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SYNTHETIC FOUNDATIONS OF CEVIAN GEOMETRY, IV: THE TCC-PERSPECTOR THEOREM

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Abstract. In this paper we give a completely synthetic proof of the TCC-perspector theorem, that the isogonal conjugate $\gamma(H)$ of the generalized orthocenter H (defined in Part III of this series of papers), with respect to a triangle ABC and a point P, is the perspector of the tangential triangle of ABC and the circumcevian triangle (both with respect to the circumcircle) of the isogonal conjugate $\gamma(Q)$, where Q is the complement of the isotomic conjugate P' of the point P.

1. Introduction.

In the first three parts of this series of papers ([12], [14], [15]), we have studied generalizations of some classical configurations for an ordinary triangle ABC and a variable point P in the extended Euclidean plane, using a mixture of affine and projective methods. As in previous papers, we assume P does not lie on the sides of ABC or its anticomplementary triangle $K^{-1}(ABC)$, where K is the complement map for ABC. Letting ι denote the isotomic map for ABC, we defined $P' = \iota(P)$ and $Q = K(P') = K \circ \iota(P)$, the latter point being the isotomcomplement of P and a generalization of the incenter. We also studied what we called the generalized orthocenter H and generalized circumcenter O associated to ABC and P. The point H is defined as the intersection of the lines through A, B, C which are parallel, respectively, to QD, QE, QF, where DEF is the cevian triangle of P for ABC (the diagonal triangle of the quadrangle $ABCP$, with $D = AP \cdot BC, E = BP \cdot CA, F = CP \cdot AB$). The point O is defined similarly, as the intersection of the lines through the midpoints D_0, E_0, F_0 of the sides BC, CA, AB, which are parallel to QD, QE, QF. It is easy to see that $O = K(H)$. We proved various relationships between the circumconic $\tilde{C}_O$ of ABC whose center is O, the nine-point conics $\mathcal{N}_H$ and $\mathcal{N}_{P'}$ of the quadrangles $ABCH$ and $ABCP'$ with

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respect to the line at infinity, and the inconic $\mathcal{I}$ which is tangent to the sides of ABC at the points D, E, F. In particular, if T_P denotes the affine map taking ABC to DEF, and $T_{P'}$ denotes the affine map taking ABC to $D_3E_3F_3$, the cevian triangle for P', many of these relationships can be expressed in terms of T_P and $T_{P'}$, in combination with the complement map. For example, the circumconic $\tilde{\mathcal{C}}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$ is the nine-point conic for the quadrangle $Q_aQ_bQ_cQ$, where $Q_aQ_bQ_c = T_{P'}^{-1}(ABC)$ is the anticevian triangle of the point Q with respect to ABC, and its center is given by $O = T_{P'}^{-1} \circ K(Q)$. To take another example, we also showed that the affine map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety or translation taking the circumconic $\tilde{\mathcal{C}}_O$ to the inconic $\mathcal{I}$. Its center S is a generalization of the classical insimilicenter.

In this part of our series on cevian geometry we define the generalized isogonal map γ_P for an ordinary triangle ABC and a point P not on the sides of ABC or $K^{-1}(ABC)$. All the relationships holding between the classical orthocenter and circumcenter also hold for the generalized notions, but with γ_P replacing the usual isogonal map γ for ABC. Thus, $\gamma_P(O) = H$ and $\gamma_P(l_\infty) = \tilde{\mathcal{C}}_O$, i.e. the P-isogonal image of the line at infinity is the circumconic $\tilde{\mathcal{C}}_O$. These relationships reduce to the classicial ones when $P = Ge$ is the Gergonne point of ABC.

We also define the corresponding notions of pedal triangle and pedal conic (with respect to P) for a point R_1 and its image $\gamma_P(R_1) = R_2$, and show that γ_P is a reciprocal conjugation, in the sense of [6]. It turns out that the pole of this reciprocal conjugation, defined by Dean and van Lamoen [6], is the isotomcomplement of our generalized orthocenter H. In addition, our development ties together and gives a synthetic context for the papers [5], [6], [7], and [8].

The main result of this paper is the TCC-Perspector Theorem in Section 4, which says that the (classical) isogonal conjugate $\gamma(H)$ is the perspector of the tangential triangle (T) of ABC and the circumcevian triangle (CC) of the isogonal conjugate $\gamma(Q)$ of Q, both taken with respect to the circumcircle of ABC. This relates the points H and Q, which are the generalizations of the orthocenter and incenter, respectively, to the tangential and circumcevian triangles in the classical sense. All but a few of our results are aimed at proving this theorem.

Along the way we also prove various other results, such as the relation $\gamma_P(P) = S$ for the center of the map M mentioned above (see [15], Theorem 3.4), and the relations $\gamma_P = \delta_P \circ \iota \circ \delta_P = \delta_H$ between γ_P and the map δ_P defined in Part I ([12], Corollary 3.6). We also prove a natural generalization of Simson's Theorem (Theorem 3.7) in this context. The proof of this theorem, as well as the proofs for the properties of the pedal triangles and pedal conics, are heavily projective, and give alternate approaches to the classical theorems that they generalize. Finally, we discuss the point $\gamma_P(G)$, which is a generalization of the symmedian point, in Theorem 4.6.

As in our previous papers, we refer to [12], [14], and [15] as I, II, and III, respectively. The generalized orthocenter and circumcenter were first defined in [13], and that paper also contains proofs of the affine formulas for

H and O. All of our proofs are synthetic, except for the proof of Proposition 2.9, in which we make very limited use of barycentric coordinates to relate γ_P to the discussion in [6]. We will give a synthetic proof of this connection elsewhere. We do, however, give a synthetic proof of Corollary 2.10, using the relationship proved in Theorem 2.11. We do not make use of the results of 2.9 and 2.10 anywhere else in the paper.

2. The generalized isogonal map.

Given a point P with cevian triangle DEF inscribed in triangle ABC, we consider the affine reflections

$$h_a \text{ about the line } AQ \text{ in the direction } a = EF,$$

$$h_b \text{ about the line } BQ \text{ in the direction } b = DF,$$

$$h_c \text{ about the line } CQ \text{ in the direction } c = DE.$$

Thus, $h_a(X) = X'$, where XX' is parallel to EF and $XX' \cdot AQ$ is the midpoint of XX'. In this situation the lines AX and AX' are harmonic conjugates with respect to AQ and the line l_a through A which is parallel to EF. From I, Theorem 3.9 and I, Corollary 3.11(a), the line $l_a = T_{P'}^{-1}(BC)$.

Definition 2.1. *We define the* **generalized isogonal map** γ_P *as follows: for a given* $X \neq A, B, C$, *let* $\gamma_P(X)$ *be the intersection of the lines* $Ah_a(X)$, $Bh_b(X)$, *and* $Ch_c(X)$. *(See Figure 1.)*

If G is the centroid, then $\gamma_G = \iota$ is the isotomic map, while if $P = Ge$ is the Gergonne point, Q is the incenter, and γ_P is the isogonal map. This definition depends on the following theorem.

Theorem 2.2. *For any* $R \neq A, B, C$, *the lines* $Ah_a(R), Bh_b(R), Ch_c(R)$ *are concurrent.*

Proof. We first state Ceva's theorem in terms of cross-ratios. Let $AR \cdot BC = R_1, BR \cdot AC = R_2, CR \cdot AB = R_3$. We have the cross-ratios

$$A(BC, RQ) = (BC, R_1D_2) = \frac{BR_1}{R_1C} \div \frac{BD_2}{D_2C},$$

$$B(CA, RQ) = (CA, R_2E_2) = \frac{CR_2}{R_2A} \div \frac{CE_2}{E_2A},$$

$$C(AB, RQ) = (AB, R_3F_2) = \frac{AR_3}{R_3B} \div \frac{AF_2}{F_2B}.$$

The condition of Ceva's theorem is therefore equivalent to

$$A(BC, RQ) \cdot B(CA, RQ) \cdot C(AB, RQ) = 1.$$

On the other hand, if $R_a = h_a(R), R_b = h_b(R), R_c = h_c(R)$, and $S_1 = AR \cdot EF, S_a = AR_a \cdot EF$, then

$$A(BC, R_aQ) = (FE, S_aA_0) = \frac{FS_a}{S_aE} = \frac{h_a(FS_a)}{h_a(S_aE)}$$
$$= \frac{S_1E}{FS_1} = \frac{1}{(FE, S_1A_0)} = \frac{1}{A(BC, RQ)},$$

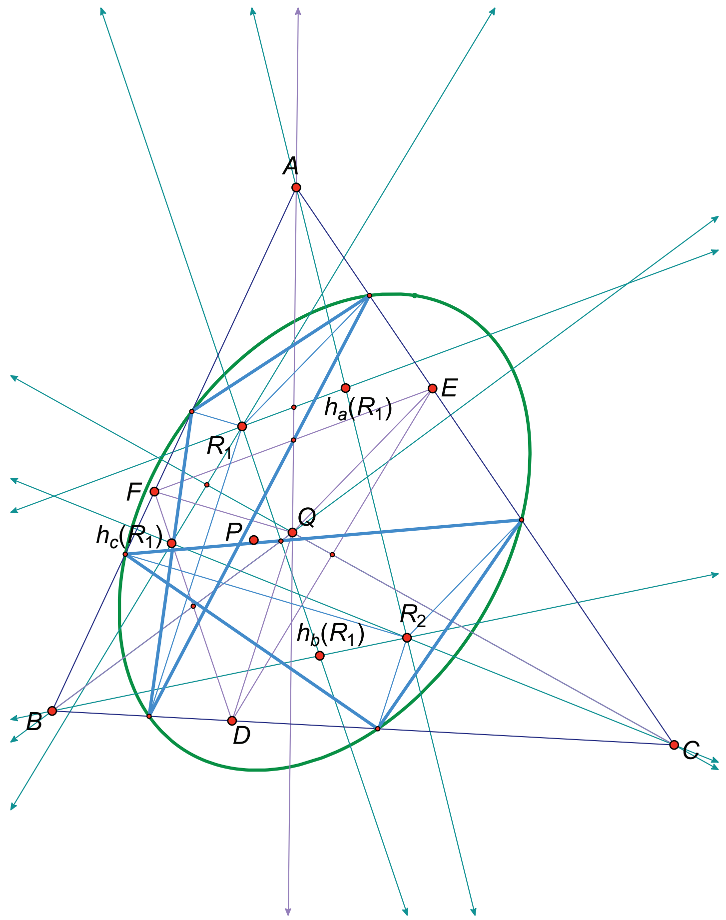

FIGURE 1. Pedal triangles and pedal conic for R_1 and $R_2 = \gamma_P(R_1)$

and similarly

$$B(CA, R_bQ) = \frac{1}{B(CA, RQ)} \quad \text{and} \quad C(AB, R_cQ) = \frac{1}{C(AB, RQ)}.$$

It follows that $1 = A(BC, R_aQ) \cdot B(CA, R_bQ) \cdot C(AB, R_cQ)$, which implies that the lines AR_a, BR_b, CR_c are concurrent. □

It is clear that the map γ_P fixes the point Q. It is also not difficult to verify that γ_P fixes the vertices of the anticevian triangle $Q_aQ_bQ_c = T_{P'}^{-1}(ABC)$ of Q with respect to ABC. This is because the sides of the anticevian triangle of Q are parallel to the sides of triangle DEF (see I, Theorem 3.9), so the side through B, for instance, is fixed by the map h_b. Since the side through B intersects CQ at the vertex Q_c, and CQ is fixed by h_c, it is clear that γ_P fixes $Q_c = Q_cQ_a \cdot CQ$. The same argument applies to the other vertices. It is easy to see that these four points are the only fixed points of γ_P.

