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CHAINS OF TANGENT CIRCLES INSCRIBED IN CURVILINEAR TRIANGLES

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Abstract. We find the characteristics (radii and coordinates of centers) for chains of tangent circles inscribed in some curvilinear triangles. Also we extend the well-known Pappus' problem to the curvilinear triangles.

1. Introduction

The most famous problem of chain of tangent circles inscribed in the curvilinear triangles is the Pappus' problem (see [1] – [3]). It states that for each circle of the Pappus chain $\{\delta_n\}_{n=1}^{\infty}$ (see Figure 1) ratio of the distance from its centers to the axis of Archimedes arbelos to its radii is double number of the circle, namely

$$\frac{d_1}{R_1} = 2, \quad \frac{d_2}{R_2} = 4, \quad \frac{d_3}{R_3} = 6, \quad \dots, \quad \frac{d_n}{R_n} = 2n, \quad n \in \mathbb{N}.$$

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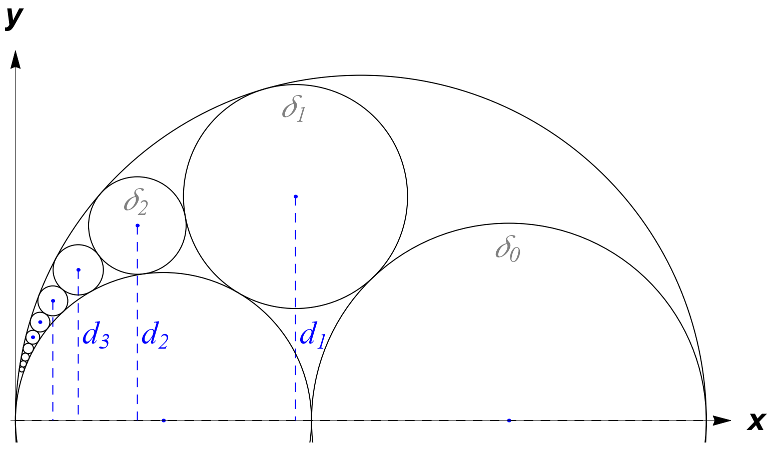

Figure 1.

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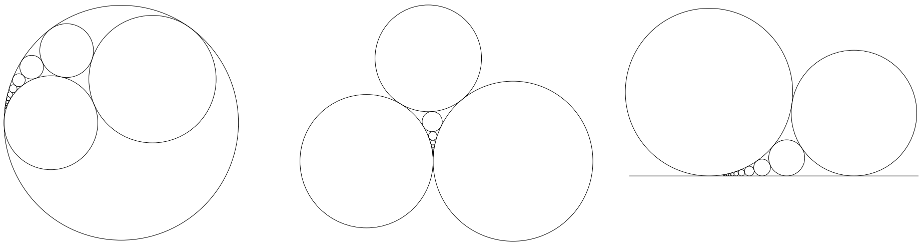
We study the chains of tangent circles inscribed in the curvilinear triangles, which are obtained by the following modifications of the classical Archimedes arbelos: a shift of central axis of base circles with respect to the third circle (see Figure 2 A), or the transition from the inner touch of base circles to an external touch (see Figure 2 B), wherein one of them may degenerate into a straight line (see Figure 2 C).



A B C

Figure 2.

The main purpose of the paper is to show that the extension of the Pappus' problem to the curvilinear triangles is quite possible. To achieve this aim explicit expressions for radii and coordinates of centers for the systems of tangent circles inscribed in the curvilinear triangles were obtained.

2. Main results

Below, in each case, the coordinate system is imposed as shown in the figures.

Let us consider the chain of tangent circles constructed according to the following principle: there are two circles $\alpha((r;0),r)$ and $\beta((R;0),R)$ $(R > r)$ on the plane that touch internally. We inscribe the chain of pairwise tangent circles into the region between the circles α and β. The first one is ξ_0. It has radius r_0. The second is ξ_1. It is tangent to ξ_0 and both α and β, and so on (see Figure 3). Let x_n, y_n and r_n denote the abscissa, the ordinate and the radius of the circle $\xi_n, n \in \mathbb{N}_0$.

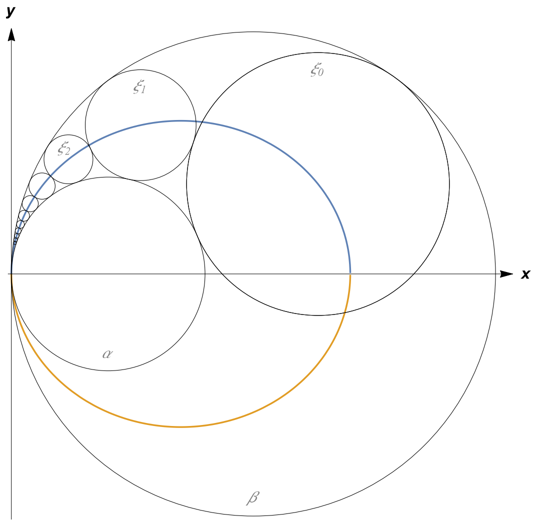


Figure 3.

