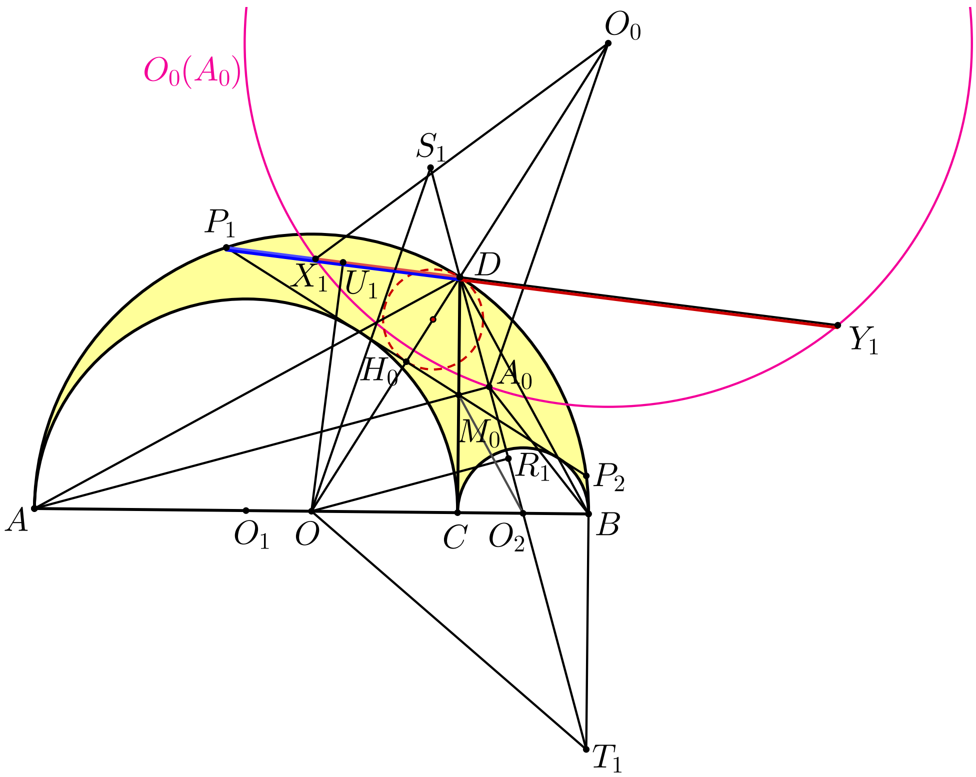

Figure 6.

Since O_2, M_0 are the midpoints of CB, CD, respectively, O_2M_0 is parallel to BD, from BD is perpendicular to AD, O_2M_0 is perpendicular to AD.

It follows that M_0 is the orthocenter of triangle ADO_2. Hence AA_0 is perpendicular to DO_2 at A_0.

Since $CD = 2\sqrt{ab}$ and $CO_2 = b$,

$$O_2A_0 + DA_0 = DO_2 = \sqrt{O_2C^2 + CD^2} = \sqrt{b^2 + 4ab}. \tag{8}$$

Note that $AO_2 = 2a + b$, $AD = \sqrt{AC.AB} = 2\sqrt{a(a + b)}$, we have

$$\begin{aligned} &O_2A_0^2 - DA_0^2 = O_2A^2 - DA^2 = (2a + b)^2 - 4a(a + b) = b^2 \\ \Longrightarrow &(O_2A_0 + DA_0)(O_2A_0 - DA_0) = b^2. \end{aligned}$$

Combining with (8), we obtain

$$O_2A_0 - DA_0 = \frac{b^2}{O_2A_0 + DA_0} = \frac{b^2}{\sqrt{b^2 + 4ab}}. \tag{9}$$

Since (8) and (9), it follows

$$DA_0 = \frac{2ab}{\sqrt{b^2 + 4ab}} \text{ và } O_2A_0 = \frac{b^2 + 2ab}{\sqrt{b^2 + 4ab}}. \tag{10}$$

Let R_1 be the projection from O onto DO_2. Applying the Thales' theorem with the note (10), we have $\frac{O_2R_1}{O_2A_0} = \frac{O_2O}{O_2A}$. Since $O_2O = a$,

$$O_2R_1 = O_2A_0\frac{O_2O}{O_2A} = \frac{b^2 + 2ab}{\sqrt{b^2 + 4ab}} \cdot \frac{a}{2a + b} = \frac{ab}{\sqrt{b^2 + 4ab}}. \tag{11}$$

