A CONSTANT ANGLE RULED SURFACES

AHMAD TAWFIK ALI

Abstract. In this paper, the constant angle ruled surfaces (CARSs) generated by Frenet frame (tangent, normal and binormal) vectors, in Euclidean 3-space are investigated. Also, CARSs parallel to tangent of general helix, normal of slant helix and \( \psi_3 \) of slant-slant helix are studied. We proved that all CARSs are developable.

1. INTRODUCTION

A constant angle surface in Euclidean three space \( \mathbb{E}^3 \) is a surface whose tangent planes make a constant angle with a fixed vector field of the ambient space. These surfaces generalize the concept of general helix, that is, curves whose tangent lines make a constant angle with a fixed vector of \( \mathbb{E}^3 \) [14]. General helix is characterized by the fact that the ratio

\[
\frac{\tau(s)}{\kappa(s)} = \sigma_0
\]

is constant along the curve, where \( \kappa \) and \( \tau \) are the curvature and the torsion of the curve, respectively [15]. Izumiya and Takeuchi [10] defined a slant helix in Euclidean 3-space \( \mathbb{E}^3 \) by saying that the normal lines make a constant angle with a fixed direction. They characterize a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix

\[
\sigma_1 = \frac{\sigma'(s)}{\kappa(s) \left(1 + \frac{\sigma''(s)}{\kappa(s)}\right)^{3/2}}
\]

is a constant function. Recently, Ali [3] generalized the definition of a general helix and slant helix and named it \( k \)-slant helix. He proved that the curve is a \( k \)-slant helix if and only if the geodesic curvature of the spherical image of \( \psi_k \) indicatrix of the curve \( \psi \)

Keywords and phrases: Constant angle surface, Ruled surfaces, Slant helices, \( k \)-slant helices.

Received: 09.11.2017. In revised form: 24.02.2018. Accepted: 12.02.2018.
(3) \[ \sigma_k = \frac{\sigma'_{k-1}(s)}{\kappa(s) \sqrt{1 + \sigma'^2_0(s)} \sqrt{1 + \sigma'^2_1(s)} \cdots \left(1 + \sigma'^2_{k-1}(s)\right)^{3/2}}, \]

is a constant function, where \( \psi_{k+1} = \frac{\psi'_k(s)}{\|\psi'_k(s)\|} \), \( \psi_0(s) = \psi(s) \), \( \sigma_0(s) = \frac{\tau(s)}{\kappa(s)} \) and \( k \in \{0, 1, 2, \ldots\} \). It is worth noting that, the straight lines, plane curves, general helices, slant helices and slant-slant helices are a special subclasses of curves from the family of \( k \)-slant helices.

Ruled surfaces are surfaces generated by moving a straight line continuously in the space and are one of the most important topics of differential geometry \([4, 17]\). A constant angle surfaces are models to describe some phenomena in physics of interfaces in liquids crystals and of layered fluids \([8]\). Munteanu and Nistor \([11]\) obtained a classification of all surfaces in Euclidean 3-space for which the unit normal makes a constant angle with a fixed vector direction being the tangent direction to \( R \). Moreover in \([12]\) it is also classified certain special ruled surfaces in \( E^3 \) under the general theorem of characterization of constant angle surfaces.

In this paper, CARS, whose tangent planes parallel to some vectors such as: tangent of general helix, normal of slant helix and \( \psi_3 \) of slant-slant helix are studied. Moreover, we study the characterizations of a flat, minimal, II-minimal and II-flat CARSs. Therefore, we proved that all CARSs are developable.