Theorem 2.1. *(see corollary 3.2 and theorem 3.3) The characteristics of the chain of tangent circles $\{\xi_n\}$, $n \in \mathbb{N}_0$ are given by the following formulas:*

$$x_n = \frac{Rrr_0(R+r)}{Rr(R-r) + 2(R-r)\sqrt{Rrr_0(R-r-r_0)}n + (R-r)^2 r_0 n^2},$$

$$y_n = \frac{2Rrr_0(R-r)n + 2Rr\sqrt{Rrr_0(R-r-r_0)}}{Rr(R-r) + 2(R-r)\sqrt{Rrr_0(R-r-r_0)}n + (R-r)^2 r_0 n^2},$$

$$r_n = \frac{Rrr_0}{Rr + 2\sqrt{Rrr_0(R-r-r_0)}n + (R-r)r_0 n^2},$$

$$\frac{y_n}{r_n} = \frac{y_0}{r_0} + 2n. \tag{1}$$

Remark 2.1. *Equality (1) generalizes the Pappus' problem for the chain of tangent circles $\{\xi_n\}$, $n \in \mathbb{N}_0$.*

Now, we consider the chain of tangent circles constructed according to the following principle: there are three circles $\alpha((r;0),r)$, $\beta((-R;0),R)$ and ω_0 on the plane that touch externally. We inscribe the chain of pairwise tangent circles into the region between the circles α, β and ω_0. The first one is ω_1. It is tangent to ω_0 and both α and β. The second is ω_2. It is tangent to ω_1 and both α and β, and so on (see Figure 4). Let x_n, y_n and r_n denote the abscissa, the ordinate and the radius of the circle $\omega_n, n \in \mathbb{N}_0$.

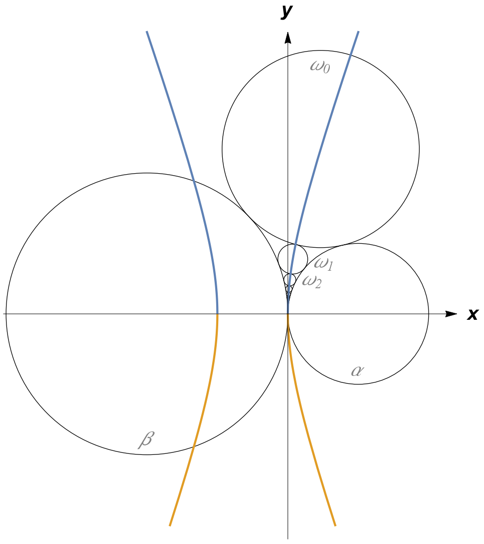


Figure 4.

Theorem 2.2. *(see corollary 4.2 and theorem 4.3). The characteristics of the chain of tangent circles $\{\omega_n\}$, $n \in \mathbb{N}_0$ are given by the following formulas:*

$$x_n = \frac{Rrr_0(R-r)}{Rr(R+r) + 2(R+r)\sqrt{Rrr_0(R+r+r_0)}n + (R+r)^2 r_0 n^2},$$

$$y_n = \frac{2Rrr_0(R+r)n + 2Rr\sqrt{Rrr_0(R+r+r_0)}}{Rr(R+r) + 2(R+r)\sqrt{Rrr_0(R+r+r_0)}n + (R+r)^2 r_0 n^2},$$

$$r_n = \frac{Rrr_0}{Rr + 2\sqrt{Rrr_0(R+r+r_0)}n + (R+r)r_0n^2},$$

$$\frac{y_n}{r_n} = \frac{y_0}{r_0} + 2n. \tag{2}$$

Remark 2.2. *Equality (2) generalizes the Pappus' problem for the chain of tangent circles* $\{\omega_n\}$, $n \in \mathbb{N}_0$.

And now, we consider the chain of tangent circles constructed according to the following principle: there are two circles $\alpha((0;R),R)$ and ζ_0 on the plane that touch externally. This circles have common tangent a. We inscribe the chain of pairwise tangent circles into the region between the circles α, β and the straight line a. The first one is ζ_1. It is tangent to ζ_0 and both α and a. The second is ζ_2. It is tangent to ζ_1 and both α and a, and so on (see Figure 5). Let x_n, y_n and r_n denote the abscissa, the ordinate and the radius of the circle ζ_n, $n \in \mathbb{N}_0$.