As an application we first show the following.

Proposition 2.3. *If S is the center of the map M (III, Theorem 3.4), then $\gamma_P(P) = S$.*

Remark. This verifies the assertion made after III, Theorem 3.4, that S coincides with the point $\gamma_P(P)$.

Proof. We note first that $\mathsf{M}^{-1}(D) = T_{P'}^{-1} \circ K \circ T_P^{-1}(D) = T_{P'}^{-1} \circ K(A) = T_{P'}^{-1}(D_0)$, which is the midpoint of the side Q_bQ_c of the anticevian triangle of Q containing the point A (I, Corollary 3.11 and III, Section 2). If $\mathsf{M}^{-1}(D) \neq A$, this implies that $A\mathsf{M}^{-1}(D) = Q_bQ_c$ is parallel to EF (I, Theorem 3.9). Therefore, $\mathsf{M}(A\mathsf{M}^{-1}(D)) = \mathsf{M}(A)D$ is parallel to EF as well, since M is a homothety or translation (III, Theorem 3.4). Thus $\mathsf{M}(A)D$ lies on $EF \cdot l_\infty$. However, both D and $\mathsf{M}(A)$ lie on the inconic $\mathcal{I}$ (III, Theorem 3.4). Hence, the midpoint of $\mathsf{M}(A)D$ lies on AQ, which is the conjugate diameter for the direction EF (by I, Theorem 2.4). This gives that $h_a(D) = \mathsf{M}(A)$, so $h_a(AP) = h_a(AD) = A\mathsf{M}(A) = AS$. If $\mathsf{M}(B) \neq E$ and $\mathsf{M}(C) \neq F$ we also get $h_b(BP) = BS$ and $h_c(CP) = CS$. Hence, $\gamma_P(P) = S$. (Note that S is never a vertex for the points P that we are considering. If $S = A$, for example, then $A = \mathsf{M}(A)$ lies on the inconic, which is impossible, since the inconic can't be tangent to a side of ABC at a vertex.)

If $\mathsf{M}(A) = D$, but $\mathsf{M}(B) \neq E$ and $\mathsf{M}(C) \neq F$, then we still have $\gamma_P(P) = h_b(BP) \cdot h_c(CP) = BS \cdot CS = S$. On the other hand, if $D = \mathsf{M}(A)$ then $\mathsf{M}(A)$, being symmetrically defined with respect to P and P' ($\mathsf{M} = T_P \circ K^{-1} \circ T_{P'} = T_{P'} \circ K^{-1} \circ T_P$ by III, Proposition 3.12 and Lemma 5.2 in the Appendix below), lies on both inconics $\mathcal{I} = \mathcal{I}_P = \mathsf{M}(\tilde{\mathcal{C}}_O)$ and $\mathcal{I}' = \mathcal{I}_{P'} = \mathsf{M}(\tilde{\mathcal{C}}_{O'})$. Since it also lies on BC, which is tangent to both inconics, $\mathsf{M}(A)$ equals both $D = D_1$ and D_3, so P and P' lie on the median AD_0. If, in addition, $E = \mathsf{M}(B)$, then in the same way $E = E_3$, implying that $P = P' = G$, in which case $Q = G$ and $\gamma_P(P) = \gamma_P(G) = G = S$. □

We also note the following relationship between the map γ_P and the map δ_P from I, Corollary 3.6. Let ι represent the isotomic map for ABC. Recall from I, Theorem 3.8 that the point X is the perspector of triangles ABC and $A_3B_3C_3$, and is the center of the map $\mathcal{S} = T_P \circ T_{P'}$.

Proposition 2.4. *(a) For any point P not on the sides of ABC or its anticomplementary triangle, $\gamma_P = \delta_P \circ \iota \circ \delta_P$.*

(b) The center X of the map $\mathcal{S} = T_P \circ T_{P'}$ satisfies $\gamma_P(X) = Q'$.

(c) The point S satisfies $\delta_P(S) = \iota(Q')$.

Proof. Let P_1 and P_2 be points distinct from the vertices of ABC, let D_1, D_2 be their traces on BC, and let D_1', D_2' be the traces of $\delta_P(P_1)$ and $\delta_P(P_2)$ on BC. Then $h_a(AP_1) = AP_2$ if and only if A_0 is the midpoint of $T_P(D_1'D_2')$, and this holds if and only if D_0 is the midpoint of $D_1'D_2'$. The corresponding relations will hold for all the sides of ABC if and only if $\gamma_P(P_1) = P_2$, and also if and only if $\delta_P(P_2)$ is the isotomic conjugate of $\delta_P(P_1)$. Hence, using the fact that δ_P is an involution, $\gamma_P(P_1) = P_2 = \delta_P \circ \iota \circ \delta_P(P_1)$. This proves (a). Part (b) is immediate from this and I, Corollary 3.6, since $\gamma_P(X) = \delta_P \circ \iota \circ \delta_P(X) = \delta_P \circ \iota(P') = \delta_P(P) = Q'$. For part (c), we have $\delta_P(S) = \iota \circ \delta_P \circ \gamma_P(S) = \iota \circ \delta_P(P) = \iota(Q')$ from Proposition 2.3. □

Proposition 2.4(b) shows that X is the generalized isogonal conjugate of the generalized Mittenpunkt: if P is the Gergonne point, then $Q' = X(9)$ is the Mittenpunkt, and $X = X(57)$ in Kimberling's *Encyclopedia* [10]. This also provides a new proof of the fact that the $X(7)$-ceva conjugate of the incenter (see I, Theorem 3.10) is the isogonal conjugate of the Mittenpunkt.

Definition 2.5. *Assume that the point P does not lie on $\iota(l_\infty)$, so that Q is an ordinary point. The vertices of the* **pedal triangle for a point R with respect to** *P are the intersections with the sides BC, AC, AB of lines through R parallel, respectively, to QD, QE, QF. (See Figures 1 and 3.)*

Theorem 2.6. *If the points R_1 and $R_2 = \gamma_P(R_1)$ are ordinary points, different from any of the points in the set $\{Q, Q_a, Q_b, Q_c\}$, then their pedal triangles with respect to P are inscribed in a common conic, called the* **pedal conic** *for R_1 and R_2 with respect to P.*

Proof. (See Figures 1 and 2.) Let D_1, D_2 be the points on BC such that R_1D_1 and R_2D_2 are parallel to QD, and F_1, F_2 be the points on AB such that R_1F_1 and R_2F_2 are parallel to QF. Furthermore, let

$$G = AB \cdot R_1D_1, H = BC \cdot R_1F_1, J = AB \cdot R_2D_2, K = BC \cdot R_2F_2;$$

and set $S = h_b(R_2)$ (on BR_1), $D' = h_b(D_2)$ (on AB), $J' = h_b(J) = BC \cdot SD'$.

Case 1. The points G, H, J, K are ordinary. Then $SD' \parallel h_b(QD) = QF$, since $BQ \cdot DF = B_0$ is the midpoint of DF (I, Theorem 2.4). It follows that $BR_1F_1 \sim BSD'$ and $BR_1H \sim BSJ'$, and therefore

$$\text{(1)} \qquad \frac{R_1F_1}{R_1H} = \frac{R_1F_1/BR_1}{R_1H/BR_1} = \frac{SD'/BS}{SJ'/BS} = \frac{SD'}{SJ'} = \frac{h_b(R_2D_2)}{h_b(R_2J)} = \frac{R_2D_2}{R_2J}.$$

We also have that $R_1D_1H \sim R_2D_2K$ so that $(R_1D_1) \cdot (R_2K) = (R_2D_2) \cdot (R_1H)$. Putting this together with (1) shows that $(R_1D_1) \cdot (R_2K) = (R_1F_1) \cdot (R_2J)$, and thus

$$\frac{R_1D_1}{R_1F_1} = \frac{R_2J}{R_2K}.$$

Since $R_1D_1 \parallel R_2J$ and $R_1F_1 \parallel R_2K$, we have that $R_1D_1F_1 \sim R_2JK$ by the *SAS* criterion for similarity. This implies that $D_1F_1 \parallel JK$ and therefore that triangles $R_1D_1F_1$ and R_2JK are perspective from the point $D_1J \cdot F_1K$ on the line R_1R_2, by the converse of Desargues' theorem. From this we see that the axis of the projectivity taking F_1JF_2 to D_1KD_2 is the line R_1R_2, since $JD_2 \cdot F_2K = R_2$. Note that $R_2 \neq D_1J \cdot F_1K$, since otherwise $D_1 = D_2, F_1 = F_2$, hence $R_1 = R_2$, contrary to the hypothesis that R_1 is not a fixed point of γ_P. Therefore, the cross-join $F_1D_2 \cdot F_2D_1$ lies on the line R_1R_2.

Case 2. One of the points G, H, J, K is infinite. If G is infinite, for example, then $QD \parallel AB$, and using the map h_b it follows that $QF \parallel BC$, so that all four points are infinite. In this case we have $AB \cdot R_1D_1 = J = AB \cdot R_2D_2, BC \cdot R_1F_1 = K = BC \cdot R_2F_2$. Considering the projectivity taking F_1JF_2 to D_1KD_2, it is easy to see that the axis is the line R_1R_2, since $D_1J \cdot F_1K = R_1$ and $JD_2 \cdot F_2K = R_2$. Hence, the cross-join $F_1D_2 \cdot F_2D_1$ lies on R_1R_2.

By similar reasoning applied to the other vertices, we have that $D_1E_2 \cdot D_2E_1$ and $E_1F_2 \cdot E_2F_1$ also lie on the line R_1R_2, where R_1E_1 and R_2E_2 are parallel to QE with E_1 and E_2 on AC. But then the converse of Pascal's theorem shows

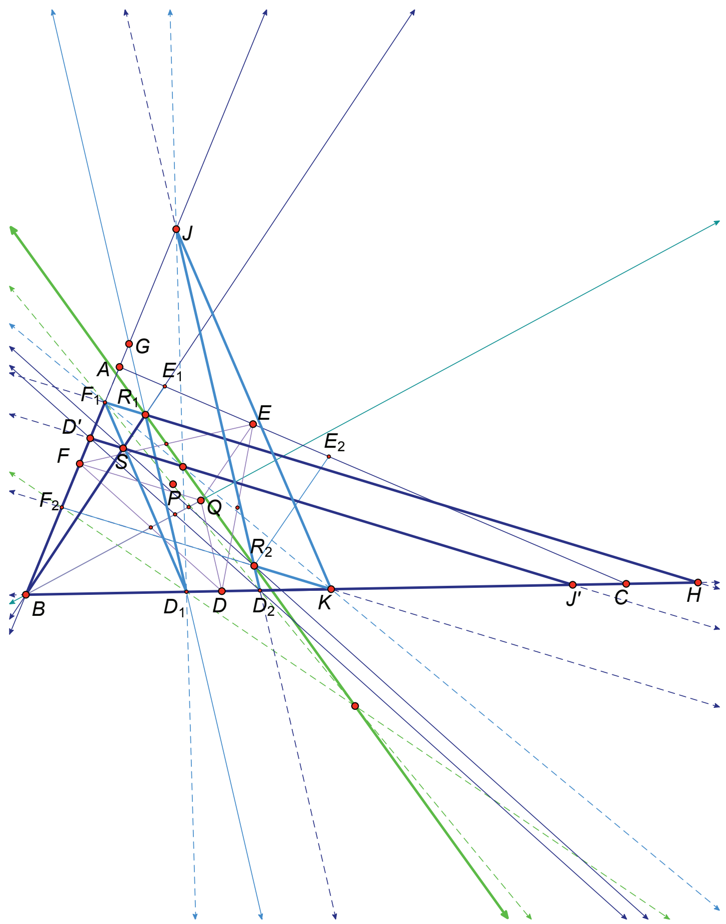

FIGURE 2. Proof of Theorem 2.6

that the vertices of the hexagon $D_1E_2F_1D_2E_1F_2$ lie on a conic. This proves the theorem. □