2. Basic Concepts

In Euclidean 3-space \( E^3 \), it is well known that to each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields \( t, n \) and \( b \) are respectively, the tangent, the principal normal and the binormal vector fields \([13]\). We consider the usual metric in Euclidean 3-space \( E^3 \), that is

\[ \langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2, \]

where \((x_1, x_2, x_3)\) is a rectangular coordinate system of \( E^3 \). If \( X = (x_1, x_2, x_3) \) and \( Y = (y_1, y_2, y_3) \) are arbitrary vectors in \( E^3 \), we define the vector product of \( X \) and \( Y \) as the following:

\[ X \wedge Y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1). \]

Let \( \psi = \psi(s) : I \subset \mathbb{R} \to E^3 \) be an arbitrary curve in \( E^3 \). The curve \( \psi \) is said to be a unit speed (or parameterized by the arc-length parameter \( s \)) if \( \langle \psi'(s), \psi'(s) \rangle = 1 \) for any \( s \in I \). Let \( \{t(s), n(s), b(s)\} \) be the moving frame of the curve \( \psi \) which satisfies the the Frenet equations:

\[ \begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}, \]

where the prime ‘ denotes the derivative with respect to the \( s \)-parameter and \( \kappa \) and \( \tau \) are the curvature and torsion of \( \psi \), respectively.
Let the position vector of the surface \( M \) in the standard form of Euclidean space \( E^3 \) is 
\[
\Psi(s, v) = (x_1(s, v), x_2(s, v), x_3(s, v)).
\]
Then the standard unit normal vector field \( U \) on the surface can be defined by:
\[
U = \frac{\Psi_s \wedge \Psi_v}{\|\Psi_s \wedge \Psi_v\|},
\]
where \( \Psi_s = \frac{\partial \Psi(s,v)}{\partial s} \) and \( \Psi_v = \frac{\partial \Psi(s,v)}{\partial v} \). Then the first \( I \) and second \( II \) fundamental forms of the surface \( M \) are given by, respectively
\[
I = Eds^2 + 2Fdvds + Gdv^2,
\]
\[
II = eds^2 + 2fdvds + gdv^2,
\]
where
\[
\begin{align*}
E &= \langle \Psi_s, \Psi_s \rangle, & F &= \langle \Psi_s, \Psi_v \rangle, & G &= \langle \Psi_v, \Psi_v \rangle, \\
e &= \langle \Psi_{ss}, U \rangle, & f &= \langle \Psi_{sv}, U \rangle, & g &= \langle \Psi_{vv}, U \rangle.
\end{align*}
\]
On other hand, the Gaussian curvature \( K \) and the mean curvature \( H \) are given by, respectively
\[
K = \frac{eg - f^2}{EG - F^2},
\]
\[
H = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}.
\]
From Brioschi’s formula in a Euclidean 3-space, we are able to compute \( K_{II} \) of a surface by replacing the components of the first fundamental form \( E, F \) and \( G \) by the components of the second fundamental form \( e, f \) and \( g \) respectively. Consequently, the second Gaussian curvature \( K_{II} \) of a surface is defined by [7]:
\[
K_{II} = \frac{1}{(eg - f^2)^2} \begin{vmatrix}
-\frac{1}{2}e_{vv} + f_{sv} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & \frac{1}{2}e_v \\
f_v - \frac{1}{2}g_s & e & f \\
\frac{1}{2}g_v & f & g
\end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & \frac{1}{2}g_s \\
\frac{1}{2}e_v & e & f \\
\frac{1}{2}g_v & f & g
\end{vmatrix}.
\]
Having in mind the usual technique for computing the second mean curvature \( H_{II} \) by using the normal variation of the area functional for the surfaces in \( E^3 \) one gets [16]:
\[
H_{II} = H + \frac{1}{4} \Delta_{II} \ln(K)
\]
where \( H \) and \( K \) denote the mean, respectively Gaussian curvatures of surface and \( \Delta_{II} \) is the Laplacian for functions computed with respect to the second fundamental form \( II \) as metric. The second mean curvature \( H_{II} \) can be equivalently expressed as
\[
H_{II} = H + \frac{1}{2\sqrt{\det(II)}} \sum_{i,j} \frac{\partial}{\partial u^i} \left[ \sqrt{\det(II)} h^{ij} \frac{\partial}{\partial u^j} (\ln \sqrt{K}) \right],
\]
where \( (h_{ij}) \) denotes the associated matrix with its inverse \( (h^{ij}) \), the indices \( i, j \) belong to \( \{1, 2\} \) and the parameters \( u^1, u^2 \) are \( s, v \) respectively.

