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AN INEQUALITY RELATED TO THE LENGTHS AND AREA OF A CONVEX QUADRILATERAL

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Abstract. In this paper we give an inequality related to the lengths and the area of a convex quadrilateral and its proof.

1. Introduction

There are many famous inequalities related to the lengths and area of a triangle, for examples with a triangle we have Pedoe's inequality [9], Weitzenbock's inequality [10], Ono's inequality [8], Blundon's inequality [3]. In a quadrilateral we have many important inequalities, such as Yun's inequality [6], Josefsson's inequality [6],[7]; some other inequalities of a quadrilateral you can see in [4], [1], [2]. In this paper we give a nice inequality related to the lengths and area of a convex quadrilateral in theorem 1.1 as follows:

Theorem 1.1. *Let a, b, c, d be the lengths of the sides of a convex quadriateral $ABCD$ with the area S, the following inequality hold:*

$$\frac{1}{3+\sqrt{3}}(ab+ac+ad+bc+bd+cd)-\frac{1}{2(\sqrt{3}+1)^2}(a^2+b^2+c^2+d^2)\geq S \tag{1}$$

Equality hold if only if $ABCD$ is a square.

To prove Theorem 1.1, first we give a stronger inequality as follows:

Theorem 1.2. *If a, b, c and d be the lengths of the sides of a convex quadrilateral $ABCD$, the following inequality hold:*

$$2(a+b+c+d)^2 \geq \left(5-\sqrt{3}\right)\left(a^2+b^2+c^2+d^2\right)+4\sqrt{3}\left(\sqrt{3}+1\right)\sqrt{abcd} \tag{2}$$

Equality hold if only if $a = b = c = d$.

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Proof.

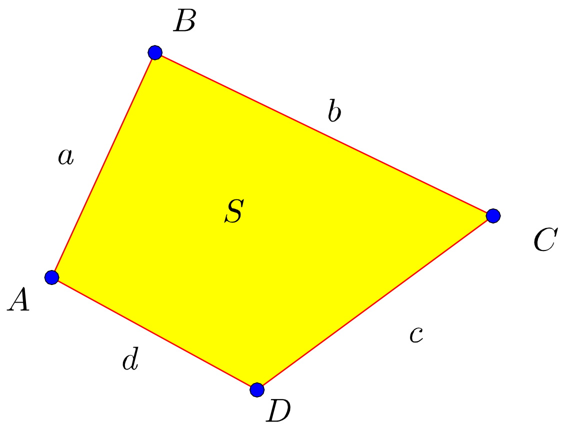

Figure 1

In order to prove (2), we need to establish few things first. Since $ABCD$ is the convex quadrilateral, we have $0 < a, b, c, d < \frac{a+b+c+d}{2}$. Because both members of (1) are homogenous of second degree, we may assume $a + b + c + d = 4$. Thus, $0 < a, b, c, d < 2$. By Maclaurin's inequality we have:

$$\frac{ab + ac + ad + bc + bd + cd}{6} \leq \left(\frac{a + b + c + d}{4}\right)^2$$

i.e.

$$ab + ac + ad + bc + bd + cd \leq 6.$$

So that there exist $t \in [0, 1)$ such that:

$$ab + ac + ad + bc + bd + cd = 6(1 - t^2). \tag{3}$$

From (3)

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 &= (a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd) \\ &= 16 - 12(1 - t^2) = 4(1 + 3t^2). \end{aligned}$$

Thus (1) is equivalent to

$$8 \geq \left(5 - \sqrt{3}\right)(1 + 3t^2) + \sqrt{3}\left(\sqrt{3} + 1\right)\sqrt{abcd} \Leftrightarrow$$

$$\sqrt{3}(\sqrt{3} + 1) - 3(5 - \sqrt{3})t^2 \geq \sqrt{3}\left(\sqrt{3} + 1\right)\sqrt{abcd} \Leftrightarrow$$

$$1 - \sqrt{3}\left(3\sqrt{3} - 4\right)t^2 \geq \sqrt{abcd}. \tag{4}$$

Define the polynomial $P : (0, \infty) \to \mathbb{R}$ as $P(x) = (x - a)(x - b)(x - c)(x - d), \forall x > 0$. By Viete's theorem combined with (3), we have: $P(x) = x^4 - 4x^3 + 6(1 - t^2)x^2 - 4sx + p, \forall x > 0$, where $4s = abc + abd + acd + bcd$ and $p = abcd$.