Corollary 2.7. *The projectivity $\pi : F_1JF_2 \mapsto D_1KD_2$ takes the point $G = AB \cdot R_1D_1$ to the point $H = BC \cdot R_1F_1$. Thus $(F_1F_2, GJ) = (D_1D_2, HK)$.*

Proof. The proof of the theorem shows that the point $D_1J \cdot F_1K$ lies on the line R_1R_2. Reversing the roles of R_1 and R_2 gives that $D_2G \cdot F_2H$ lies on R_1R_2. But this line is the axis of π, and this implies that G on AB must map to H on BC. □

Corollary 2.8. *If the points D_1, E_1, F_1 for the ordinary point R_1 are collinear, then the point $R_2 = \gamma_P(R_1)$ is an infinite point; i.e., the lines $Ah_a(R_1)$, $Bh_b(R_1)$, and $Ch_c(R_1)$ are parallel.*

Proof. If both points R_1 and R_2 are ordinary, then the line D_1E_1 would intersect the pedal conic in three points, impossible unless two of the points D_1, E_1, F_1 are equal. But the latter is also impossible, since these points lie on

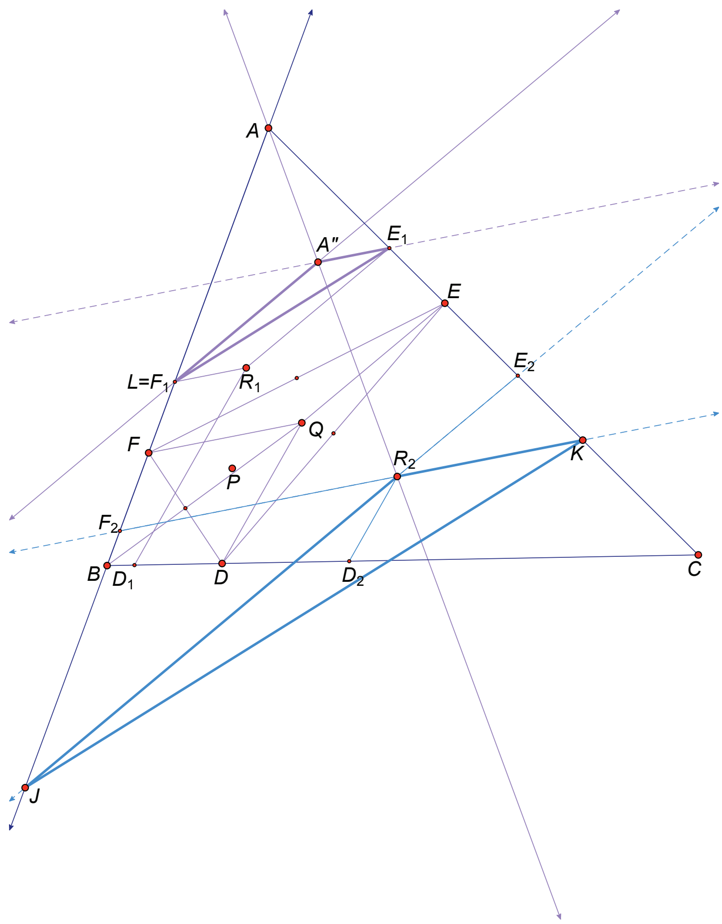

Figure 3. Proof of Proposition 2.9

three distinct lines through R_1. Note that the directions QD, QE, QF are all distinct, because they are conjugate to the directions of the sides BC, AC, AB with respect to the inconic. □

Remark. Theorem 2.6 generalizes the classical result that the pedal triangles of isogonal conjugates are inscribed in a common circle. See Proposition 3.8 below.

In the following two results (Prop. 2.9 and Cor. 2.10), we make very limited use of barycentric coordinates to identify the map γ_P with a construction of Dean and Lamoen [6]. We will give a synthetic proof of this connection in another paper.

Proposition 2.9. *The mapping γ_P is a reciprocal conjugation, in the sense of Dean and van Lamoen* [6], *with pole $P_0 = Q^2$, the point whose barycentric coordinates, with respect to ABC, are the squares of the barycentric coordinates of Q.*

Proof. (See Figure 3.) We use the fact that $R_1D_1F_1 \sim R_2JK$ from the proof of Theorem 2.6. Switching to the vertex A, this similarity becomes $R_1E_1F_1 \sim R_2JK$, where

$$J = AB \cdot R_2E_2, K = AC \cdot R_2F_2.$$

In particular, we have that $E_1F_1 \parallel JK$ as in the proof of Theorem 2.6. Let the line $A''E_1$ parallel to QF intersect the line AR_2 in the point A'', and let the line $A''L$ parallel to QE intersect AB in the point L. Then $A''E_1 \parallel R_2K$ and $A''L \parallel R_2J$. We also have

$$\frac{A''E_1}{A''L} = \frac{R_2K}{R_2J}$$

from the similarities $AA''E_1 \sim AR_2K$ and $AA''L \sim AR_2J$. It follows that $A''LE_1 \sim R_2JK$, so that $LE_1 \parallel JK$ and hence $LE_1 \parallel E_1F_1$. This implies that $LE_1 = E_1F_1$ and $L = F_1$ since L lies on AB. Thus, $R_1E_1A''F_1$ is a parallelogram, with A'' on the line $AR_2 = A\gamma_P(R_1)$. This shows that $R_2 = \gamma_P(R_1)$ can be found using the construction in [6], Prop. 1. By the results of [6], the mapping γ_P is given in barycentric coordinates by

$$\gamma_P(r,s,t) = \left(\frac{l}{r}, \frac{m}{s}, \frac{n}{t}\right), \tag{2}$$

where $lmn \neq 0$. Since $Q = (r,s,t)$ is a fixed point of γ_P we can take $(l,m,n) = (r^2,s^2,t^2) = Q^2$. Thus, Q^2 is the pole of the reciprocal conjugation γ_P. □

Corollary 2.10. *The point $Q^2 = \gamma_P(G)$ is the isotomcomplement of the generalized orthocenter H of the point P with respect to ABC.*

Proof. In the construction of the proposition, the lines QD, QE, QF are parallel to the cevians of the generalized orthocenter $H = (f,g,h)$ for P. Dean and van Lamoen state that in this case the pole of the corresponding reciprocal conjugation is the point $P_0 = (f(g+h), g(f+h), h(f+g))$. But these are just the coordinates of the isotomcomplement of H. This proves the corollary. □

We will now give a synthetic proof of the relationship of the last corollary, in the form $\gamma_P(G) = K \circ \iota(H)$. This proof uses the relationship $\gamma_P(O) = H$, which we will prove in Proposition 3.9 below. (Also see Theorem 4.6.)

Theorem 2.11. *If the point H is not a vertex of ABC and P does not lie on $\iota(l_\infty)$, then $\gamma_P = \delta_H$.*

Proof. First note that if H is on one of the sides of ABC, then it has to be a vertex. If H is on BC, say, but is not B or C, then $BA = h_b(BC) = h_b(BH) = BO$, and similarly $CA = h_c(CB) = h_c(CH) = CO$, so $O = A$ and $H = K^{-1}(A)$, which is not on BC; contradiction. Also, H cannot lie on a side of the anticomplementary triangle $K^{-1}(ABC)$. If it did, then $K(H) = O$ would lie on one of the sides of ABC, and since the ordinary point O is the center of the circumconic $\tilde{C}_O$, this would force O to be a midpoint of a side, implying again that $H = K^{-1}(O)$ is a vertex.

From I, Corollary 3.6 we have that $\delta_P(P) = Q' = K(P)$, for points P that are not on the sides of ABC or $K^{-1}(ABC)$. Hence, $\delta_H(H) = K(H) = O$. We will also use the fact that $\gamma_P(H) = O$ from Proposition 3.9. This shows that the maps γ_P and δ_H share the point pair (H,O).

Now consider the involution π_A on the line BC given by $\pi_A(D^*) = AT_H(D^*) \cdot BC$, where T_H maps ABC to the cevian triangle of H. This is one of three involutions which are used to define the map δ_H. (See the discussions following Lemma 3.4 and Theorem 3.5 in Part I.) It is clear that π_A switches B and C, and also the pair $D_H = AH \cdot BC$ and $D_O = AO \cdot BC$, since $\delta_H(H) = O$. Since H is not on AB or AC, these are two different pairs of points, so the involution π_A is uniquely determined.

We claim that this is the same involution which is induced by γ_P on the line BC. This follows from the fact that the involution π'_A given by $\pi'_A(D^*) = Ah_a(D^*) \cdot BC$, where h_a is affine reflection in AQ in the direction of EF, also switches the pairs (B, C) and (D_H, D_O), since $\gamma_P(H) = O$. Hence the two involutions must be the same: $\pi_A = \pi'_A$. Similar statements are true for the other sides: $\pi_B = \pi'_B$ and $\pi_C = \pi'_C$. But $\delta_H(X)$ is the intersection of $A\pi_A(AX \cdot BC)$, $B\pi_B(BX \cdot AC)$, and $C\pi_C(CX \cdot AB)$. Since the same is true for the map $\gamma_P(X)$, with π replaced by $\pi' = \pi$, we have $\gamma_P(X) = \delta_H(X)$. □

Corollary 2.12. *If P does not lie on $\iota(l_\infty)$ or any of the conics $\overline{C}_A, \overline{C}_B, \overline{C}_C$ of* [16] *(so that $H \neq A, B, C$), then $\gamma_P(G) = K \circ \iota(H)$ is the isotomcomplement of the generalized orthocenter H.*

Proof. We have $\gamma_P(G) = \delta_H(G) = K \circ \iota(H)$, by I, Corollary 3.6, with $P = H$. □

3. Generalized isogonal relationships.

Proposition 3.1. *Let ψ be an involution on the line at infinity, which is induced by the polarity for a central conic, and let A, B, C be non-collinear points. Then there is a unique conic through A, B, C whose polarity induces ψ.*

Remark. We only need the proposition for central conics, since pedal conics are only defined when P is not on the Steiner circumellipse $\iota(l_\infty)$ and Q is ordinary.