Since (10) and (11), it follows $DA_0 = 2O_2R_1$. Let S_1 be the point symmetric to A_0 across D, and let T_1 be the point symmetric to D across O_1. Then

$$\begin{aligned} R_1S_1 &= R_1D + DS_1 = DA_0 + (DO_2 - O_2R_1) = DA_0 - O_2R_1 + DO_2 \\ &= O_2R_1 + DO_2 = O_2R_1 + O_2T_1 \\ &= R_1T_1. \end{aligned}$$

It follows that OR_1 is the perpendicular bisector of S_1T_1. Hence $OS_1 = OT_1$.

Applying the property of rotation about center, we have $A_0O_0 = OS_1 = OT_1$, and BT_1 is parallel and equal to DC, so BT_1 is perpendicular to OB. Applying the Pythagorean theorem, we have

$$O_0A_0 = OT_1 = \sqrt{T_1B^2 + BO^2} = \sqrt{4ab + (a + b)^2} = \sqrt{a^2 + b^2 + 6ab}. \tag{12}$$

Since (7) we have $DP_1 = DC = 2\sqrt{ab}$. Hence

$$\cos\angle H_0DP_1 = \frac{DH_0}{DP_1} = \frac{2t}{DP_1} = \frac{2ab}{(a + b)2\sqrt{ab}} = \frac{\sqrt{ab}}{a + b}. \tag{13}$$

Applying the Cosine's law to triangle O_0DX_1, we have

$$\begin{aligned} O_0X_1^2 &= O_0D^2 + X_1D^2 - 2O_0D.X_1D.\cos\angle O_0DX_1 \\ &= O_0D^2 + X_1D^2 + 2O_0D.X_1D.\cos\angle H_0DP_1 \\ &= (a + b)^2 + X_1D^2 + 2(a + b).X_1D.\cos\angle H_0DP_1. \end{aligned}$$

Note that $O_0X_1 = O_0A_0$. Combining with (12) and (13), we deduce that

$$a^2 + b^2 + 6ab = (a+b)^2 + X_1D^2 + 2(a+b).X_1D\frac{\sqrt{ab}}{a+b}$$
$$= (a+b)^2 + X_1D^2 + 2\sqrt{ab}.X_1D.$$

It follows $X_1D^2 + 2\sqrt{ab}X_1D - 4ab = 0$. From this, we get

$$X_1D = (-1+\sqrt{5})\sqrt{ab}. \tag{14}$$

Since (14), it follows

$$X_1P_1 = DP_1 - X_1D = 2\sqrt{ab} - (-1+\sqrt{5})\sqrt{ab} = (3-\sqrt{5})\sqrt{ab}. \tag{15}$$

Since (14) and (15), it follows

$$\frac{X_1D}{X_1P_1} = \frac{(-1+\sqrt{5})\sqrt{ab}}{(3-\sqrt{5})\sqrt{ab}} = \frac{1+\sqrt{5}}{2} = \varphi.$$

This means that X_1 divides DP_1 in the golden ratio.

On the other hand, applying the Power-of-a-point theorem with the note that $O_0D = OD = a+b$ and (12), (14), we have

$$DX_1.DY_1 = O_0A_0^2 - DO_0^2 = 4ab \Longrightarrow Y_1D = \frac{4ab}{DX_1} = (1+\sqrt{5})\sqrt{ab}.$$

It follows

$$\frac{Y_1D}{DP_1} = \frac{(1+\sqrt{5})\sqrt{ab}}{2\sqrt{ab}} = \varphi \text{ and } \frac{Y_1P_1}{Y_1D} = \frac{\varphi+1}{\varphi} = \varphi.$$

Hence point Y_1 divides segment P_1D in the golden ratio. □

From the configuration of theorem 8, we obtain a pair of Archimedean circles as follows

Theorem 9. *The circle $O_0(A_0)$ meets the semi-circle (O) at U_1 and U_2. Then the circles that are tangent with P_1P_2 and their centers are U_1 and U_2, respectively are Archimedean (see figure 7).*

Proof. Let V_1 be the projection from U_1 onto OO_0.