Lemma 1.1. *Let a, b, c and d be the lengths of the sides of a convex quadrilateral $ABCD$ with perimeter equal to 4 then*

$$a^2 + b^2 + c^2 + d^2 < 8. \tag{5}$$

Proof. Denote $u = a - 1$, $v = b - 1$, $w = c - 1$ and $z = d - 1$. Then $-1 < u, v, w, z < 1$ and $u + v + w + z = 0$. The inequality (5) is equivalent with $u^2 + v^2 + w^2 + z^2 < 4$, which is true since $0 \le u^2, v^2, w^2, z^2 < 1$. This completes the proof of Lemma 1.1. Note that Lemma 1.1 combined to (4) gives us that $t \in [0, \frac{1}{\sqrt{3}})$.

Lemma 1.2. *If $0 \le t < \frac{1}{3}$ then $p \le (1 - t)^3(1 + 3t)$.*

Proof. Consider the function $f : (0, \infty) \to R, f(x) = \frac{P(x)}{x}$. Since f admits four positive roots, the from Rolle's theorem, f' admits at least 3 positive roots. But

$$f'(x) = \frac{3x^4 - 8x^3 + 6(1 - t^2)x^2 - p)}{x^2}, \ \forall x > 0.$$

From the obove consideration, the polynomial $g(x) = 3x^4 - 8x^3 + 6(1 - t^2)x^2 - p$ admits at least three roots in $(0, \infty)$. We have:

$$g'(x) = 12x \left(x^2 - 2x + (1 - t^2)\right),$$

thus the critical points of g are $1 - t \ge 1 + t$ and f. We form the function g the Rolle's sequence $0_+ < 1 - t \le 1 + t < \infty$. Since $f(0_+) = -\infty < 0, f(\infty) = +\infty > 0$ and f has three positive roots, the from the consequence of Rolle's theorem, we obtain

$$f(1 - t) \ge 0 \Rightarrow p \le (1 - t)^3(1 + 3t).$$

This completes the proof of Lemma 1.2.

Lemma 1.3. *Let a, b, c and d be real numbers situated in the interval $[0, 2]$ such that $a+b+c+d = 4$ and $ab + ac + ad + bc + bd + cd = 6(1 - t^2)$. If $\frac{1}{3} \le t \le \frac{1}{\sqrt{3}}$ then:*

$$p \le \frac{4\left(2 - \sqrt{2(9t^2 - 1)}\right)\left(1 + \sqrt{2\left(9t^2 - 1\right)}\right)}{27}. \tag{6}$$

Moreover $p = \frac{4\left(2-\sqrt{2(9t^2-1)}\right)\left(1+\sqrt{2(9t^2-1)}\right)}{27}$ iff $(a, b, c, d) = \left(2, \frac{2-\sqrt{2(9t^2-1)}}{3}, \frac{2-\sqrt{2(9t^2-1)}}{3}, \frac{2+\sqrt{2(9t^2-1)}}{3}\right)$ and permutations.

Proof. Let k be a real number with $0 < k < \frac{2}{3}$. We consider the real numbers a, b, c and d situated in the interval $[0, 2]$ such that $a + b + c + d = 4$ and $a^2 + b^2 + c^2 + d^2 = 2(3k^2 - 4k + 4)$. Then,

$$abcd \le 4k^2(1 - k).$$

First let's remark that if $(a, b, c, d) = (k, k, 2 - 2k, 2)$ and their permutations, then

$$a + b + c + d = 4, a^2 + b^2 + c^2 + d^2 = 2(3k^2 - 4k + 4)$$

and $abcd = 4k^2(1-k)$. Let α, β be positive real numbers with $\alpha > \beta$. We consider the non negative real numbers u, v, w and t such $u+v+w+t = 2\alpha+\beta$ and $u^2+v^2+w^2+t^2 = 2\alpha^2+\beta^2$. Then

$$uvw + uvt + uwt + vwt \geq \alpha^2\beta.$$

The result is known, hence we don't need to prove it here.