Proof. Let ψ be the involution on l_∞ induced by the polarity for a conic $\mathcal{C}$. If L and M are the midpoints of AB and AC, and the points at infinity on AB and AC are R and S, then the lines $L\psi(R)$ and $M\psi(S)$ are distinct diameters of $\mathcal{C}$, since otherwise $LM = \psi(R)\psi(S)$ would the line at infinity. Thus, their intersection is the center of $\mathcal{C}$ (an ordinary point because $\psi(R) \neq \psi(S)$). If the center is not collinear with any two of the vertices of ABC, the conic is determined by the vertices and two of their reflections through the center, which makes 5 points. If the center is collinear with two of the points, say B and C, then it has to be the midpoint of BC. If the point at infinity on BC is T, the tangent lines at B and C are $B\psi(T)$ and $C\psi(T)$. Then $\mathcal{C}$ is determined by the 3 vertices and the two tangent lines. See [3]. □

Proposition 3.2. *If the point P does not lie on $\iota(l_\infty)$, the pedal conic $\mathcal{P}$ of an ordinary point R_1 and its ordinary P-isogonal conjugate R_2 induces the same involution ψ on l_∞ as the inconic $\mathcal{I}$. The center of $\mathcal{P}$ is the midpoint M of the segment R_1R_2.*

Proof. (See Figure 4.) Let R_2 be any point on the line $l = Ah_a(R_1)$ and M the midpoint of segment R_1R_2. Let E_1, F_1 and E_2, F_2 be the intersections with AC and AB of the lines through R_1 and R_2 parallel to QE, QF. We show that the conic $\mathcal{C}_M$ on the points E_1, F_1, F_2 with center M induces ψ. If R_1 is fixed, then as R_2 varies on the line l, the point M varies on a line l' parallel to l. The line l' lies on the midpoint of E_1F_1, since M is equal to this midpoint when R_2 is the point A'' in the proof of Proposition 2.9; for $R_1E_1A''F_1$ is a parallelogram with diagonals $R_1A'' = R_1R_2$ and E_1F_1 (see Figure 3). Hence, the direction of the line l' is conjugate to the direction of the line E_1F_1 for any of the conics $\mathcal{C}_M$, since E_1F_1 is a chord on each of these conics. Furthermore, if F_3 is the midpoint of segment F_1F_2, then it is easy to see that $MF_3 \parallel QF$. Hence, the directions of AB (= F_1F_2) and QF are conjugate for any conic $\mathcal{C}_M$. (If R_1 lies on AQ and $R_2 = R_1$, then if $\mathcal{C}_M$ is the conic with center $M = R_1$ which is tangent to AB at $F_1 = F_2$ and tangent to AC at $E_1 = E_2$, the same conclusions hold.) Since the involutions induced on l_∞ by these conics share two pairs of conjugate points, they must all be the same involution ψ_1.

We now show this involution coincides with ψ. To do this, consider the point $R_2 = h_a(R_1)$ and its corresponding midpoint M. With this choice of R_2, M lies on the line AQ, by definition of the affine reflection h_a. Now the map h_a interchanges the directions of QE and QF, since AQ lies on the midpoint of EF (I, Theorem 2.4), so it maps triangle $E_1R_1F_1$ to $F_2R_2E_2$. This implies that $F_2 = h_a(E_1)$ and $E_2 = h_a(F_1)$. Thus $E_1F_2 \parallel EF$, and the midpoint of E_1F_2 lies on $AQ = AM$. Hence, EF and AQ represent conjugate directions for both involutions ψ_1 and ψ (Q is the center of $\mathcal{I}$, lying on E and F). But we have already shown that ψ_1 has the conjugate pair of directions AB and QF, and this pair is shared by ψ, since AB is tangent to the conic $\mathcal{I}$ at F. This proves that $\psi_1 = \psi$.

Once again, let R_2 be any point on the line $Ah_a(R_1)$. If E_3 is the midpoint of E_1E_2, then $ME_3 \parallel QE$. It follows from what we have proved that ME_3 and E_1E_2 (= AC) are conjugate directions for the conic $\mathcal{C}_M$, which implies that E_2 must also lie on $\mathcal{C}_M$. This shows that there is a unique conic on the points E_1, F_1, E_2, F_2 which induces the involution ψ and whose center is the midpoint M of R_1R_2. Now let $R_2 = \gamma_P(R_1)$. By the above argument, there are conics $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$, all with center M, and all inducing the same involution ψ, lying on the point sets $\{E_1, F_1, E_2, F_2\}$, $\{D_1, F_1, D_2, F_2\}$, and $\{D_1, E_1, D_2, E_2\}$, respectively. However, any two of these conics share four points and induce the same involution on l_∞. For example, $\mathcal{C}_1 \cap \mathcal{C}_2$ contains $\{F_1, F_2, F_1', F_2'\}$, where F_i' is the reflection of F_i through M. By Proposition 3.1, these conics must all be the same conic, the pedal conic of R_1 and R_2. This completes the proof. □

Corollary 3.3. *The maps on the line l_∞ induced by h_a, h_b, h_c commute with the involution ψ.*

Proof. For example, $h_a\psi h_a^{-1} = \psi$ follows from the fact that when $R_2 = h_a(R_1)$, the map h_a takes the point set $\{E_1, F_1, E_2, F_2\}$ to itself and fixes M. Therefore, $h_a(\mathcal{C}_M) = \mathcal{C}_M$. Alternatively, the center $EF \cdot l_\infty$ of the homology h_a is the pole of its axis AQ, with respect to the conic $\mathcal{I}$, and therefore h_a maps $\mathcal{I}$ to itself. See [3], p. 76, Exer. 4. □

Lemma 3.4. *If l is any line not lying on a vertex of triangle ABC, then $\gamma_P(l)$ is a circumconic for the triangle.*

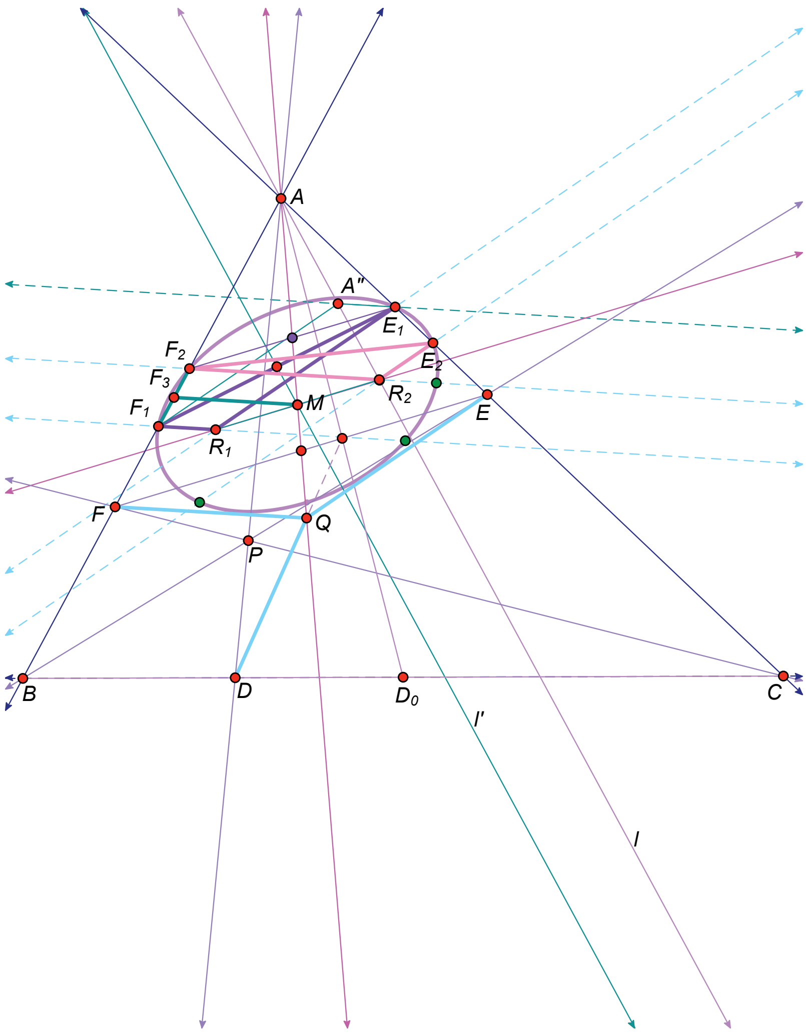

FIGURE 4. Proof of Proposition 3.2

Remark. In particular, this lemma also holds for the isotomic map $\iota = \gamma_G$.

Proof. The image $\gamma_P(l)$ is the locus of points $h_a(AR) \cdot h_b(BR)$, for points R on l. Since h_a is a projective collineation, we have that

$$h_a(AR) \;\overline{\wedge}\; AR \;\overset{l}{\overline{\wedge}}\; BR \;\overline{\wedge}\; h_b(BR).$$

($\overline{\wedge}$ denotes a projectivity.) Thus, the locus $h_a(AR) \cdot h_b(BR)$ is either a line or a conic. However, if R lies on a side of ABC, then $\gamma_P(R)$ is the opposite vertex. Thus, $\gamma_P(l)$ lies on the vertices of ABC and is therefore a conic. □

Proposition 3.5. *For any point P not on the sides of ABC or $K^{-1}(ABC)$, we have that $\gamma_P(l_\infty) = \tilde{\mathcal{C}}_O$.*

Proof. If $P = G$, then $P' = Q = G$, $\gamma_P = \iota$ is the isotomic map, and $O = T_G^{-1}(K(G)) = K^{-1}(K(G)) = G$, so the conic $\tilde{\mathcal{C}}_O = \iota(l_\infty)$ is the Steiner circumellipse. Now assume $P \neq G$. From the previous lemma we know that $\gamma_P(l_\infty)$ is a circumconic of ABC. We will show that $\gamma_P(l_\infty)$ lies on the midpoints of the sides of the anticevian triangle of Q with respect to ABC, which is $Q_aQ_bQ_c = T_{P'}^{-1}(ABC)$. This will show that $\gamma_P(l_\infty)$ has six points in common with $\tilde{\mathcal{C}}_O$, by III, Theorem 2.4. It suffices to show that $\gamma_P(EF \cdot l_\infty)$ is the midpoint of Q_bQ_c. Let $Y = EF \cdot l_\infty$. We know that the map h_a fixes the line $AY = Q_bQ_c$, since this line lies on A and is parallel to EF (I, Theorem 3.9). Furthermore, h_b takes the line BY to its harmonic conjugate with respect to BQ and $B(DF \cdot l_\infty)$. The section of this harmonic set of lines by $AY = Q_bQ_c$ is a harmonic set of points. Now BQ lies on Q_b, while $B(DF \cdot l_\infty)$ lies on Q_c. Since $BY \parallel Q_bQ_c$, it follows that $h_b(BY)$ intersects Q_bQ_c at the midpoint of Q_bQ_c, which coincides with the intersection $AY \cdot h_b(BY) = Ah_a(Y) \cdot Bh_b(Y) = \gamma_P(Y)$. A similar argument shows that $\gamma_P(DF \cdot l_\infty)$ is the midpoint of Q_aQ_c and $\gamma_P(DE \cdot l_\infty)$ is the midpoint of Q_aQ_b. To finish the proof we just have to check that at least two of these midpoints do not coincide with the vertices A, B, or C. If A is the midpoint of Q_bQ_c, then $T_{P'}(A) = D_3$ is the midpoint of $T_{P'}(Q_bQ_c) = BC$ (I, Corollary 3.11), which implies that the point P' lies on the median of ABC through A. If two midpoints coincide with vertices, then $P = P' = G$ is the centroid. □

Combined with Corollary 2.8, Proposition 3.5 shows that any point R_1 for which

$$R_1D_1 \parallel QD, \ R_1E_1 \parallel QE, \ R_1F_1 \parallel QF,$$

where D_1, E_1, F_1 are collinear points lying on the respective sides BC, AC, AB, must lie on the circumconic $\tilde{\mathcal{C}}_O$, since $R_2 = \gamma_P(R_1)$ lies on l_∞. This is just the converse of the generalized Simson theorem. See [1], p. 140. To prove Simson's theorem in this situation we first prove the following property. (Cf. Prop. 288 in [1].)