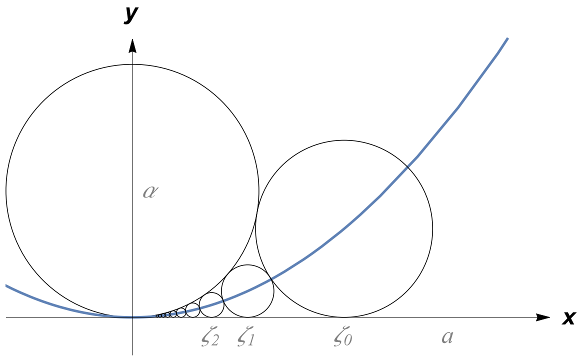


Figure 5.

Theorem 2.3. *(see corollary 5.1 and theorem 5.3). The characteristics of the chain of tangent circles* $\{\zeta_n\}$, $n \in \mathbb{N}$ *are given by the following formulas:*

$$x_n = \frac{2R\sqrt{r_0}}{\sqrt{R} + \sqrt{r_0}n},$$

$$r_n = y_n = \frac{Rr_0}{\left(\sqrt{R} + \sqrt{r_0}n\right)^2},$$

$$\frac{x_n}{y_n} = \frac{x_n}{r_n} = \frac{x_0}{r_0} + 2n. \tag{3}$$

Remark 2.3. *Equality (3) generalizes the Pappus' problem for the chain of tangent circles* $\{\zeta_n\}$, $n \in \mathbb{N}_0$.

3. PROOF OF THE THEOREM 2.1

Lemma 3.1. *Let* $R > r$. *If circle* ξ *externally touches upon the circle* $\alpha((r;0),r)$ *and internally touches upon the circle* $\beta((R;0),R)$, *then center of the circle* ξ *lies on the ellipse*

$$\frac{\left(x - \frac{R+r}{2}\right)^2}{\left(\frac{R+r}{2}\right)^2} + \frac{y^2}{\left(\sqrt{Rr}\right)^2} = 1. \tag{4}$$

Proof. Let the center of the circle ξ is $(x; y)$. Since the circle ξ touches upon the circles $\alpha((r; 0), r)$ and $\beta((R; 0), R)$, then

$$\rho((r; 0), (x; y)) - r = R - \rho((R; 0), (x; y)). \tag{5}$$

Here $\rho((x_1; y_1), (x_2; y_2))$ is distance between two points with the coordinates $(x_1; y_1)$ and $(x_2; y_2)$. From (5) we have

$$\sqrt{(r - x)^2 + y^2} + \sqrt{(R - x)^2 + y^2} = R + r.$$

Double squaring of both parts of the last equality, after simplifications we obtain the equation of the ellipse (4).

Remark 3.1. *According to the lemma 3.1 coordinates of centers of the chain of tangent circles* $\{\xi_n\}, n \in \mathbb{N}_0$ *is*

$$\left(x_n; \frac{2\sqrt{Rrx_n(R + r - x_n)}}{R + r}\right).$$

Remark 3.2. *In case if* $r_0 = R - r$ *we have classical Pappus chain.*

Let $(x - x_n)^2 + (y - y_n)^2 = r_n^2$, $n \in \mathbb{N}_0$ is the equation of the circle ξ_n. Bearing in mind elementary geometry reasons we construct the system of equation for finding characteristics of the chain of tangent circles $\{\xi_n\}$, $n \in \mathbb{N}_0$:

$$\begin{cases} r_n = R - \sqrt{(R - x_n)^2 + y_n^2}, \\ y_n = \dfrac{2\sqrt{Rrx_n(R + r - x_n)}}{R + r}, \\ r_n + r_{n-1} = \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}, \end{cases} \qquad n \in \mathbb{N}. \tag{6}$$

From system (6) we have the following equality

$$2R - \sqrt{(R - x_n)^2 + \left(\frac{2\sqrt{Rrx_n(R + r - x_n)}}{R + r}\right)^2} -$$

$$-\sqrt{(R - x_{n-1})^2 + \left(\frac{2\sqrt{Rrx_{n-1}(R + r - x_{n-1})}}{R + r}\right)^2} =$$

$$= \sqrt{(x_n - x_{n-1})^2 + \frac{4Rr}{(R + r)^2}\left(\sqrt{x_n(R + r - x_n)} - \sqrt{x_{n-1}(R + r - x_{n-1})}\right)^2}.$$

Selecting the complete squares in the last equality, we have:

$$2R - \sqrt{\left(\frac{r(R + x_n) + R(R - x_n)}{(r + R)}\right)^2} - \sqrt{\left(\frac{r(R + x_{n-1}) + R(R - x_{n-1})}{(r + R)}\right)^2} =$$

$$= \sqrt{(x_n - x_{n-1})^2 + \frac{4Rr}{(R + r)^2}\left(\sqrt{x_n(R + r - x_n)} - \sqrt{x_{n-1}(R + r - x_{n-1})}\right)^2}.$$