Now, we can write the following important definitions of surfaces:
Definition 1. [18]

(1): A regular surface is flat (developable) if and only if its Gaussian curvature vanishes identically.

(2): A regular surface for which the mean curvature vanishes identically is minimal surface.

(3): A non developable surface is called II-flat if the second Gaussian curvature vanishes identically.

(4): A non developable surface is called II-minimal if the second mean curvature vanishes identically.

A ruled surface is generated by one-parameter family of straight lines and it possesses a parametric representation

\[ \Psi(s, v) = \psi(s) + v X(s). \]  

The curve \( \psi(s) \) is called a base curve and \( X(s) \) is a director curve [19]. In particular, if \( X(s) \) is a constant vector, the ruled surface \( \Psi \) is said to be cylindrical. It is called non-cylindrical otherwise [9].

We will consider a parametric surface \( \Psi = \Psi(s, v) \) and \( \psi = \psi(s) \) be a unit speed curve in \( \mathbb{E}^3 \), where \( t = \psi'(s) \) is the unit tangent vector to \( \psi \). Consider the Frenet frame of the curve \( \psi, \{t, n, b\} \), where \( n \) and \( b \) denote the normal and binormal vectors respectively.

In this paper, we take the base curve \( \psi \) is not a straight line, i.e., the curvature \( \kappa \neq 0 \). Now, we will generalize a ruled surface (13) as the following [6]:

\[ \Psi(s, v) = c(s) + v X(s), \]

where

\[ c'(s) = \alpha t + \beta n + \gamma b, \]

where \( \alpha, \beta \) and \( \gamma \) are smooth functions of \( s \).

It is worth noting that the ruled surface (13) is a special case of the ruled surfaces (14), when \( \alpha = 1 \) and \( \beta = \gamma = 0 \). In this case \( c(s) = \int t(s) ds = \psi(s) \). For the ruled surface, there are two base curve, we will call \( \psi(s) \) is the base or original (first) base and \( c(s) \) is the new (second) base. Also, any vector \( X(s) \in \mathbb{E}^3 \) can be written as:

\[ X(s) = x_1 t + x_2 n + x_3 b, \]

where \( x_1, x_2 \) and \( x_3 \) are smooth functions of \( s \).

By a straightforward computation, we can obtain the normal vector to the surface (14) as the following:

\[ U = U_1 t + U_2 n + U_3 b, \]

where

\[ U_1 = U_{11} + U_{12} v, \quad U_2 = U_{21} + U_{22} v, \quad U_3 = U_{31} + U_{32} v, \]
and

\[
\begin{align*}
U_{11} &= x_3\beta - x_2\gamma, \\
U_{12} &= x_1x_3\kappa - (x_2^2 + x_3^2)\tau + x_3x_2' - x_2x_3', \\
U_{21} &= x_1\gamma - x_3\alpha, \\
U_{22} &= x_2x_3\kappa + x_1x_2\tau + x_1x_3' - x_3x_1', \\
U_{31} &= x_2\alpha - x_1\beta, \\
U_{32} &= x_1x_3\tau - (x_1^2 + x_2^2)\kappa + x_2x_1' - x_1x_2'.
\end{align*}
\]

(19)

3. CARs Parallel to Tangent Vector

In this section we take the normal vector $U$ of a ruled surface is parallel to the tangent $t$ of the base curve $\psi$. Therefore $\psi$ is a general helix and we have the following conditions:

\[
\tau = m\kappa, \quad U_1 = (U_{11}, U_{12}) \neq (0, 0), \quad U_{21} = U_{22} = U_{31} = U_{32} = 0.
\]

To study this, we have the following cases:

**Case (1):** $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 \neq 0$. By solving the equations $U_{21} = 0$, $U_{22} = 0$ and $U_{31} = 0$, we can get

\[
\beta = \frac{x_2}{x_1}\alpha, \quad \gamma = \frac{x_3}{x_1}\alpha, \quad x_2 = \frac{x_3x_1' - x_1x_3'}{(m_1x_1 + x_3}\kappa).
\]

(21)

If we substitute from the above equation in (19) we have the following conditions:

\[
U_{11} = 0, \quad x_1U_{12} + x_3U_{32} = 0.
\]

(22)

The condition $U_{32} = 0$ leads to $U_{12} = 0$ and therefore $U_1 = (U_{11}, U_{12}) = (0, 0)$ which is contradiction with the condition (20).