Proposition 3.6. *Assume P does not lie on $\iota(l_\infty)$. Given the point R_1 on the circumconic $\tilde{\mathcal{C}}_O$, let D_1, E_1, F_1 denote the 'feet' of the parallels dropped to BC, AC, AB from R_1 in the directions QD, QE, QF. Let A', B', C' be the second intersections of the lines R_1D_1, R_1E_1, R_1F_1 with $\tilde{\mathcal{C}}_O$. Then the lines AA', BB', CC' are parallel; i.e., triangles ABC and $A'B'C'$ are perspective from a point on l_∞.*

Proof. (See Figure 5.) Let I_1 and I_2 denote the intersections of the lines QD, QE with l_∞, and consider the involution on the conic $\tilde{\mathcal{C}}_O$ given by intersecting it with secant lines through I_1. Then R_1 maps to the second intersection A' of the conic with R_1D_1, and A maps to the second intersection of the conic with AH, where H is the generalized orthocenter for P, which is S_1, in the notation of III, Proposition 3.2, since $AH \parallel QD$. Furthermore, B maps to the reflection $\mathsf{R}_O(C)$ of C in O, which we denote by C^*. This is because the direction of BC^* is conjugate (with respect to $\tilde{\mathcal{C}}_O$) to the direction of BC, since CC^* is a diameter, and therefore $BC^* \parallel QD$. In the same way, C maps to $B^* = \mathsf{R}_O(B)$. Thus we have the involution

$$R_1ABC \ \overline{\wedge} \ A'S_1C^*B^*.$$

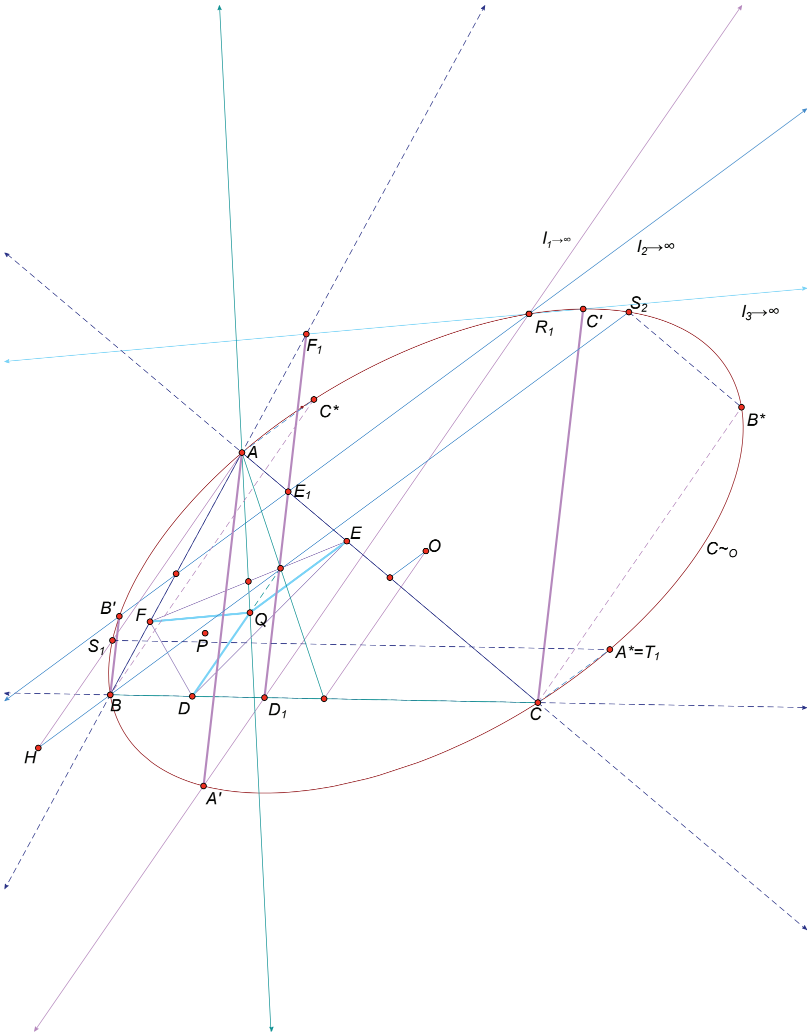

FIGURE 5. Proof of Proposition 3.6 and Theorem 3.7

There is also an involution on $\tilde{C}_O$ given by intersecting it with secant lines through I_2, for which we have

$$R_1ABC \;\overline{\wedge}\; B'C^*S_2A^*,$$

where S_2 is the second intersection of BH with the conic and $A^* = \mathsf{R}_O(A)$. This gives the projectivity

$$B'C^*S_2A^* \;\overline{\wedge}\; A'S_1C^*B^*.$$

If the axis of this projectivity is l, then $S_1A^* \cdot C^*B^*$ and $S_2B^* \cdot C^*A^*$ lie on l. But both of these points lie on l_∞, by III, Proposition 3.2 and III, Corollary 3.7 (where A^* is denoted by T_1), since $C^*B^* \parallel CB \parallel S_1A^*$ and $C^*A^* \parallel CA \parallel S_2B^*$. Hence, $l = l_\infty$. It follows from the projectivity that $A^*A' \parallel B^*B'$. In the same way, using the product of two involutions on $\tilde{C}_O$ defined by taking secant lines through the ideal points I_1 and I_3 (lying on QF), we see that $A^*A' \parallel C^*C'$. Now the lines AA', BB', CC' lie in conjugate directions to the parallel lines A^*A', B^*B', C^*C', and are therefore parallel to each other. This proves the proposition. □

Theorem 3.7. *(Simson's Theorem) Assume P does not lie on $\iota(l_\infty)$. If R_1 lies on the circumconic $\tilde{C}_O$, then the 'feet' D_1, E_1, F_1 of the parallels dropped to BC, AC, AB from R_1 in the directions QD, QE, QF, are collinear.*

Proof. From Proposition 3.6 we have that A, B, C, A', B', C', R_1 are points on a conic with ABC perspective to $A'B'C'$ from some point O_1. Looking at the hexagon $A'R_1B'BCA$, Pascal's theorem implies that $R_1A' \cdot BC = D_1, R_1B' \cdot CA = E_1$, and $AA' \cdot BB' = O_1$ are collinear. Similarly, the hexagon $A'R_1C'CBA$ gives that $R_1A' \cdot BC = D_1, R_1C' \cdot AB = F_1$, and $AA' \cdot CC' = O_1$ are collinear. Hence, the points D_1, E_1, F_1 lie on the line D_1O_1, which is parallel to AA'. □

Proposition 3.8. *If the pedal triangles of two ordinary points R_1 and R_2 are inscribed in a common conic $\mathcal{P}$, inducing the same involution on l_∞ as $\tilde{C}_O$, then $R_2 = \gamma_P(R_1)$, and $\mathcal{P}$ is the pedal conic for R_1 and R_2.*

Proof. First note that, with the notation from Theorem 2.6 (see Figure 2), the points D_1, E_1, F_1 are not collinear, since they lie on the conic $\mathcal{P}$. Theorem 3.7 shows that R_1 does not lie on the conic $\tilde{C}_O$, and therefore the point $\gamma_P(R_1)$ is ordinary, by Proposition 3.5. Now Theorem 2.6 shows that the pedal triangles of R_1 and $\gamma_P(R_1)$ are inscribed in their common pedal conic $\mathcal{P}'$, which induces the same involution on l_∞ as the given conic $\mathcal{P}$ (Proposition 3.2). By Proposition 3.1 , $\mathcal{P} = \mathcal{P}'$, and since the intersections of $\mathcal{P}$ with the sides of ABC determine the pedal triangle of R_2, it follows that $R_2 = \gamma_P(R_1)$. □

Remark. If $\mathcal{P}$ is the unique conic lying on the vertices of the pedal triangle of an ordinary point $R_1 \notin \tilde{C}_O$ inducing the same involution ψ on l_∞ as in Proposition 3.2, then the other intersections of $\mathcal{P}$ with the sides of ABC are precisely the vertices of the pedal triangle for $\gamma_P(R_1)$. This follows in the same way as in the proof just given.

The next proposition shows that the generalized orthocenter H and generalized circumcenter O stand in the same relationship as their classical counterparts, if the ordinary isogonal map γ is replaced by γ_P.

Proposition 3.9. *For any point P not on the sides of ABC or $K^{-1}(ABC)$, we have that $\gamma_P(O) = H$.*

Proof. (See Figure 5.) First assume P does not lie on the Steiner circumellipse $\iota(l_\infty)$. If the point H does not coincide with a vertex, the nine-point conic $\mathcal{N}_H$ for the quadrangle $ABCH$ is a conic on the vertices of the pedal triangles of the ordinary points H and O, and induces the same involution on l_∞ as $\tilde{C}_O$, by III, Proposition 3.6. The result follows from Proposition 3.8. If H coincides with the vertex A, for example, then $O = D_0$ is a point on the opposite side BC, so $\gamma_P(O) = A = H$. If P lies on $\iota(l_\infty)$, then the points O and H both coincide with $Q = P'$ (III, Corollary 2.3), which is fixed by γ_P. □

For the proof of the next result, recall that the point X' is the center of the map $\mathcal{S}' = T_{P'} \circ T_P$.

Proposition 3.10. *If the ordinary point P does not lie on a median of ABC, then $\gamma_P(SQ) = \mathcal{C}_P$, and the tangent to $\mathcal{C}_P$ at Q is SQ. If P does not lie on $\iota(l_\infty)$, then the line $SQ = OQ$. In particular, the pole of the line QQ' with respect to the conic $\mathcal{C}_P$ is S.*

Proof. First assume P does not lie on $\iota(l_\infty)$. By III, Theorem 3.9 we know that the point S lies on OQ, and we claim that $S \neq Q$, so that $OQ = SQ$. If $S = Q$, then since S is the center of the map M and $\mathsf{M}(O) = Q$, where Q and O are ordinary points, we conclude that $O = S = Q$, yielding $T_{P'}(O) = T_{P'}(Q) = P'$ (I, Theorem 3.7). On the other hand, $K(Q) = T_{P'}(O) = P'$ by the affine formula for O, so $K(Q) = P' = K^{-1}(Q)$, giving that $Q = G$ and P lies on a median, which is contrary to hypothesis. Hence, $OQ = SQ$. We next check that the line OQ does not lie on a vertex of ABC. If OQ lies on C, for example, then $T_{P'}(OQ) = K(Q)P'$ lies on the point $T_{P'}(C) = F_3$, i.e. $K(Q)P' = CP'$. But $K(Q)$ is the midpoint of $P'Q$, so Q and also G lies on CP', so that P' lies on the median CG. Since $S = \gamma_P(P)$ (Proposition 2.3) never coincides with a vertex of ABC (P never lies on a side of ABC), Lemma 3.4 shows that $\gamma_P(OQ)$ is a circumconic of ABC lying on the points $\gamma_P(S) = P$ and $\gamma_P(Q) = Q$. Since $P \neq Q$, this shows that $\gamma_P(OQ) = ABCPQ = \mathcal{C}_P$. To show SQ is tangent to $\mathcal{C}_P$, argue as in the proof of [16], Proposition 2.4. If the point L lies on $SQ \cap \mathcal{C}_P$, then L and $\gamma_P(L)$ are in this intersection, so either $L = Q$ or $L = \gamma_P(L)$. But the only fixed points of γ_P are Q and the vertices of the anticevian triangle of Q with respect to ABC. If L were one of these vertices, then $SQ = QL$ would lie on a vertex of ABC, which we showed above to be impossible. Therefore, $L = Q$ is the only point in $SQ \cap \mathcal{C}_P$, showing that SQ is the tangent line at Q.