Back to the problem. Putting $\alpha = 2-k$, $\beta = 2k$, $u = 2-a$, $v = 2-b$, $w = 2-c$ and $t = 2-d$ we get: $\alpha > \beta > 0$; $u, v, w, t \geq 0$; $u+v+w+t = 2\alpha+\beta$ and $u^2+v^2+w^2+t^2 = 2\alpha^2+\beta^2$. Hence according to the lemma,

$$uvw + uvt + uwt + vwt \geq \alpha^2\beta \Rightarrow abc + abd + acd + bcd \leq 2k(-k^2 - 2k + 4).$$

We assume by absurd that there exist $a, b, c, d \in [0, 2]$ with $a+b+c+d = 4$ and $a^2+b^2+c^2+d^2 = 2(3k^2 - 4k + 4)$ such that $abcd > 4k^2(1-k)$.

Define the polynomials $f, g : [0, 2] \to \mathbb{R}$, as $f(x) = (x-a)(x-b)(x-c)(x-d)$ and $g(x) = (x-k)^2\,(x - 2(1-k))\,(x-2)$, $\forall x \in [0, 2]$. Then

$$f(x) = x^4 - 4x^3 + (-3k^2 + 4k + 4)x^2 - mx + p, \forall x \in [0, 2]$$

where $m = abc + abd + acd + bcd$, and $p = abcd$ and

$$g(x) = x^4 - 3x^3 + (-3k^2 + 4k + 4)x^2 - 2k(-k^2 - 2k + 4)x + 4k^2(1-k), \forall x \in [0, 2].$$

Thus, $f(x) - g(x) = \big(2k(-k^2 - 2k + 4) - m\big)\,x + p - 4k^2(1-k) > 0$, $\forall x \in [0, 2]$. Hence, if $\gamma\,\{a, b, c, d\}$, then $f(\gamma) - g(\gamma) > 0 \Rightarrow g(\gamma) < 0$.

But $g(x) < 0$ if and only if $x \in (2(1-k), 2) \Rightarrow \{a, b, c, d\} \subset (2(1-k), 2)$. Consequently, via Rolle's theorem, the roots of f' are contained in the interval $(2(1-k), 2)$. Let y_1, y_2 and y_3 be those roots. Then

$$f'(x) = 4(x-y_1)(x-y_2)(x-y_3) \geq 0,$$

which is false, because $k < 2(1-k) < y_1, y_2, y_3$. So that our assumption was false. In conclusion, $abcd \leq 4k^2(1-k)$. Back to the lemma's 1.3 proof.

If $\frac{1}{3} < t < \frac{1}{\sqrt{3}}$, let's remark first that the exists $k \in (0, \frac{2}{3})$ such that $a^2 + b^2 + c^2 + d^2 = 2(3k^2 - 4k + 4)$.

Indeed, solving the quadratic $2(3k^2 - 4k + 4) = 4(1 + 3t^2)$, we choose the root $k = \frac{2-\sqrt{2(9t^2-1)}}{3}$ and clearly $0 < k < \frac{2}{3}$. Hence

$$abcd \leq \frac{4\left(2 - \sqrt{2(9t^2-1)}\right)^2 \left(1 + 2\sqrt{2(9t^2-1)}\right)}{27}.$$

If $t = \frac{1}{3}$, then, by the proof of the lemma 1.2, $abcd \leq (1-t)^3(1+3t)$. If $t = \frac{1}{\sqrt{3}}$, the only choices we have are $(a, b, c, d) = (2, 2, 0, 0)$ and permutations and clearly,

$$abcd \leq \frac{4\left(2 - \sqrt{2(9t^2-1)}\right)^2 \left(1 + 2\sqrt{2(9t^2-1)}\right)}{27}$$

This completes the proof of Lemma 1.3.

We are now ready to prove the inequality (2).

Case 1: $0 \le t \le \frac{1}{3}$. By the lemma 2 we have: $(1-t)^3(1+3t) \ge p$, hence suffice it to show:

$$1-\sqrt{3}(3\sqrt{3}-4)t^2 \ge \sqrt{(1-t)^3(1+3t)}$$

$$t^2(1-3t)\left(2\sqrt{3}-3-(11-6\sqrt{3})t\right) \ge 0$$

which is true since $0 \le t < \frac{1}{3}$ and $\frac{2\sqrt{3}-3}{11-6\sqrt{3}} > \frac{1}{3}$. We deduce from the above proof that in this case equality holds at $t=0 \Leftrightarrow a=b=c=d$.

Case 2: $\frac{1}{3} \le t \le \frac{1}{\sqrt{3}}$.