**Case (2):** $x_1 = 0$. In this case we have:

\[
\begin{align*}
U_{11} &= x_3\beta - x_2\gamma, \\
U_{12} &= x_3x_2' - x_2x_3' - (x_2^2 + x_3^2)\tau, \\
U_{21} &= -x_3\alpha, \\
U_{22} &= x_2x_3\kappa, \\
U_{31} &= x_2\alpha, \\
U_{32} &= -x_2^2\kappa.
\end{align*}
\]

(23)

It is easy to see that the condition $U_{32} = 0$ leads to $x_2 = 0$. Because $X = (x_1, x_2, x_3) \neq (0, 0, 0)$, then $x_3 \neq 0$ and therefore the condition $U_{21} = 0$ is satisfied only when $\alpha = 0$. In this case $U_1 \neq 0$, and we have obtained a CARs (14) which takes the form

\[
\Psi(s, v) = \int \left( \beta(s) n + \gamma(s) b \right) ds + v x_3(s) b,
\]

where $\beta$, $\gamma$ and $x_3$ are arbitrary smooth functions of $s$.

**Case (3):** $x_1 \neq 0$ and $x_2 = 0$. In this case $U_{31} = -x_1\beta$ and $U_{32} = -x_1(x_1 - m_1)x_3\kappa$. Therefore the two conditions $U_{31} = 0$ and $U_{32} = 0$ lead to $\beta = 0$ and $x_1 = m_1x_3$. Substituting in $U_1$ we have $U_{11} = U_{12} = 0$ which is
contradiction.

Case (4): $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 = 0$. In this case $U_{22} = m x_1 x_2 \kappa \neq 0$ which is contradiction.

Now, we can write the following Theorem:

**Theorem 3.1.** The ruled surface (24) is a constant angle surface if and only if the base curve $\psi$ is a general helix.

**The Proof:** $(\Rightarrow)$ From the above discussion, we have proved that, If the base curve $\psi$ is a general helix, then the ruled surface (24) is a constant angle surface.

$(\Leftarrow)$ Let the ruled surface (24) is a constant angle surface. It is easy to prove that the unit normal vector $U$ of this surface is equal to the tangent vector $t$ of the base curve $\psi$ which makes a constant angle with a fixed vector. Therefore, the base curve is a general helix and the proof is complete.

**Lemma 3.1.** The Gaussian and mean curvatures of a CARS (24) is given by:

$$K = 0, \quad H = \frac{\kappa}{2(x_3 \tau v - \beta)}.$$  

From the above lemma and theorem, we can write the following corollary:

**Corollary 3.2.** The surface (24) can be named by: Constant angle ruled binormal developable surface.

Now, we will write the position vector of a constant angle surface (24). Firstly, we introduced the following theorem:

**Theorem 3.3.** [1] The position vector $\psi$ of a general helix is expressed in the natural representation form as follows:

$$\psi(s) = \frac{n}{m} \int \left( \cos \left[ \xi \right], \sin \left[ \xi \right], m \right) ds,$$

where $\xi = \sqrt{1 + m^2} \int \kappa(s) ds$, $m = \frac{n}{\sqrt{1 - n^2}}$, $n = \cos[\phi]$ and $\phi$ is the angle between the fixed straight line $e_3$ (axis of a general helix) and the tangent vector $t$ of the curve $\psi$.