Now assume P does lie on $\iota(l_\infty)$. Then $Q = P' = O \in l_\infty$ (see III, Corollary 2.3). In this case, the map $\mathcal{S} = T_P T_{P'} = K^{-1}$ (I, Theorem 3.14) has center $X = G$. From I, Corollary 3.11, we know that $X' = T_{P'}(X) = T_{P'}(G) = G_2$ is an ordinary point. Since X' is a fixed point of $\mathcal{S}' = T_{P'} T_P$, we have that

$$\mathsf{M}(X') = T_{P'} K^{-1} T_P(X') = (T_{P'} T_P)^2(X') = X',$$

so $X' = G_2 = S$ is the center of the map M, using the fact that M is a translation or homothety with a unique ordinary fixed point (III, Theorem 3.4). By II, Lemma 2.5, we also know that G is the midpoint of segment $G_1 G_2$, where $G_1 = T_P(G)$, and $GG_1 \parallel PP'$. But $P' = Q$ in this case, so the infinite point Q lies on the line $GG_2 = GS$. Then II, Theorem 4.3 shows that $GG_1 = GS = SQ$ is an asymptote of $\mathcal{C}_P$; in other words, SQ is the tangent to $\mathcal{C}_P$ at Q. Since P is not on a median of ABC, none of the vertices lie on the line $SQ = GS = GQ$, so the same argument as in the first paragraph of the proof shows that $\gamma_P(SQ) = \mathcal{C}_P$. The last assertion of the proposition follows from the fact that the point $S = GV \cdot OQ = GV \cdot O'Q'$ is symmetric with respect to P and P', because it is the center of the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$, which is symmetric in P and P' (by III, Proposition 3.12b and Lemma 5.2 in the Appendix). Hence, SQ' is the tangent to $\mathcal{C}_P = \mathcal{C}_{P'}$ at Q'. □

Theorem 3.11. *Assume that the point P is ordinary and does not lie on a median of ABC.*

1. $X = PQ' \cdot SQ$ $(= PQ' \cdot OQ$, *if P does not lie on* $\iota(l_\infty))$.
2. *The following nine points are always collinear:*
$$X,\ T_P(P'),\ T_P(G),\ Q,\ S,\ O,\ \mathsf{M}(Q),\ T_{P'}^{-1}(G),\ \text{and } T_{P'}^{-1}(Q) = T_P^{-1}(H).$$
3. $T_P(G)$ *is the pole of PQ with respect to the conic C_P, and the polar of P is* $p = PT_P(G)$.
4. $\mathsf{M}(Q)$ *is the pole of $P'Q$ with respect to* C_P.
5. *The tangent to $T_{P'}(C_P)$ at P' is* $P'Q$.
6. *The pole of QQ' with respect to $T_{P'}^{-1}(C_P) = T_P^{-1}(C_P)$ is* G.
7. $\mathsf{M}(QQ') = K^{-1}(PP')$.
8. *The tangent to C_P at H is* $h = HT_P(P')$.

Remark. The point $T_{P'}^{-1}(G)$ in the second statement is the centroid of the anticevian triangle of Q, and the point $G_1 = T_P(G)$ is the centroid of the cevian triangle of P. Also, $\mathsf{M}(Q) = T_{P'}(P')$.

Proof. By applying T_P to $P'GQ$, we see that $T_P(P'), T_P(G), Q$ are collinear. By applying $T_{P'}^{-1}$ to $P'GQ$, we see that $Q, T_{P'}^{-1}(G), T_{P'}^{-1}(Q) = T_P^{-1}(H) = \tilde{H}$ are collinear (see III, equation (3)). Now applying $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ to $Q, T_{P'}^{-1}(G), T_{P'}^{-1}(Q)$, we see that $\mathsf{M}(Q), T_P(G), T_P(P')$ are collinear. But these are collinear with Q, since $T_P(Q) = Q$! It follows that $T_P(G)$ and $T_P(P')$ lie on the line $QM(Q) = SQ$. Applying $\mathsf{M}^{-1} = T_{P'}^{-1} \circ K \circ T_P^{-1}$, which fixes this line (S is the center of M), we see that $T_{P'}^{-1}(G)$ and $T_{P'}^{-1}(Q)$ lie on the same line. Thus, the points

$$T_P(P'), T_P(G), Q, S, O = \mathsf{M}^{-1}(Q), \mathsf{M}(Q), T_{P'}^{-1}(G), \text{ and } T_{P'}^{-1}(Q)$$

are collinear. This implies that $\mathcal{S}' = T_{P'} \circ T_P$ fixes the line $P'Q$ since $T_P(P'Q) = T_P(P')Q = T_{P'}^{-1}(Q)T_{P'}^{-1}(P') = T_{P'}^{-1}(P'Q)$, so the center of $\mathcal{S}'$, namely X', is on $P'Q$. Applying the map η shows that X is on PQ'. Finally, $\mathcal{S}(Q) = T_P \circ T_P'(Q) = T_P(P')$ lies on the line SQ, so X is on SQ as well, proving parts (1.) and (2.).

With respect to the conic C_P, the pole $p \cdot q$ of PQ lies on $q = SQ$, by Proposition 3.10. Also, since V lies on PQ, $p \cdot q$ lies on $v = GV_\infty$ (see II, p. 26). Thus, $p \cdot q$ is the intersection of GV_∞ and SQ. But we already know $T_P(G)$ lies on SQ and $T_P(G)G = G_1G = GV_\infty \parallel PP'$ by II, Lemma 2.5; hence, $T_P(G)$ is this intersection, giving (3.). This implies $PT_P(G) = p$ is the polar of P.

Now, $\mathsf{M}(Q)$ lies on $SQ = q$, so to prove (4.) we just need to show that $\mathsf{M}(Q)$ lies on p', the polar of P'. But $p' = P'T_{P'}(G)$, by (3.) applied to the point P'. Hence, applying $T_{P'}$ to the collinear points Q, G, P', we see that $\mathsf{M}(Q) = T_{P'} \circ K^{-1} \circ T_P(Q) = T_{P'}(P')$ lies on p'.

Part (5.) of the theorem follows from the fact that the tangent to C_P at Q, namely SQ, lies on $T_{P'}^{-1}(Q)$. Thus, the tangent to $T_{P'}(C_P)$ at $T_{P'}(Q) = P'$ lies on Q. Therefore, this tangent is $P'Q$.

Now the tangent to C_P at Q goes through $T_P(P')$ so the tangent to $T_P^{-1}(C_P)$ at $T_P^{-1}(Q) = Q$ goes through P', i.e. equals $P'Q = QG$. Similarly, since

$T_P^{-1}(\mathcal{C}_P) = T_{P'}^{-1}(\mathcal{C}_P)$, the tangent at Q' is $Q'G$, so the pole of QQ' with respect to this conic is G, giving (6.).

For (7.), $P'Q$ goes through G, whose polar with respect to $\mathcal{C}_P$ is VV_∞. Thus, the pole $\mathsf{M}(Q)$ of $P'Q$ lies on $VV_\infty = K^{-1}(PP')$ (see II, Proposition 2.3(e)). Similarly, $\mathsf{M}(Q')$ lies on $K^{-1}(PP')$, hence $\mathsf{M}(QQ') = K^{-1}(PP')$.

Finally, the tangent at $\tilde{H} = T_P^{-1}(H) = T_{P'}^{-1}(Q)$ to $T_P^{-1}(\mathcal{C}_P)$ is $T_{P'}^{-1}(SQ)$, since SQ is tangent to $\mathcal{C}_P$ at Q. The point P' lies on $T_{P'}^{-1}(SQ)$ since $T_{P'}(P') = \mathsf{M}(Q)$ lies on SQ. Therefore, $T_P(P')$ lies on the tangent to $\mathcal{C}_P$ at H. Note that $\tilde{H}$ is the midpoint of the segment joining P' and $K^{-1}(H)$, by III, Lemma 3.8, so $T_P(P') \neq H$. This proves (8.). □

Corollary 3.12. *If the ordinary point P does not lie on a median of ABC or on $\iota(l_\infty)$ and $H = A$, then the tangent to $\mathcal{C}_P$ at A is parallel to the line QD.*

Proof. By the theorem, we know the tangent to $\mathcal{C}_P$ at $H = A$ lies on $T_P(P')$, so we want to show that $HT_P(P') \parallel QD$. Applying the map $\lambda = T_{P'} \circ T_P^{-1}$, this is equivalent to $QT_{P'}(P') \parallel P'D_3 = AP'$, since $\lambda(D) = T_{P'}(A) = D_3$. But $QT_{P'}(P') = Q\mathsf{M}(Q) = QO = QD_0 = K(P'A)$, so these lines are indeed parallel. □

This corollary allows us to show that the result of III, Corollary 2.6 also holds when H is a vertex. In that case the conics $\mathcal{C}_P, \mathcal{C}_{P_a}, \mathcal{C}_{P_b}, \mathcal{C}_{P_c}$ all have the same tangent line at H, so their common intersection is $\{A, B, C\}$. See the discussion leading to III, Theorem 2.5.

4. Circumcevian triangles and the TCC perspector.

In this section we prove our main theorem.

Definition 4.1. *The* **circumcevian triangle** *of a point R with respect to ABC and the circumconic $\tilde{\mathcal{C}}_O$ is the triangle $A'B'C'$, where AR, BR, CR intersect $\tilde{\mathcal{C}}_O$ in the respective second points A', B', C'.*

Proposition 4.2. *Assume that P is an ordinary point, not lying on $\iota(l_\infty)$.*

a) The circumcevian triangle of Q with respect to ABC and $\tilde{\mathcal{C}}_O$ is the triangle

$$A'B'C' = T_{P'}^{-1}(R_1'R_2'R_3'),$$

where R_1', R_2', R_3' are the midpoints of the respective segments AP', BP', CP'.

b) The triangle $A'B'C'$ is perspective to the medial triangle $D_0E_0F_0$ from the point O.

c) The antipodal triangle of $A'B'C'$ on the conic $\tilde{\mathcal{C}}_O$ is the triangle $T_{P'}^{-1}(D_0E_0F_0)$, the medial triangle of the anticevian triangle for Q.

d) $A'B'C'$ is homothetic or congruent to the cevian triangle DEF of P.

Proof. From [13], Corollary 5(b) we know that D_0, R_1', A_0', and $K(Q)$ are collinear (assuming P' is ordinary). Applying the map $T_{P'}^{-1}$ gives that

$$(3) \qquad T_{P'}^{-1}(D_0),\ T_{P'}^{-1}(R_1'),\ T_{P'}^{-1}(A_0') = D_0,\ T_{P'}^{-1}K(Q) = O$$

are collinear. Furthermore, $T_{P'}^{-1}(R_1')$ is the midpoint of the segment $T_{P'}^{-1}(AP') = Q_aQ$, so $T_{P'}^{-1}(R_1')$ lies on the line AQ. Also, R_1' lies on the nine-point conic $\mathcal{N}_{P'}$

of the quadrangle $ABCP'$, so by III, Theorem 2.4, $T_{P'}^{-1}(R_1')$ lies on the conic $\tilde{C}_O$.

Now there are several cases to consider. Suppose that $T_{P'}^{-1}(R_1') \neq A$, or equivalently, $R_1' \neq T_{P'}(A) = D_3$. Then $T_{P'}^{-1}(R_1')$ is the second intersection of AQ with $\tilde{C}_O$. This proves a) in the case that none of the midpoints R_1', R_2', R_3' coincides with the respective points D_3, E_3, F_3.