After the elementary transformations of the left side, the last equality takes the following form:

$$\frac{(R - r)(x_n + x_{n-1})}{r + R} =$$

(7)

$$= \sqrt{(x_n - x_{n-1})^2 + \frac{4Rr}{(R+r)^2}\left(\sqrt{x_n(R+r-x_n)} - \sqrt{x_{n-1}(R+r-x_{n-1})}\right)^2}.$$

Double-raising both sides of the equality (7) to square, after simply transformations we obtain the quadratic equation

$$x_n^2\left(4r^2R^2x_{n-1}(r+R-x_{n-1}) + \left((R^2+r^2)x_{n-1} - Rr(R+r)\right)^2\right) +$$

$$+x_n\left(2rR\left(Rr(R+r) - (R^2+r^2)x_{n-1}\right)(R+r)\,x_{n-1} -\right.$$

$$\left.-4r^2R^2x_{n-1}(r+R)(r+R-n_{n-1})\right) + R^2r^2\,(R+r)^2\,x_{n-1}^2 = 0,$$

solving it we obtain the formula

$$(8)\quad x_n = \frac{Rr(R+r)x_{n-1}}{Rr(R+r) + 2(R-r)\sqrt{rRx_{n-1}(r+R-x_{n-1})} + (R-r)^2x_{n-1}}.$$

Remark 3.3. *Equality (8) specifies the recurrent formula for finding the abscissa of the center of the circle* ξ_n *through the abscissa of the center of the previous circle* ξ_{n-1}, $n \in \mathbb{N}$.

Theorem 3.1. *The abscissas of the centers of the system* $\{\xi_n\}$, $n \in \mathbb{N}$ *are defined by the following sequence:*

$$(9)\qquad x_n = \frac{Rr(R+r)x_0}{Rr(R+r) + 2(R-r)\sqrt{rRx_0(r+R-x_0)}n + (R-r)^2x_0n^2}.$$

Proof. We use the method of mathematical induction. Let $a = rR(R+r)$, $b = R - r$, $c = -Rr$. We need to prove the formula

$$(10)\qquad x_n = \frac{ax_0}{a + 2b\sqrt{x_0}\sqrt{a+cx_0}\,n + b^2x_0n^2},$$

if we know the following recurrent formula

$$(11)\qquad x_n = \frac{ax_{n-1}}{a + 2b\sqrt{x_{n-1}}\sqrt{(a+cx_{n-1})} + b^2x_{n-1}}.$$

Basis. From (10) and (11) we can find x_1. These expressions are equal. Inductive step. Let the formula (10) is true for $n = k$. According to the formula (11)

$$(12)\qquad x_{k+1} = \frac{ax_k}{a + 2b\sqrt{x_k}\sqrt{(a+cx_k)} + b^2x_k}.$$

Substituting x_k in the formula (12) from the formula (10), after simplification we have:

$$x_{k+1} = (ax_0)/\left(a + 2b\sqrt{x_0}\sqrt{a+cx_0}k + b^2x_0k^2 +\right.$$

$$\left.+\,2b\sqrt{x_0}\sqrt{a + 2b\sqrt{x_0}\sqrt{a+cx_0}k + b^2x_0k^2 + cx_0} + b^2x_0\right).$$

Selecting the complete square, we have:

$$x_{k+1} =$$

$$= \frac{ax_0}{a + 2b\sqrt{x_0}\sqrt{a+cx_0}k + b^2x_0k^2 + 2b\sqrt{x_0}\sqrt{(b\sqrt{x_0}k + \sqrt{a+cx_0})^2} + b^2x_0} =$$

$$= \frac{ax_0}{a + 2b\sqrt{x_0}\sqrt{a+cx_0}k + b^2x_0k^2 + 2b\sqrt{x_0}(b\sqrt{x_0}k + \sqrt{a+cx_0}) + b^2x_0} =$$

$$= \frac{ax_0}{a + 2b\sqrt{x_0}\sqrt{a+cx_0}k + 2b\sqrt{x_0}\sqrt{a+cx_0} + b^2x_0k^2 + 2b^2x_0k + b^2x_0} =$$

$$= \frac{ax_0}{a + 2b\sqrt{x_0}\sqrt{a+cx_0}(k+1) + b^2x_0(k+1)^2}.$$

Corrolary 3.1. *The abscissas of the centers of the system* $\{\xi_n\}$, $n \in \mathbb{N}$ *are defined by the following sequence:*

$$x_n = \frac{Rr(R+r)x_0}{Rr(R+r) + (R^2-r^2)y_0n + (R-r)^2x_0n^2}.$$

Proof. The validity of this statement follows from the remark 3.1.