The moving Frenet frame of the general helix (25) can be obtained as follows:

$$\begin{align*}
    t &= \frac{n}{m} \left( \cos \left[ \xi \right], \sin \left[ \xi \right], m \right), \\
    n &= \left( -\sin \left[ \xi \right], \cos \left[ \xi \right], 0 \right), \\
    b &= \frac{n}{m} \left( -m \cos \left[ \xi \right], -m \sin \left[ \xi \right], 1 \right).
\end{align*}$$

Then, in the standard frame, the position vector $\Psi(s, v) = (\Psi_1, \Psi_2, \Psi_3)$ of a constant angle surface (24) takes the following form:

$$\Psi(s, v) = \begin{cases}
    \Psi_1 = -\int (\beta(s) \sin \left[ \xi \right] + n\gamma(s) \cos \left[ \xi \right]) ds - n x_3(s) v \cos \left[ \xi \right], \\
    \Psi_2 = \int (\beta(s) \cos \left[ \xi \right] - n\gamma(s) \sin \left[ \xi \right]) ds - n x_3(s) v \sin \left[ \xi \right], \\
    \Psi_3 = \frac{m}{n} \left( \int \beta(s) ds + v x_3(s) \right).
\end{cases}$$
4. CARS PARALLEL TO NORMAL VECTOR

In this section we take the normal vector $U$ of a ruled surface is parallel to the normal $n$ of the base curve $\psi$. Therefore, the base curve $\psi$ must be a slant helix and the following conditions are satisfied:

$$\tau = \frac{m \kappa(s) \int \kappa(s) ds}{\sqrt{1 - m^2(\int \kappa(s) ds)^2}}, \quad (U_{21}, U_{22}) \neq (0, 0), \quad U_{11} = U_{12} = U_{31} = U_{32} = 0. \quad (28)$$

Then there exist some cases as follows:

**Case (1):** $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 \neq 0$. From (19), by solving the equations $U_{11} = 0$, $U_{12} = 0$ and $U_{31} = 0$ we get

$$x_1 = \frac{(x_2^2 + x_3^2) \tau + x_2 x_3' - x_3 x_2'}{x_3 \kappa}, \quad \alpha = \left(\frac{x_1}{x_2}\right) \beta, \quad \gamma = \left(\frac{x_3}{x_2}\right) \beta. \quad (29)$$

Substitute from the above equation in (19) again, we have

$$U_{21} = 0, \quad x_2 U_{22} + x_3 U_{32} = 0. \quad (30)$$

The condition $U_{32} = 0$ leads to $(U_{21}, U_{22}) = (0, 0)$ which is contradiction with (28).

**Case (2):** $x_1 = 0$. In this case $U_{32} = -x_2^2 \kappa = 0$, which leads to $x_2 = 0$. Therefore $U_{12} = -x_3^2 \tau = 0$ is satisfied when $X = 0$ or $\tau = 0$ which is contradiction.

**Case (3):** $x_1 \neq 0$ and $x_2 = 0$. In this case we have

$$\begin{cases} U_{11} = x_3 \beta, \\ U_{12} = x_3 (x_1 \kappa - x_3 \tau), \\ U_{21} = x_1 \gamma - x_3 \alpha, \\ U_{22} = x_1 x_3' - x_3 x_1', \\ U_{31} = -x_1 \beta, \\ U_{32} = x_1 (x_3 \tau - x_1 \kappa). \end{cases} \quad (31)$$

If $x_3 = 0$, then $U_{32} = -x_2^2 \kappa$ contradiction. Then $x_3 \neq 0$ and the solution of the equations $U_{11} = 0$, $U_{12} = 0$, $U_{31}$ and $U_{32} = 0$, is:

$$\beta = 0, \quad x_1 = \left(\frac{\tau}{\kappa}\right) x_3.$$  

Without loss of generality, we can put $x_3(s) = \kappa(s) \lambda(s)$. Now, the CARS (14) takes the form

$$\Psi(s, v) = \int \left(\alpha(s) t + \gamma(s) b\right) ds + \lambda(s) \tau(s) t + \kappa(s) b, \quad (32)$$

where $\kappa$, $\alpha$, $\gamma$ and $\lambda$ are arbitrary smooth functions of $s$.