Note that if $T_{P'}^{-1}(R_1') = A$, then $R_1' = D_3$ is collinear with D_0 and $K(Q)$, so Q lies on the side $K^{-1}(D_0D_3) = K^{-1}(BC)$ of the anticomplementary triangle of ABC, as long as $D_0 \neq D_3$. If $R_1' = D_3 = D_0$, then it is easy to see that $P' = K^{-1}(A)$ is a vertex of the anticomplementary triangle $K^{-1}(ABC)$ and $P = P'$, which is excluded. Thus, at most one of the points R_1', R_2', R_3' can coincide with their counterparts D_3, E_3, F_3. Suppose that $R_1' = D_3$. As the center of the conic $\mathcal{N}_{P'}$ lying on the points D_3 and D_0, $K(Q)$ is the midpoint of D_0D_3, since we know it is an ordinary point on the line BC. Therefore $O = T_{P'}^{-1}(K(Q))$ is the midpoint of $T_{P'}^{-1}(D_3D_0) = AT_{P'}^{-1}(D_0)$, implying that $\tilde{A} = T_{P'}^{-1}(D_0)$ is the reflection of A in the point O, lying on $\tilde{C}_O$. Now III, Corollary 3.5 says that the tangent l to $\tilde{C}_O$ at $\tilde{A}$ is parallel to BC. Since A and $\tilde{A}$ are opposite points on $\tilde{C}_O$, the tangent line to $\tilde{C}_O$ at A is also parallel to BC, so this tangent is $K^{-1}(BC) = AQ$. Thus, in this case, AQ only intersects $\tilde{C}_O$ in the point $T_{P'}^{-1}(R_1') = A$, and $T_{P'}^{-1}(R_1'R_2'R_3')$ is again the circumcevian triangle of Q.

Part b) follows from the observation that $T_{P'}^{-1}(R_1'), D_0$, and O are collinear, with similar statements for the other vertices. Part c) follows from (3) and the fact that D_0, E_0, F_0 lie on the nine-point conic $\mathcal{N}_{P'}$ (with respect to l_∞) of the quadrangle $ABCP'$, so the points $T_{P'}^{-1}(D_0), T_{P'}^{-1}(E_0), T_{P'}^{-1}(F_0)$, which are the midpoints of the sides of the anticevian triangle of Q, lie on $\tilde{C}_O = T_{P'}^{-1}(\mathcal{N}_{P'})$. As above, the points R_1', R_2', R_3' are distinct from the points D_0, E_0, F_0; if $R_1' = D_0$, for example, then $R_1' = D_0 = D_3$. Therefore, $T_{P'}^{-1}(D_0E_0F_0)$ is the antipodal triangle of $A'B'C'$ on $\tilde{C}_O$. Denoting the half-turn about O by R_O, we have

$$\begin{aligned} A'B'C' &= \mathsf{R}_O T_{P'}^{-1}(D_0E_0F_0) = \mathsf{R}_O T_{P'}^{-1} K(ABC) \\ &= \mathsf{R}_O T_{P'}^{-1} K T_P^{-1}(DEF) = \mathsf{R}_O \mathsf{M}^{-1}(DEF), \end{aligned}$$

where $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is a homothety or translation, by III, Theorem 3.4. This shows that corresponding sides of $A'B'C'$ and DEF are parallel, so these triangles are homothetic or congruent, by the converse of Desargues' Theorem. □

Definition 4.3. *The* **tangential triangle** *of ABC with respect to the circumconic $\tilde{C}_O$ is the triangle whose sides are tangent to $\tilde{C}_O$ at the points A, B, C.*

Theorem 4.4. *Assume that P is an ordinary point, not lying on $\iota(l_\infty)$. The point O is the perspector of the circumcevian triangle of Q and the tangential triangle of ABC with respect to the conic $\tilde{C}_O$.*

Proof. First assume H is not a vertex, so that O is distinct from the points D_0, E_0, F_0. The point O is the center of the conic $\tilde{C}_O$, and D_0 is the midpoint of the chord BC on this conic, so OD_0 lies on the pole U of the line BC and $b = UB$ and $c = UC$ are the tangents to $\tilde{C}_O$ from U. If V and W are the poles of AC and AB, then UVW is the tangential triangle of ABC, and UVW is perspective to $D_0E_0F_0$ from the point O. The assertion then follows from Proposition 4.2b). If H is a vertex, say $H = A$, then $O = D_0$ and V and W still lie on OB' and OC', respectively. In this case B and C are antipodal points on $\tilde{C}_O$, so the tangents at those points are parallel and meet at $U \in l_\infty$. To show that UVW and $A'B'C'$ are perspective from O, we need to show that OA' lies on U. By Proposition 4.2c), $\tilde{A} = T_{P'}^{-1}(D_0)$ and A' are antipodal points on $\tilde{C}_O$. The tangents at A' and $\tilde{A}$ are parallel to BC, so the pole of the line OA' is the point at infinity on BC. Thus, B, C, and the pole of OA' are collinear, which implies that OA' and the tangents at (i.e. polars of) B and C are concurrent. □

Lemma 4.5. *If γ denotes the isogonal map for triangle ABC, the map $\gamma \circ \gamma_P$ is a projective collineation fixing the vertices of ABC.*

Proof. Let ρ_a, ρ_b, ρ_c denote the reflections in the angle bisectors of angles A, B, and C. The definitions of the maps γ and γ_P imply that $\gamma \circ \gamma_P(R)$ is the intersection of the lines $A\rho_a h_a(R), B\rho_b h_b(R), C\rho_c h_c(R)$, for any point R which is not on a side of ABC. Now, if R is a point on the side AB, for example, other than a vertex, this intersection is just $AB \cdot C\rho_c h_c(R)$. Also, for the vertex A, for example, this intersection can naturally be considered to be A itself, since the intersection $B\rho_b h_b(A) \cdot C\rho_c h_c(A) = BA \cdot CA = A$, and $A\rho_a h_a(A) = A$. Therefore, we can extend the definition of the map $\gamma \circ \gamma_P$ to coincide with the intersection of $A\rho_a h_a(R), B\rho_b h_b(R), C\rho_c h_c(R)$ for all points R.

Let the point R vary on a line l, which does not lie on A or B. As in the proof of Lemma 3.4 we have the sequence of projectivities

$$\rho_a h_a(AR) \;\bar{\wedge}\; AR \;\overset{l}{\bar{\bar{\wedge}}}\; BR \;\bar{\wedge}\; \rho_b h_b(BR).$$

Hence, we have a projectivity $\phi : \rho_a h_a(AR) \;\bar{\wedge}\; \rho_b h_b(BR)$ between the pencil on A and the pencil on B. If R is on the line AB, then $\rho_a h_a(AR) = \rho_a h_a(AB) = \rho_a(AC) = AB = \rho_b h_b(BA)$. The common line AB in the pencils on A and B is invariant, so ϕ is a perspectivity. (See [3], p. 35.) Hence $\gamma \circ \gamma_P(R) = \rho_a h_a(AR) \cdot \rho_b h_b(BR)$ varies on a line. If l lies on $R = C$, then we have $\gamma \circ \gamma_P(C) = C$.

This argument shows that $\gamma \circ \gamma_P(l)$ is a line whenever l is not a side of ABC, and that $\gamma \circ \gamma_P$ fixes the vertices. If R varies on AB, then since $\rho_a h_a(AB) = \rho_b h_b(AB) = AB$, it is clear that $\gamma \circ \gamma_P(R) = AB \cdot \rho_c h_c(CR)$ varies on AB, so the sides are invariant lines. Thus, $\gamma \circ \gamma_P$ is a collineation. It is also easy to see this map is a projective collineation, for if R varies on a line not through A, for example, then $\gamma \circ \gamma_P(R)$ varies on a line m, and

$$R \;\bar{\wedge}\; AR \;\bar{\wedge}\; \rho_a h_a(AR) \;\bar{\wedge}\; \gamma \circ \gamma_P(R)$$

is a projectivity between l and m. This proves the lemma. □

We are now ready to prove the main result of this paper. Recall from [15] that the generalized orthocenter H for P with respect to ABC is the intersection of the lines through the vertices which are parallel, respectively, to the lines QD, QE, QF.

The TCC-Perspector Theorem. *Assume P is an ordinary point and does not lie on $\iota(l_\infty)$. If $H \neq A, B, C$ is the generalized orthocenter for P with respect to triangle ABC, then the isogonal conjugate $\gamma(H)$ is the perspector of the tangential triangle of ABC and the circumcevian triangle of $\gamma(Q)$, both taken with respect to the circumcircle of ABC. In short, $\gamma(H)$ is the TCC-perspector of $\gamma(Q)$ with respect to ABC.*

Proof. Apply the collineation $\gamma \circ \gamma_P$ to the result of Theorem 4.4. Then $\gamma \circ \gamma_P(O)$ is the perspector of the circumcevian triangle of $\gamma \circ \gamma_P(Q) = \gamma(Q)$ and the tangential triangle of $\gamma \circ \gamma_P(ABC) = ABC$ with respect to the conic $\gamma \circ \gamma_P(\tilde{\mathcal{C}}_O)$. But Proposition 3.5 shows that $\gamma \circ \gamma_P(\tilde{\mathcal{C}}_O) = \gamma(\mathfrak{l}_\infty)$ is just the circumcircle of ABC (!), so $\gamma(H)$ is the TCC-perspector of $\gamma(Q)$, as claimed. □

This theorem yields the following formula for the TCC-perspector $T(Q)$ of a point Q:

$$T(Q) = \gamma \circ K^{-1} \circ T_R^{-1} \circ K \circ \gamma(Q),$$

where $R = K^{-1}(\gamma(Q))$. This is reminiscent of the formula that was proved in Part I for the cyclocevian conjugate of a point P.

In our final result we give an alternate characterization of the point $\gamma_P(G)$, which is a kind of supplement to the TCC-Perspector Theorem.

Theorem 4.6. *The perspector of ABC and the tangential triangle abc of ABC with respect to the circumconic $\tilde{\mathcal{C}}_O$ is $\gamma_P(G)$.*

Proof. As in Theorem 4.4, let $U = b \cdot c$, where b and c are the polars of B and C with respect to $\tilde{\mathcal{C}}_O$. It is enough to show that $\gamma_P(G)$ lies on AU; similar arguments for the other two sides imply the theorem. From the definition of γ_P, we just need to show that the lines AU and AG are harmonic conjugates with respect to the lines AQ and $l_a = Q_bQ_c = T_{P'}^{-1}(BC)$. Let $A'B'C'$ be the circumcevian triangle of Q with respect to ABC and $\tilde{\mathcal{C}}_O$, and let $\tilde{A} = T_{P'}^{-1}(D_0)$. Parts b) and c) of Proposition 4.2 show that the line $A'\tilde{A}$ lies on D_0 and O, hence also on U (OD_0 and UD_0 both lie in the conjugate direction to BC with respect to $\tilde{\mathcal{C}}_O$). Taking the section $\{U, D_0, A', \tilde{A}\}$ of the four lines AU, AG, AQ, Q_bQ_c by the line $A'\tilde{A}$ (using the fact that $AQ \cap \tilde{\mathcal{C}}_O = \{A, A'\}$), it is enough to show that the points U and D_0 are harmonic conjugates with respect to A' and $\tilde{A}$. But A' and $\tilde{A}$ are on $\tilde{\mathcal{C}}_O$ by Proposition 4.2c) and D_0 on BC is conjugate to $U = b \cdot c$, so this follows from the involution of conjugate points on $A'\tilde{A}$. □

5. Appendix: two lemmas.

Lemma 5.1. *For any ordinary point R, the point $K(R)$ is the midpoint of $T_P(R)$ and $T_{P'}(R)$.*

Remark. This lemma is easy to prove using barycentric coordinates. See [17]. It requires some ingenuity to prove this synthetically.