Theorem 3.2. *The ordinates of the centers and the radii of the system* $\{\xi_n\}$, $n \in \mathbb{N}$ *are defined by the following sequences:*

$$y_n = \frac{2rR\left(\sqrt{rRx_0(R+r-x_0)} + (R-r)nx_0\right)}{Rr(R+r) + 2(R-r)\sqrt{rRx_0(r+R-x_0)}n + (R-r)^2x_0n^2}, \tag{13}$$

$$r_n = \frac{Rr(R-r)x_0}{Rr(R+r) + 2(R-r)\sqrt{rRx_0(r+R-x_0)}n + (R-r)^2x_0n^2} \tag{14}$$

or (taking into account equality $y_0 = \frac{2\sqrt{Rrx_0(R+r-x_0)}}{R+r}$*, see remark 3.1)*

$$y_n = \frac{Rr(R+r)y_0 + 2Rr(R-r)x_0n}{Rr(R+r) + (R^2-r^2)y_0n + (R-r)^2x_0n^2},$$

$$r_n = \frac{rR(R-r)x_0}{Rr(R+r) + (R^2-r^2)y_0n + (R-r)^2x_0n^2}.$$

Proof. To prove this fact, it is sufficient to substitute formula (9) into the system (6) and use remark 3.1.

Corrolary 3.2. *The characteristics of the chain of tangent circles* $\{\xi_n\}$, $n \in \mathbb{N}_0$ *obey the following formulas:*

$$\frac{x_n}{r_n} = \frac{R+r}{R-r}, \tag{15}$$

$$\frac{y_n}{r_n} = \frac{2\sqrt{Rrx_0(R+r-x_0)}}{(R-r)x_0} + 2n = \frac{R+r}{R-r}\frac{y_0}{x_0} + 2n = \frac{y_0}{r_0} + 2n,$$

$$\frac{y_n}{x_n} = \frac{y_0}{x_0} + \frac{2(R-r)}{(R+r)}n.$$

Theorem 3.3. *The characteristics of the chain of tangent circles* $\{\xi_n\}$, $n \in \mathbb{N}_0$ *are given by the following formulas:*

$$x_n = \frac{Rrr_0(R+r)}{Rr(R-r) + 2(R-r)\sqrt{Rrr_0(R-r-r_0)}n + (R-r)^2r_0n^2},$$

$$y_n = \frac{2Rrr_0(R-r)n + 2Rr\sqrt{Rrr_0(R-r-r_0)}}{Rr(R-r) + 2(R-r)\sqrt{Rrr_0(R-r-r_0)}n + (R-r)^2r_0n^2},$$

$$r_n = \frac{Rrr_0}{Rr + 2\sqrt{Rrr_0(R - r - r_0)}n + (R - r)r_0 n^2}.$$

Proof. To prove this fact, it is sufficient to substitute formula (15) (in case if $n = 0$) into the formulas (9), (13) and (14).

4. PROOF OF THE THEOREM 2.2

Lemma 4.1. *Let $R \neq r$. If circle ω externally touches upon the circles $\alpha((r;0), r)$ and $\beta((-R;0), R)$, then center of the circle ω lies on the hyperbola*

$$\frac{\left(x + \frac{R-r}{2}\right)^2}{\left(\frac{R-r}{2}\right)^2} - \frac{y^2}{\left(\sqrt{Rr}\right)^2} = 1. \tag{16}$$

Proof. Let the center of the circle ω is $(x;y)$. As the circle ω externally touches on the circles $\alpha((r;0), r)$ and $\beta((R;0), R)$, then

$$\rho((r;0),(x;y)) - r = \rho((-R;0),(x;y)) - R. \tag{17}$$

From (17) we have

$$\sqrt{(x+R)^2 + y^2} - \sqrt{(x-r)^2 + y^2} = R - r.$$

Double squaring of both parts of the last equality, after simplifications, we obtain the equation of the hyperbola (16).

Remark 4.1. *Assume that $R > r$, then $x_0 > 0$. According to the lemma 4.1 coordinates of centers of the chain of tangent circles $\{\omega_n\}, n \in \mathbb{N}_0$ is*

$$\left(x_n; \frac{2\sqrt{rRx_n(x_n + R - r)}}{R - r}\right).$$

Let $(x - x_n)^2 + (y - y_n)^2 = r_n^2$, $n \in \mathbb{N}_0$ is the equation of the circle ω_n. Bearing in mind elementary geometry reasons we construct the system of equation for finding the characteristics of the chain of tangent circles $\{\omega_n\}$, $n \in \mathbb{N}_0$:

$$\begin{cases} r_n = \sqrt{(r - x_n)^2 + y_n^2} - r, \\ y_n = \dfrac{2\sqrt{rRx_n(x_n + R - r)}}{R - r}, \\ r_n + r_{n-1} = \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}, \end{cases} \qquad n \in \mathbb{N}. \tag{18}$$