**Case (4):** $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 = 0$. In this case $U_{12} = -x_2^2 \tau$ contradiction also.

Then, we have the following theorem:
Theorem 4.1. The ruled surface (32) is a constant angle surface if and only if the base curve \( \psi \) is a slant helix.

The proof of this theorem is similar than the proof of theorem (3.1).

Lemma 4.1. The Gaussian and mean curvatures of a CARS (32) is given by:

\[
K = 0, \quad H = \frac{\kappa^2 + \tau^2}{2[\gamma \tau - \kappa \alpha - \lambda(\kappa \tau' - \tau \kappa')]v}.
\]

From the above lemma and theorem, we can write the following corollary:

Corollary 4.2. The surface (32) can be named by: Rectifying constant angle ruled developable surface.

Now, we will write the position vector of a constant angle surface (32). Firstly, we introduced the following theorem:

Theorem 4.3. [2] The position vector \( \psi = (\psi_1, \psi_2, \psi_3) \) of a slant helix is computed in the natural representation form:

\[
\psi_1(s) = \frac{n}{m} \int \left( \int \kappa(s) \cos t \, ds, \int \kappa(s) \sin t \, ds, m \int \kappa(s) \, ds \right) \, ds,
\]

where \( t = \frac{1}{n} \arcsin(m \int \kappa(s) \, ds) \), \( m = \frac{n}{\sqrt{n^2 - n^2}}, n = \cos[\phi] \) and \( \phi \) is the angle between the fixed straight line (axis of a slant helix) and the principal normal vector \( n \) of the curve.

By a straightforward computations, we can find the moving Frenet frame of the slant helix (33) as the following:

\[
\begin{align*}
\mathbf{t} &= \frac{n}{m} \left( \int \kappa(s) \cos t \, ds, \int \kappa(s) \sin t \, ds, m \int \kappa(s) \, ds \right), \\
\mathbf{n} &= \frac{n}{m} \left( \cos t, \sin t, m \right), \\
\mathbf{b} &= \frac{m}{n} \left( - \int \tau(s) \cos t \, ds, \int \tau(s) \sin t \, ds, m \int \tau(s) \, ds \right).
\end{align*}
\]

Therefore, in the standard frame, the position vector \( \Psi(s, v) = (\Psi_1, \Psi_2, \Psi_3) \) of a constant angle surface (32) takes the following form:

\[
\begin{align*}
\Psi_1 &= \frac{n}{m} \left[ \int (\alpha(s) \int \kappa(s) \cos t \, ds - \gamma(s) \int \tau(s) \cos t \, ds) \, ds \\
&\quad + \lambda(s)(\int \kappa(s) \cos t \, ds - \kappa(s) \int \tau(s) \cos t \, ds) v \right], \\
\Psi_2 &= \frac{n}{m} \left[ \int (\alpha(s) \int \kappa(s) \sin t \, ds + \gamma(s) \int \tau(s) \sin t \, ds) \, ds \\
&\quad + \lambda(s)(\int \kappa(s) \sin t \, ds + \kappa(s) \int \tau(s) \sin t \, ds) v \right], \\
\Psi_3 &= \frac{n}{m} \left[ \int (\alpha(s) \int \kappa(s) ds + \gamma(s) \int \tau(s) ds) \, ds \\
&\quad + \lambda(s)(\int \kappa(s) ds + \kappa(s) \int \tau(s) ds) v \right].
\end{align*}
\]