We have

$$O_0V_1 + OV_1 = OO_0 = 2OD = 2(a+b). \tag{16}$$

Note that $U_1O_0 = O_0A_0 = \sqrt{a^2+b^2+6ab}$ (since (12)), $DH_0 = 2t$ and $U_1O = a+b$. We have

$$V_1O_0^2 - V_1O^2 = U_1O_0^2 - U_1O^2 = a^2 + b^2 + 6ab - (a+b)^2 = 4ab.$$

It follows $OO_0(O_0V_1 - OV_1) = 4ab$. From this, we get

$$O_0V_1 - OV_1 = \frac{4ab}{OO_0} = \frac{4ab}{2(a+b)} = \frac{2ab}{a+b} = 2t. \tag{17}$$

Since (16) and (17), it follows

$$O_0V_1 = a+b+t \text{ and } OV_1 = a+b-t.$$

Hence

$$V_1H_0 = |OV_1 - H_0O| = |a+b-t-(a+b)| = t.$$

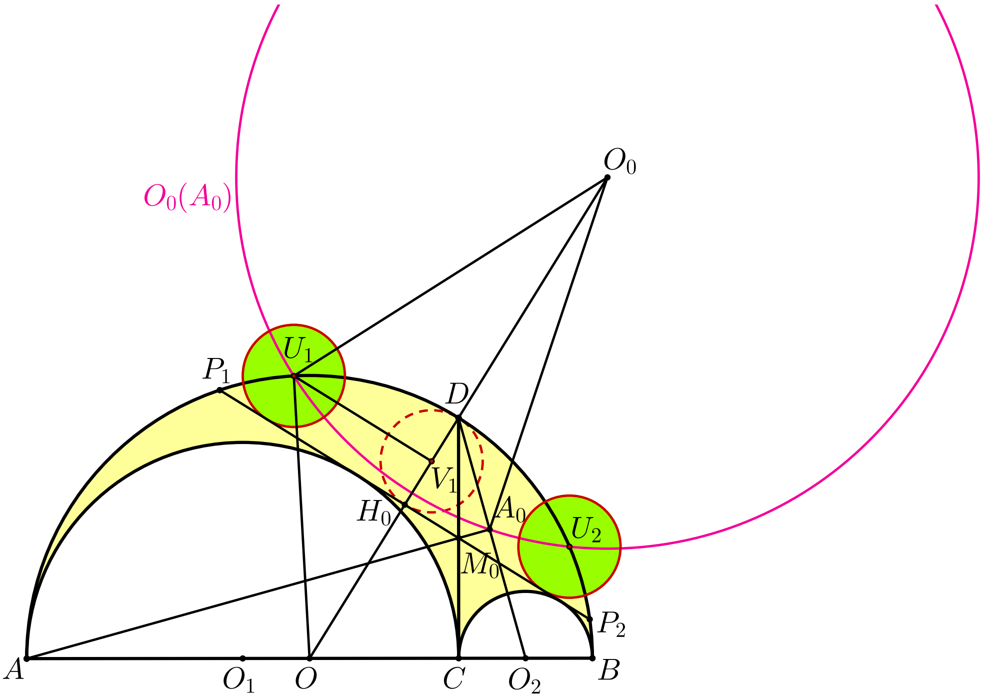

Figure 7.

This prove that the distance from U_1 to P_1P_2 is equal to t.

Similarly, the distance from U_2 to P_1P_2 is also equal to t.

It follows that the circles are tangent with P_1P_2 and their centers are U_1 and U_2, respectively such that their radii are equal to t, so they are Archimedean. The theorem is proved. $\square$

References

[1] T. O. Dao, Q. D. Ngo, and P. Yiu, *Golden sections in an isosceles triangle and its circumcircle*, Global Journal of Advanced Research on Classical and Modern Geometries, **5 (2016)**, 93—97.

[2] C. W. Dodge, T. Schoch, P. Y. Woo and P. Yiu, *Those ubiquitous Archimedean circles*, Math. Mag., **72 (1999)**, 202-–213.

[3] F. M. van Lamoen, *Online catalogue of Archimedean circles*, available at http://home.kpn.nl/lamoen/wiskunde/Arbelos/Catalogue.htm.

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