From system (18) we have the following equality

$$\sqrt{(r - x_n)^2 + \frac{4rRx_n(x_n + R - r)}{(R - r)^2}} + \sqrt{(r - x_{n-1})^2 + \frac{4rRx_{n-1}(x_{n-1} + R - r)}{(R - r)^2}} =$$

$$= 2r + \sqrt{(x_n - x_{n-1})^2 + \frac{4rR}{(R - r)^2}\left(\sqrt{x_n(x_n + R - r)} - \sqrt{x_{n-1}(x_{n-1} + R - r)}\right)^2}.$$

Selecting the complete squares in the last equality, we have:

$$\sqrt{\left(\frac{Rx_n+r(x_n+R-r)}{R-r}\right)^2}+\sqrt{\left(\frac{Rx_{n-1}+r(x_{n-1}+R-r)}{R-r}\right)^2}=$$

$$2r+\sqrt{(x_n-x_{n-1})^2+\frac{4rR}{(R-r)^2}\left(\sqrt{x_n(x_n+R-r)}-\sqrt{x_{n-1}(x_{n-1}+R-r)}\right)^2}.$$

After the elementary transformations of the left side, the last equality takes the following form:

$$\frac{(r+R)(x_n+x_{n-1})}{R-r}=$$

(19)

$$=\sqrt{(x_n-x_{n-1})^2+\frac{4rR}{(R-r)^2}\left(\sqrt{x_n(x_n+R-r)}-\sqrt{x_{n-1}(x_{n-1}+R-r)}\right)^2}.$$

Double-raising both sides of the equality (19) to square, after simply transformations we obtain the quadratic equation

$$\left(\left(rR(R-r)-(r^2+R^2)x_{n-1}\right)^2-4r^2R^2x_{n-1}(x_{n-1}+R-r)\right)x_n^2+$$
$$+\left(2rR(rR(r-R)+(r^2+R^2)x_{n-1})(r-R)x_{n-1}-\right.$$
$$\left.-4r^2R^2(R-r)x_{n-1}(x_{n-1}+R-r)\right)x_n+rR(R-r)x_{n-1}=0,$$

solving it we achieve the formula

(20)

$$x_n=\frac{rR(R-r)x_{n-1}}{rR(R-r)+2(r+R)\sqrt{rRx_{n-1}(x_{n-1}+R-r)}+(r+R)^2x_{n-1}}.$$

Remark 4.2. *Equality (20) specifies the recurrent formula for finding the abscissa of the center of the circle* ω_n *through the abscissa of the center of the previous circle* ω_{n-1}*,* $n\in\mathbb{N}$*.*

Theorem 4.1. *The abscissas of the centers of the system* $\{\omega_n\}$, $n\in\mathbb{N}$ *are defined by the following sequence:*

$$x_n=\frac{rR(R-r)x_0}{rR(R-r)+2(r+R)\sqrt{rRx_0(x_0+R-r)}\,n+(r+R)^2x_0n^2}. \tag{21}$$

Proof. Let $a=rR(R-r)$, $b=r+R$, $c=rR$. We need to prove the formula

$$x_n=\frac{ax_0}{a+2b\sqrt{x_0}\sqrt{a+cx_0}\,n+b^2x_0n^2},$$

if we know the following recurrent formula

$$x_n=\frac{ax_{n-1}}{a+2b\sqrt{x_{n-1}}\sqrt{(a+cx_{n-1})}+b^2x_{n-1}}.$$

Further it is sufficient to repeat the proof of the theorem 3.1.

Corrolary 4.1. *The abscissas of the centers of the system* $\{\omega_n\}$, $n\in\mathbb{N}$ *are defined by the following sequence:*

$$x_n=\frac{rR(R-r)x_0}{rR(R-r)+(R^2-r^2)y_0n+(r+R)^2x_0n^2}.$$

Proof. The validity of this statement follows from remark 4.1.

Theorem 4.2. *The ordinates of the centers and the radii of the system* $\{\omega_n\}$, $n \in \mathbb{N}$ *are defined by the following sequences:*

$$(22) \qquad y_n = \frac{2rR\left(\sqrt{rRx_0(x_0+R-r)}+(r+R)nx_0\right)}{rR(R-r)+2(r+R)\sqrt{rRx_0(x_0+R-r)}\,n+(r+R)^2x_0n^2},$$

$$(23) \qquad r_n = \frac{rR(r+R)x_0}{rR(R-r)+2(r+R)\sqrt{rRx_0(x_0+R-r)}\,n+(r+R)^2x_0n^2}$$

or (taking into account equality $y_0 = \dfrac{2\sqrt{rRx_0(x_0+R-r)}}{R-r}$*, see remark 4.1)*

$$y_n = \frac{rR(R-r)y_0+2rR(r+R)nx_0}{rR(R-r)+(R^2-r^2)y_0n+(r+R)^2x_0n^2},$$

$$r_n = \frac{rR(R+r)x_0}{rR(R-r)+(R^2-r^2)y_0n+(r+R)^2x_0n^2}.$$

Proof. To prove this theorem, it is enough to substitute formula (21) into the system (18) and use remark 4.1.