5. CARS parallel to \( \psi_3 \)

In this section we take the normal vector \( U \) of a ruled surface is parallel to the normal \( \psi_3 = \frac{-\kappa \mathbf{t} + \tau \mathbf{b}}{\sqrt{\kappa^2 + \tau^2}} \) of the base curve \( \psi \). Therefore \( \psi \) is a slant-slant
helix and we have the following conditions:

\[
\begin{cases}
\sigma'(s) = m \kappa(s) \sqrt{1 + f'^2(s) \left(1 + \sigma'^2(s)\right)^{3/2}}, \\
U_1 \neq 0, \ U_3 \neq 0, \ U_{31} / U_{11} = U_{32} / U_{12} = -\frac{\tau}{\kappa}, \ U_{21} = U_{22} = 0,
\end{cases}
\]

where \(\sigma(s) = \frac{f'(s)}{\kappa(s) \left(1 + f'^2(s)\right)^{3/2}}\) and \(f(s) = \frac{\tau(s)}{\kappa(s)}\). We will study all cases as the following:

**Case (1):** \(x_1 \neq 0, \ x_2 \neq 0 \) and \(x_3 \neq 0\). If we solving the equations \(U_{21} = 0\) and \(U_{22} = 0\) we can get

\[
x_2 = \frac{x_3 x'_1 - x_1 x'_3}{x_1 \tau - x_3 \kappa}, \quad \alpha = \left(\frac{x_1}{x_3}\right) \gamma.
\]

Substitute from the above equation in the condition \(U_{31} / U_{11} = -\frac{\tau}{\kappa}\) and solve it, we have

\[
\beta = \left(\frac{x_3 x'_1 - x_1 x'_3}{x_1 (x_1 \tau + x_3 \kappa)}\right) \frac{\alpha}{\tau}.
\]

On other hand, the condition \(U_{32} / U_{12} = -\frac{\tau}{\kappa}\) leads to

\[
\left(\frac{x_3 \tau - x_1 \kappa}{x_1 \tau + x_3 \kappa}\right)^2 \left[x_1 x_3 (x_3 \kappa^3 - x_1 \tau^3) - (x_3 \kappa' + x_1 \tau') (x_3 x'_1 - x_1 x'_3)ight.
\]

\[
- x_3 (x_3^2 - 2x_1^2) \kappa^2 \tau + \tau [x_3 (x_1 x'' - 2x_1'^2) + x_1 (2x'_1 x'_3 - x_1 x''_3)]
\]

\[
+ \kappa [x_1 (x_1^2 - 2x_3^2) \tau^2 + 2x_1 x_3^2 - x_3^2 x''_1 - x_3 (2x'_1 x'_3 + x_1 x''_3)] = 0.
\]

It is easy to see that

\[
x_1 = \left(\frac{\tau}{\kappa}\right) x_3
\]

is one of the solutions of the above equation. As a conclusion, from (37), (38) and (40), we have

\[
\begin{cases}
x_1 = \left(\frac{x_3}{\kappa}\right) \tau, \\
x_2 = \left(\frac{x_3}{\kappa}\right) \left(\frac{\kappa \tau' - \tau \kappa'}{\kappa^2 + \tau^2}\right), \\
\alpha = \left(\frac{\gamma}{\kappa}\right) \tau, \\
\beta = \left(\frac{\gamma}{\kappa}\right) \left(\frac{\kappa \tau' - \tau \kappa'}{\kappa^2 + \tau^2}\right).
\end{cases}
\]

Without loss of generality, we can put \(x_3(s) = \kappa(s) \lambda(s)\) and \(\gamma(s) = \kappa(s) \mu(s)\). Now, the CARS (14) takes the form:

\[
\Psi(s, v) = \int \left[\tau \mathbf{t} + \left(\frac{\kappa \tau' - \tau \kappa'}{\kappa^2 + \tau^2}\right) \mathbf{n} + \kappa \mathbf{b}\right] \lambda \, ds + v \left[\tau \mathbf{t} + \left(\frac{\kappa \tau' - \tau \kappa'}{\kappa^2 + \tau^2}\right) \mathbf{n} + \kappa \mathbf{b}\right] \mu,
\]

where \(\lambda\) and \(\mu\) are arbitrary smooth functions of \(s\).
Case (2): $x_1 = 0$. In this case we have
\[
\begin{aligned}
U_{11} &= x_3 \beta - x_2 \gamma, \\
U_{12} &= x_3 x_2' - x_2 x_3' - (x_2^2 + x_3^2) \tau, \\
U_{21} &= -x_3 \alpha, \\
U_{22} &= x_2 x_3 \kappa, \\
U_{31} &= x_2 \alpha, \\
U_{32} &= -x_2^2 \kappa.
\end{aligned}
\]
(43)

From the equation $U_{22} = 0$, we get two cases:

(2.1): $x_2 = 0$. In this case we have
$U_3 = (U_{31}, U_{32}) = 0$ which is contradiction with (36).