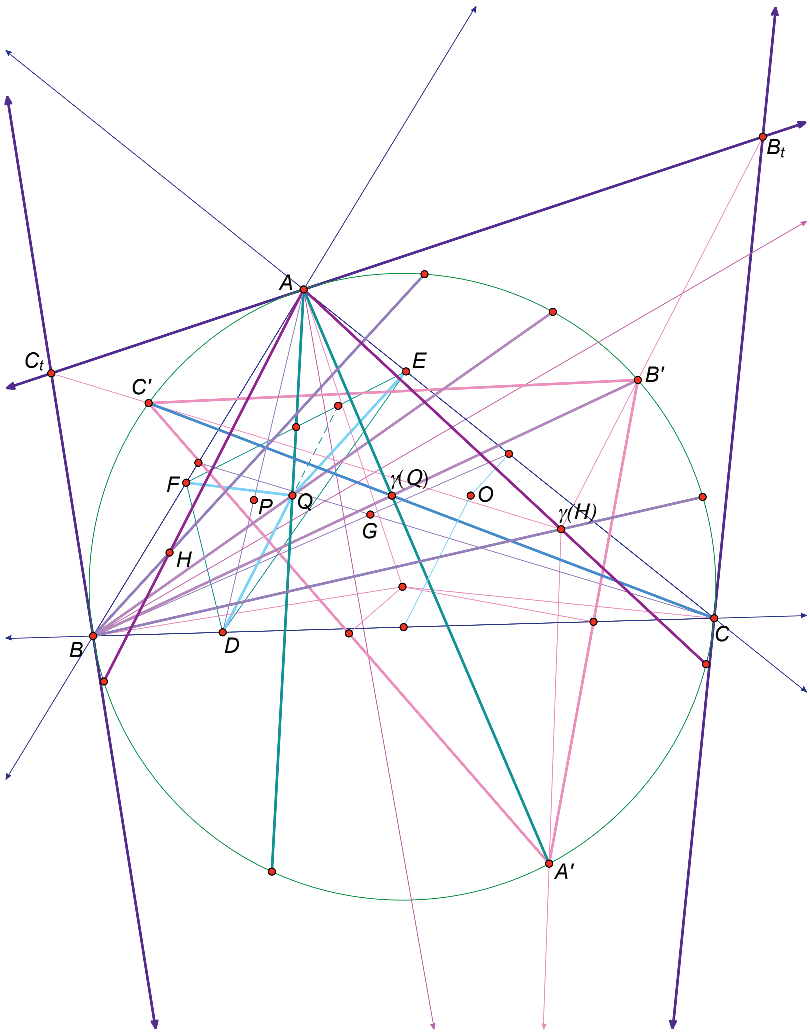

FIGURE 6. TCC-perspector $\gamma(H)$ of circumcevian triangle $A'B'C'$ of $\gamma(Q)$ and tangential triangle $A_tB_tC_t$. (Pink rays from A and B are angle bisectors.)

Proof. We shall use the fact that if A_1 and A_2 are affine maps, then the mapping T which takes a point R to the midpoint of the segment joining $\mathsf{A}_1(R)$ to $\mathsf{A}_2(R)$ is affine. If this is the case, then with $\mathsf{A}_1 = T_P$ and $\mathsf{A}_2 = T_{P'}$, we know that $\mathsf{T}(A) = D_0, \mathsf{T}(B) = E_0$, and $\mathsf{T}(C) = F_0$. Hence the affine mapping T agrees with the affine mapping K on three non-collinear points, so they must be the same map.

Let A_1 and A_2 be arbitrary affine maps, and for the purposes of this proof let A, B, C be three collinear points, with $A_i = \mathsf{A}_i(A), B_i = \mathsf{A}_i(B), C_i = \mathsf{A}_i(C)$. We will prove that T is affine. We know that A_i, B_i, C_i are collinear, for $i = 1, 2$, and the ratios $\frac{|A_1B_1|}{|B_1C_1|} = \frac{|A_2B_2|}{|B_2C_2|}$, since affine maps preserve ratios along a line. The affine map $T = \mathsf{A}_2 \circ \mathsf{A}_1^{-1}$ induces a projectivity from $l_1 = A_1B_1$ to $l_2 = A_2B_2$. We show first that the midpoints I, J, K of A_1A_2, B_1B_2, and C_1C_2 are collinear (assuming $l_1 \neq l_2$; otherwise the assertion is trivial). Let $F = l_1 \cdot l_2$. The axis m of this projectivity is the join of $T^{-1}(F)$ on l_1 and $T(F)$ on l_2. (See [3], p. 37.) Let M_1 and M_2 be the midpoints of the segments $FT^{-1}(F)$ and $T(F)F$, respectively, so that $T(M_1) = M_2$.

If $T(F) = F = T^{-1}(F)$ is an ordinary point, this is easy, because then the triangles A_1FA_2, B_1FB_2, and C_1FC_2 are all similar (by SAS), so I, J, K are all collinear with F. If F lies on l_∞, then it is also easy to see, because in that case the line IJ, which is parallel to l_1 and l_2, cuts C_1C_2 in the same ratio as it cuts A_1A_2 and B_1B_2, so I, J, K are also collinear in this case. (If T is not injective on AB and $I = J$, then it is easy to see that $A_1IB_1 \cong A_2IB_2$ by a half-turn, which implies easily that $\mathsf{T}(AB) = I$, i.e., that T maps the whole line AB to the point I.)

Now assume F is ordinary and $FT(F)T^{-1}(F)$ is a triangle. Then $n = M_1M_2$ is parallel to the axis $m = T(F)T^{-1}(F)$. Moreover, I, J and K are distinct. We will show the points I, J, K lie on the line n. It is enough to prove this for the point I. We redefine I as the point $I = n \cdot A_1A_2$ and have to show I is the midpoint of segment A_1A_2. Let $S = M_1A_2 \cdot M_2A_1$ be the cross-join for the points M_1, A_1 and their images M_2, A_2 under the projectivity induced by T. The point S lies on the axis m. We know that

$$\frac{|M_2F|}{|FA_2|} = \frac{|M_1T^{-1}(F)|}{|T^{-1}(F)A_1|} = \frac{|M_2S|}{|SA_1|}$$

by the fact that T is affine and the similarity of the triangles $M_2A_1M_1$ and $SA_1T^{-1}(F)$. Hence, $\frac{|M_2F|}{|FA_2|} = \frac{|M_2S|}{|SA_1|}$. It follows that triangle FM_2S is similar to $A_2M_2A_1$ and therefore $SF \parallel A_1A_2$. Then using the quadrangle FM_1SM_2, with diagonal points A_1 and A_2 and I on M_2M_1 shows that I is the harmonic conjugate of $A_1A_2 \cdot SF = A_1A_2 \cdot l_\infty$, so is the midpoint of A_1A_2. Conversely, given the ordinary point $I \neq M_1, M_2, M_1'$ on the line $n = M_1M_2$, where M_1' is the midpoint of segment M_1M_2, let R be the harmonic conjugate of I with respect to M_1 and M_2. Then the ordinary point R is also distinct from M_1, M_2, M_1'. If l is the line through I parallel to FR, then l intersects the line l_1 in a point A_1 and l_2 in a point A_2 such that

$$\frac{|M_1A_1|}{|M_1F|} = \frac{|M_1I|}{|M_1R|} = \frac{|M_2I|}{|M_2R|} = \frac{|M_2A_2|}{|M_2F|} = \frac{|M_2A_2|}{|M_2T(F)|},$$

from which we conclude that $T(A_1) = A_2$ and I is the midpoint of A_1A_2. Since $\mathsf{T}\mathsf{A}_1^{-1}$ maps $T^{-1}(F), M_1, F$ to M_1, M_1', M_2, respectively, we see that T maps line AB onto line n.

Thus, the mapping T taking R to the midpoint of $\mathsf{A}_1(R)$ and $\mathsf{A}_2(R)$ has the property that it maps every line either to a line or a point. In the former case, if the map $T : l_1 \to l_2$ (with $l_1 \neq l_2$) has $F = l_1 \cdot l_2$ as a fixed point, then the mapping $\mathsf{T} : AB \to IJ$ is projective. If F is ordinary, this holds because, as in the third paragraph of the proof above, the triangles A_1FA_2, B_1FB_2, and C_1FC_2 are similar, so $A_1A_2 \parallel B_1B_2 \parallel C_1C_2$. If $O = A_1A_2 \cdot l_\infty$ is the point at infinity on these lines, then we have $A_1B_1C_1 \overset{O}{\barwedge} IJK$ is projective, so

$ABC \;\overline{\wedge}\; IJK$ is projective as well. If T is a bijection, then since T transforms one range projectively, it is a projective collineation (see [3], p. 50), and since it takes ordinary points to ordinary points, it fixes the line l_∞ and must be an affine map. If the point F is infinite, then the lines A_1A_2, B_1B_2, and C_1C_2 are either parallel or concurrent, and the same conclusion follows.

In the case under consideration, namely $\mathsf{A}_1 = T_P$ and $\mathsf{A}_2 = T_{P'}$, it is not difficult to see that T is a bijection on the whole plane. Since T maps the vertices A, B, C (going back to our usual notation) to the distinct points D_0, E_0, F_0, the above argument shows that every point on the sides of the medial triangle $D_0E_0F_0$ is in the image of T. This implies easily that the map T is $1-1$. If, for example, $\mathsf{T}(U) = \mathsf{T}(V) = L$ for ordinary points $U \neq V$, then T maps the whole line UV to the point L. If S is any point for which $\mathsf{T}(S) = L' \neq L$, then T maps the lines US and VS to the line LL'. Since UVS is a triangle, then every point of the plane maps to some point on LL', which contradicts the fact that the points D_0, E_0, F_0 are in the image of T. Thus, T is $1-1$ and maps lines to lines. Now the fact that $D_0E_0F_0$ is a triangle implies that every ordinary point is on a line joining two points in the image of T, so T is surjective.

Furthermore, the map $T = T_{P'}T_P^{-1} = \lambda$ always has a fixed point. If P does not lie on a median of triangle ABC or on the Steiner circumellipse $\iota(l_\infty)$, such a fixed point is the center Z of the cevian conic $\mathcal{C}_P$, by II, Theorem 4.1. If $P \in \iota(l_\infty)$, but does not lie on a median of ABC, the same conclusion holds by II, Theorem 4.3. And if P lies on a median, say on AG, where G is the centroid, then the point $F = D_0 = AG \cdot BC$ is a fixed point of the map $T = \lambda$. Hence, T induces a projectivity on every line. This completes the proof of the lemma. □

The result we need this for is the following, which weakens the hypothesis of III, Proposition 3.12.

Lemma 5.2. *For any point P, not on the sides of triangles ABC or $K^{-1}(ABC)$, the two maps $T_P \circ K^{-1}$ and $T_{P'} \circ K^{-1}$ commute with each other. In particular, the map $\mathsf{M} = T_P \circ K^{-1} \circ T_{P'}$ is symmetric in P and $P' = \iota(P)$.*

Proof. The assertion is equivalent to $\mathsf{M} = T_PK^{-1}T_{P'} = T_{P'}K^{-1}T_P$, or to

$$T_{P'}^{-1}T_PK^{-1} = K^{-1}T_PT_{P'}^{-1}.$$

By Lemma 5.1, for any ordinary point R, R is the midpoint of $K^{-1}T_P(R)$ and $K^{-1}T_{P'}(R)$. Replacing R by $T_{P'}^{-1}(Y)$, we know that $T_{P'}^{-1}(Y)$ is the midpoint of $K^{-1}T_PT_{P'}^{-1}(Y)$ and $K^{-1}(Y)$. On the other hand, Y is the midpoint of $T_PK^{-1}(Y)$ and $T_{P'}K^{-1}(Y)$, so applying $T_{P'}^{-1}$ gives that $T_{P'}^{-1}(Y)$ is the midpoint of $T_{P'}^{-1}T_PK^{-1}(Y)$ and $K^{-1}(Y)$. It follows that $T_{P'}^{-1}T_PK^{-1}(Y) = K^{-1}T_PT_{P'}^{-1}(Y)$ for any ordinary point Y, and this implies the assertion. □

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