Corrolary 4.2. *The characteristics of the chain of tangent circles* $\{\omega_n\}$, $n \in \mathbb{N}_0$ *obey by the following formulas:*

$$(24) \qquad \frac{x_n}{r_n} = \frac{R-r}{R+r},$$

$$\frac{y_n}{r_n} = \frac{2\sqrt{rRx_0(x_0+R-r)}}{(r+R)x_0}+2n = \frac{R-r}{R+r}\frac{y_0}{x_0}+2n = \frac{y_0}{r_0}+2n,$$

$$\frac{y_n}{x_n} = \frac{y_0}{x_0}+\frac{2(R+r)}{(R-r)}n.$$

Theorem 4.3. *The characteristics of the chain of tangent circles* $\{\omega_n\}$, $n \in \mathbb{N}_0$ *are given by the following formulas:*

$$(25) \qquad x_n = \frac{Rrr_0(R-r)}{Rr(R+r)+2(R+r)\sqrt{Rrr_0(R+r+r_0)}n+(R+r)^2r_0n^2},$$

$$(26) \qquad y_n = \frac{2Rrr_0(R+r)n+2Rr\sqrt{Rrr_0(R+r+r_0)}}{Rr(R+r)+2(R+r)\sqrt{Rrr_0(R+r+r_0)}n+(R+r)^2r_0n^2},$$

$$(27) \qquad r_n = \frac{Rrr_0}{Rr+2\sqrt{Rrr_0(R+r+r_0)}n+(R+r)r_0n^2}.$$

Proof. To prove this assertion, it is sufficient to substitute formula (24) (in case if $n = 0$) into the formulas (21), (22) and (23).

Remark 4.3. *In case if* $R = r$ *(see Figure 6) the theorem 4.3 and the corollary 4.2 for the system* $\{\omega_n\}$, $n \in \mathbb{N}_0$ *have the the following form:*

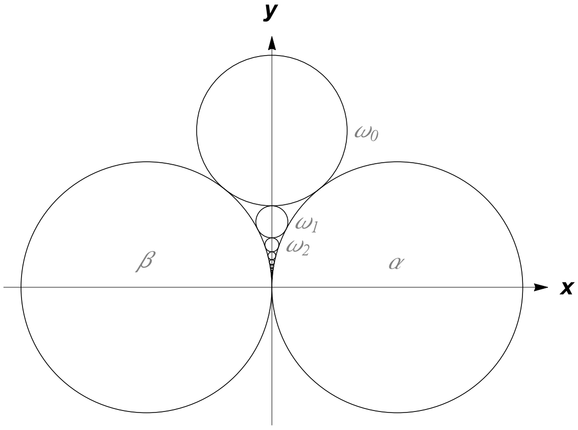


Figure 6.

Theorem 4.4. *Let $R = r$. The characteristics of the chain of tangent circles $\{\omega_n\}$, $n \in \mathbb{N}_0$ are given by the following formulas:*

$$y_n = \frac{R\sqrt{r_0^2 + 2r_0R} + 2Rr_0n}{R + 2\sqrt{r_0^2 + 2r_0R}\,n + 2r_0n^2}, \tag{28}$$

$$r_n = \frac{Rr_0}{R + 2\sqrt{r_0(2R + r_0)}n + 2r_0n^2}, \tag{29}$$

$$\frac{y_n}{r_n} = \frac{\sqrt{r_0^2 + 2r_0R}}{r_0} + 2n = \frac{y_0}{r_0} + 2n.$$

Proof. For proving the formulas (28) and (29), it is sufficient to pass to the limit as $r \to R$ in the formulas of theorem 4.3.

5. PROOF OF THE THEOREM 2.3

Remark 5.1. *Note that in the introduced coordinate system for the circle $\zeta_n, n \in \mathbb{N}$ equity $r_n = y_n$ is performed (see Figure 5).*

Lemma 5.1. *Let $R > 0$. If circle ζ externally touches on circle $\alpha((0; R), R)$ and the x-axis, then center of the circle ζ lies on the parabola*

$$y = \frac{x^2}{4R}. \tag{30}$$

Proof. Let the center of the circle ζ is $(x; y)$. As the circle ζ externally touches upon the circles $\alpha((0; R), R)$ and x-axis, then

$$\rho((0; R), (x; y)) - R = y. \tag{31}$$

From (31) we have

$$\sqrt{x^2 + (y - R)^2} = R + y.$$

Raising both parts of the last equality to square, after simplifications, we obtain the equation of the parabola (30).