(2.2): $x_3 = 0$. In this case the condition $U_{32} / U_{12} = -\frac{\tau}{\kappa}$ leads to $\kappa^2 + \tau^2 = 0$
which is contradiction again.

Case (3): $x_1 \neq 0$ and $x_2 = 0$. In this case we have
\[
\begin{aligned}
U_{11} &= x_3 \beta, \\
U_{12} &= x_3 (x_1 \kappa - x_3 \tau), \\
U_{21} &= x_1 \gamma - x_3 \alpha, \\
U_{22} &= x_1 x_3' - x_3 x_1', \\
U_{31} &= -x_1 \beta, \\
U_{32} &= x_1 (x_3 \tau - x_1 \kappa).
\end{aligned}
\]
(44)

Solving the equations $U_{21} = 0$ and $U_{22} = 0$, we get $x_3 = c_1 x_1$ and $\gamma = c_1 \alpha$,
where $c_1$ is an arbitrary constant of integration. Substituting in the condition
$U_{32} / U_{12} = -\frac{\tau}{\kappa}$, we have $\tau = c_1 \kappa$ which is contradiction with the base curve
is a slant-slant helix.

Case (4): $x_1 \neq 0$, $x_2 \neq 0$ and $x_3 = 0$. In this case $U_{22} = x_1 x_2 \tau$ leads
to $\tau = 0$ which is contradiction also.

Then, we can get the following theorem:

**Theorem 5.1.** The ruled surface (42) is a constant angle surface if and
only if the base curve $\psi$ is a slant-slant helix.

The proof of this theorem is similar than the proof of theorem (3.1).

**Lemma 5.1.** The Gaussian and mean curvatures of a CARS (32) is given
by:
\[
K = 0, \quad H = -\frac{\sqrt{1 - m^2 (\int \xi(s) ds)^2}}{2m \xi(s) \lambda(s) \nu},
\]
where
\[
\begin{aligned}
\kappa(s) &= \xi(s) \cos \left( \sqrt{\frac{1 - m^2 (\int f(s) ds)^2}{m}} \right), \\
\tau(s) &= \xi(s) \sin \left( \sqrt{\frac{1 - m^2 (\int f(s) ds)^2}{m}} \right).
\end{aligned}
\]
(45)
where $\xi(s)$ is an arbitrary function of $s$.

It is worth noting that: the curvature and torsion in (45) are the intrinsic equations of slant-slant helix [5]. From the above lemma and theorem, we can write the following corollary:

**Corollary 5.2.** The surface (42) can be named by: Slant-slant constant angle ruled developable surface.

Now, by the method explained in the above two sections, we can obtain the position vector of a constant angle surface (42).

### 6. Conclusion

Finally, from the Lemmas 3.1, 4.1 and 5.1, we can write the following important general lemma and remark:

**Lemma 6.1.** CARS parallel to the vector

$$
\psi_{k+1} = \frac{\psi'_k(s)}{\|\psi'_k(s)\|}, \quad \psi_0(s) = \psi(s), \quad k \in \{0, 1, 2, \ldots\}.
$$

is a developable surface.

**Remark 1.** CARS parallel to the vector $\psi_{k+1}$ can be named by: $k$-slant constant angle ruled developable surface.

### References


DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KING ABDULAZIZ UNIVERSITY
PO BOX 80203, JEDDAH, SA

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
AL-AZHAR UNIVERSITY
NASR CITY 11884, CAIRO, EGYPT

E-mail address: ahmadtawfik95@gmail.com