The fact that $0 < a,b,c,d < 2$ combined with the Lemma 1.3 give us that:

$$\frac{4\left(2-\sqrt{2(9t^2-1)}\right)^2\left(1+\sqrt{2(9t^2-1)}\right)}{27} \ge p$$

Suffice it to show that:

$$1-\sqrt{3}(3\sqrt{3}-4)t^2 \ge \sqrt{\frac{4\left(2-\sqrt{2(9t^2-1)}\right)^2\left(1+\sqrt{2(9t^2-1)}\right)}{27}}$$

Which is true after strightward calculations. This completes the proof of Theorem 1.2.

2. PROOF OF THE THEOREM 1.1

By famous Brahmagupta's formula and Bretschneider's formula (see [5]), we know that among all convex quadrilaterals $ABCD$ with the lengths of the side a,b,c and d then the cyclic ones have the lagest area. So we only need to prove the Theorem 1.1 with $ABCD$ is the convex cyclic quadrilateral.

By the Brahmagupta's formula we have that the area of convex cyclic quadrilateral $ABCD$ is $S=\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where $s=\frac{a+b+c+d}{2}$. So the inequality (1) $\Leftrightarrow$

$$\frac{4}{\sqrt{3}}(ab+ac+ad+bc+bd+cd) \ge (\sqrt{3}-1)(a^2+b^2+c^2+d^2)+4(\sqrt{3}+1)\sqrt{(s-a)(s-b)(s-c)(s-d)} \Leftrightarrow$$

$$(7)\quad 2(a+b+c+d)^2 \ge (5-\sqrt{3})(a^2+b^2+c^2+d^2)+4\sqrt{3}(\sqrt{3}+1)\sqrt{(s-a)(s-b)(s-c)(s-d)}$$

As above, without loss of generality, we may assume that $a+b+c+d=4$. Denote $x=2-a$, $y=2-b, z=2-c$ and $w=2-d$. Then $0 < x,y,z,w < 2$. Moreover, $a=2-x$, $b=2-y$, $c=2-z$, $d=2-w$ and $x+y+z+w=8-(a+b+c+d)=4$ so that x,y,z and w are the lengths of the sides of a covex quadrilateral $XYZW$.

Noted that $a^2+b^2+c^2+d^2=(2-x)^2+(2-y)^2+(2-z)^2+(2-w)^2=16-4(x+y+z+w)+x^2+y^2+z^2+w^2=x^2+y^2+z^2+w^2$. Thus, (7) be come:

$$(8)\qquad 2(x+y+z+w)^2 \ge (5-\sqrt{3})(x^2+y^2+z^2+w^2)+4\sqrt{3}(\sqrt{3}+1)\sqrt{xyzw}$$

Equality in (8) holds if only if $a=b=c=d$ and $ABCD$ is a cyclic quadrilateral. The inequality (8) is true according to the (2). This complete the proof of Theorem 1.1.

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References

[1] Alsina, C. and Nelsen, R., *When Less is More: Visualizing Basic Inequalities*, Mathematical Association of America, 2009.
[2] Andreescu, T., Andrica, D., *360 Problems for Mathematical Contests*, GIL Publishing House, 2003.
[3] Andrica, D., Barbu, C. and Minculete, N., *A Geometric way to generate Blundon Type inequalities*, Acta Universitatis Apulensis, **31 (2012)**, 96-106.
[4] Bottema, O., *Geometric Inequalities*, Wolters-Noordhoff Publishing, The Netherlands, 1969, 129-132.
[5] Brahmagupta's formula, https://en.wikipedia.org/wiki/Brahmagupta\s_formula.
[6] Josefsson, M., *Five Proofs of an Area Characterization of Rectangles*, Forum Geometricorum, **13 (2013)**, 17-21.
[7] Josefsson, M., *A few inequalities in quadrilateral*, International Journal of Geometry, **4 (1) (2015)**, 11-15.
[8] Ono, T., Problem 4417, Intermed. Math., **21 (1914)**, 146.
[9] Pedoe, D., *An Inequality Connecting Any Two Triangles*, The Mathematical Gazette, **25 (267) (1941)**, 310-311.
[10] Stoica, E., Minculete, N. and Barbu, C., *New aspects of Ionescu–Weitzenbock's inequality*, Balkan Journal of Geometry and Its Applications, **21 (2) 2016**, 95-101.

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