Let $(x-x_n)^2+(y-y_n)^2=r_n^2$, $n\in\mathbb{N}_0$ is the equation of the circle ζ_n. According to elementary geometry reasons we construct the system of equation for finding characteristics of the chain of tangent circles $\{\zeta_n\}$, $n\in\mathbb{N}_0$:

$$\begin{cases} r_n=\sqrt{x_n^2+(y_n-R)^2}-R, \\ y_n=\dfrac{x_n^2}{4R}, \\ r_n+r_{n-1}=\sqrt{(x_n-x_{n-1})^2+(y_n-y_{n-1})^2}, \end{cases} \qquad n\in\mathbb{N}. \tag{32}$$

From system (32) we have the following equality

$$\sqrt{x_n^2+\left(\frac{x_n^2}{4R}-R\right)^2}+\sqrt{x_{n-1}^2+\left(\frac{x_{n-1}^2}{4R}-R\right)^2}-2R=$$
$$=\sqrt{(x_n-x_{n-1})^2+\left(\frac{x_n^2}{4R}-\frac{x_{n-1}^2}{4R}\right)^2}.$$

Selecting the complete squares in the last equality, we have:

$$\sqrt{\frac{(4R^2+x_n^2)^2}{16R^2}}+\sqrt{\frac{\left(4R^2+x_{n-1}^2\right)^2}{16R^2}}-2R=$$
$$=\sqrt{(x_n-x_{n-1})^2+\left(\frac{x_n^2}{4R}-\frac{x_{n-1}^2}{4R}\right)^2}.$$

After simplification we have:

$$\frac{x_n^2+x_{n-1}^2}{4R}=\sqrt{(x_n-x_{n-1})^2+\left(\frac{x_n^2}{4R}-\frac{x_{n-1}^2}{4R}\right)^2}.$$

Double-raising both sides of the last equality to square, after simply transformations we obtain the quadratic equation

$$(x_{n-1}^2-4R^2)x_n^2+8R^2x_{n-1}x_n-4R^2x_{n-1}^2=0,$$

solving it we obtain the formula

$$x_n=\frac{2Rx_{n-1}}{2R+x_{n-1}}. \tag{33}$$

Remark 5.2. *Equality (33) specifies a recurrent formula for finding the abscissa of the center of the circle ζ_n through the abscissa of the center of the previous circle ζ_{n-1}, $n\in\mathbb{N}$.*

Theorem 5.1. *The abscissas of the centers of the system $\{\zeta_n\}$, $n\in\mathbb{N}$ are defined by the following sequence:*

$$x_n=\frac{2Rx_0}{2R+x_0n}. \tag{34}$$

Proof. We apply the method of mathematical induction. It is necessary to prove the formula (34) knowing the recurrent formula (33).
Basis. From (33) and (34) we can find x_1. These expressions are equal.
Inductive step. We need to prove the formula:

$$x_{k+1} = \frac{2Rx_0}{2R + x_0(k+1)}.$$

We have:

$$x_{k+1} = \frac{2Rx_k}{2R + x_k} = \frac{2R \cdot \frac{2Rx_0}{2R+x_0k}}{2R + \frac{2Rx_0}{2R+x_0k}} = \frac{4R^2x_0}{4R^2 + 2Rx_0(k+1)} = $$
$$= \frac{2Rx_0}{2R + x_0(k+1)}.$$

Theorem 5.2. *The radii and the ordinates of the centers of the system* $\{\zeta_n\}$, $n \in \mathbb{N}$ *are defined by the following sequences:*

$$r_n = y_n = \frac{Rx_0^2}{(2R + nx_0)^2}. \tag{35}$$

Proof. For proving this assertion, it is sufficient to substitute formula (34) into the system (32) and use remark 5.1.

Theorem 5.3. *The coordinates of the centers and the radii of the system* $\{\zeta_n\}$, $n \in \mathbb{N}$ *are defined by the following sequences:*

$$x_n = \frac{2R\sqrt{r_0}}{\sqrt{R} + \sqrt{r_0}n}, \tag{36}$$

$$r_n = y_n = \frac{Rr_0}{\left(\sqrt{R} + \sqrt{r_0}n\right)^2}. \tag{37}$$

Proof. To prove this assertion, it is sufficient to substitute formula $x_0 = 2\sqrt{Rr_0}$ (see equality (30)) into the equalities (34) and (35).

Corrolary 5.1. *The characteristics of the chain of tangent circles* $\{\zeta_n\}$, $n \in \mathbb{N}_0$ *are given by the following formula:*

$$\frac{x_n}{y_n} = \frac{x_n}{r_n} = 2 \cdot \frac{\sqrt{R}}{\sqrt{r_0}} + 2n = \frac{x_0}{r_0} + 2n.$$

Remark 5.3. *Formulas (36) and (37) can be obtained from formulas (25) – (27) by passing to the limit as* $r \to \infty$ *and rotating of the coordinate system. In this case one of three tangent circles degenerates into the straight line.*

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