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PROJECTIVE GEOMETRY AND ORBITAL MECHANICS

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Abstract Due to the fact that orbits are conic sections, projective geometry gives a description of orbits based on projective hyperquadrics properties. This paper is a first approach to this method using basic linear algebra and geometry results and shows the relation between conic metric invariants and orbital energy and some applications to basic problems in orbital mechanics.

1. Introduction

Definition 1.1. Given a n+1 dimensional vector space $V_{n+1}(K)$ and a binary equivalence relation $\sim$ in which $\forall u, v \in V_{n+1}(K), u \sim v \Leftrightarrow u = \lambda v$ with $\lambda \in K$. The quotient space defines the n dimensional projective space $P_n(V_{n+1}(K))$.

Hyperquadrics (conics in a two dimensional projective space) can be defined in the projective space and so, use the linear algebra of this space to study their geometrical properties and applications to basic problems in orbital mechanics.

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2. The orbit

Definition 2.1. A hyperquadric, $C = \pi(f)$, [4] in the projective space $P_n(V_{n+1}(K))$ is a point of a projective space $P_{\frac{n+3}{2}}(S_2(V_{n+1}(K)))$ which comes from the projection of the bilinear symmetric forms over the vector space $V_{n+1}(K)$; where n is the dimension of the projective space, $f \in S_2(V_{n+1}(K))$ and π is the canonical projective map.

If n=2 the hyperquadric is a conic and needs 5 points to be determined. So given 5 different positions of an object, its orbit is determined.

Definition 2.2. A point is autoconjugate [4] with a hyperquadric if it comes from the projection of an isotrope vector with the metric; i.e. $f(v, v) = 0$ with $v \in V_{n+1}(K)$ and $f \in S_2(V_{n+1}(K))$

Definition 2.3. The set of all points in the projective space which are autoconjugate with a hyperquadric draws the dotted hyperquadric, ImC, of the hyperquadric, $C = \pi(f)$ and is the projection of the vector subspace of all isotrope vectors with the metric.

Matricially the general equation of a conic is:

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

where x_i are the homogeneous (or projective) coordinates [4][3] of the point $M = \pi(v); v \in V_3(K)$ and π is the projective map.

Once the conic matrix has been determined it is straightforward to compute the affine elements of the conic and from here, any orbital parameter in the orbit plane.

Definition 2.4. In $P_n(V_{n+1}(K))$ the conjugation map [4][3] with respect a hyperquadric $C = \pi(f)$ establish a correspondence between points and hyperplanes of $P_n(V_{n+1}(K))$ in the way that if $C = \pi(f)$ is irreducible, the correspondence is biunique and is called polarity [4][3][1].

For conics, the polar of a point is a line and if $M \in ImC$, its polar is the tangent line to the conic at M:

$$\begin{pmatrix} m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

where m_i are the homogeneous coordinates [4][3] of M.

Because the velocity vector of an object is tangent to its path at each point, the normalized direction vector of the tangent line to the conic at each point in affine coordinates [4][3] gives the velocity direction of the object and its modulus is given by Newton's vis-viva equation [5][2][1]:

$$v = \sqrt{\mu \left(\frac{2}{((x - x_0)^2 + (y - y_0)^2)^{\frac{1}{2}}} - \frac{1}{a} \right)} \tag{1}$$

where (x_0, y_0) are the affine coordinates of the planet center of mass (one of the focus of the orbit), (x,y) are the affine coordinates of the satellite, a is the semi-major axis of the orbit and μ is the gravitational parameter [5][2][1]. So, the velocity vector can be computed in any cartesian coordinate frame in the orbit plane, using basic results of linear algebra and geometry, avoiding the use of a specific conic parameterization and differential calculus.

Definition 2.5. Given an irreducible hyperquadric $C = \pi(f) \in P_n(V_{n+1}(K))$ the tangent hyperplanes at all its points are points in the dual projective space $P_n^*(V_{n+1}^*(K))$ and they form a hyperquadric $C^* = \pi(f^*) \in P_n^*(V_{n+1}^*(K))$ called the tangential, dual or contravariant hyperquadric [4].

So if the matrix equation of C is $XAX^t = 0$ the equation of C^* is $UA^{-1}U^t = 0$ with $UX^t = 0$, where A is the hyperquadric matrix, X is the matrix of the homogeneous coordinates of $M \in ImC$ in $P_n(V_{n+1}(K))$ and U the matrix of the homogeneous coordinates of $M^* \in ImC^*$ in $P_n^*(V_{n+1}^*(K))$.

Due to the fact that projective conics are metrics, the geometric information of the orbit, given in any frame in the orbit plane, can be stored in a single 3x3 symmetric matrix and if A is the matrix of a given orbit, its tangential conic can be immediately computed: A^{-1}.

3. Metric invariants, conic classification and energy

Proposition 3.1. *There are three metric invariants I_1, I_2, I_3 [4][3] for conics, where:*

$$I_3 = det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{31} & a_{23} & a_{33} \end{pmatrix}$$

$$I_2 = Adj(a_{33})$$

$$I_1 = a_{11} + a_{22}$$

Proposition 3.2. *(Conic classification)*
For a real ellipse:
$I_3 \neq 0$
$I_2 > 0$
$I_1 \cdot I_3 < 0$
if $a_{11} = a_{22}$ and $a_{22} = 0$, it is a circle.
For a hyperbola:
$I_3 \neq 0$
$I_2 < 0$
if $a_{11} + a_{22} = 0$ and $a_{ii} \neq 0$, it is an equilateral hyperbola.
For a parabola:
$I_3 \neq 0$
$I_2 = 0$

Lemma 3.3. *For an ellipse or a hyperbola, the reduced conic equation is* [4][3]*:*

$$c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 = 0 \quad (2)$$

Lemma 3.4. *The coefficients* c_{11} *and* c_{22} *can be computed solving the equation* [4] $x^2 - I_1 x + I_2 = 0$

Theorem 3.5. *(Total mechanical energy of the orbit) The total mechanical energy of an elliptical or hyperbolic orbit expressed in terms of metric invariants is:*

$$E = \frac{-\mu \cdot I_2}{2\sqrt{-c_{ii} \cdot I_3}} \tag{3}$$

where $i = 1$ *if* $c_{11} > c_{22}$ *or* $i = 2$ *if* $c_{11} < c_{22}$ *and* μ *is the gravitational parameter* [5][2][1].

Proof. The total mechanical energy for an elliptical or hyperbolic orbit is [5][2][1]

$$E = \frac{-\mu}{2a} \tag{4}$$

In terms of the reduced conic equation (2) and supposing the case $c_{11} < c_{22}$ (the case $c_{11} > c_{22}$ is analogous), the semi-major axis is:

$$a = \sqrt{\frac{-c_{33}}{c_{11}}} \tag{5}$$

and the metric invariants I_3 and I_2 are:

$$I_3 = c_{11} \cdot c_{22} \cdot c_{33} \ \ and \ \ I_2 = c_{11} \cdot c_{22}$$

so

$$a^2 = \frac{\frac{I_3}{I_2}}{\frac{I_2}{c_{22}}} = \frac{c_{22} \cdot I_3}{I_{22}^2} \tag{6}$$

Inserting equation (6) in (4), gives equation (3). □

Corollary 3.6. *For a parabolic orbit the total mechanical energy is:*

$$E = I_2 \tag{7}$$

Proof. It follows for the fact that for parabolic orbits the total mechanical energy is zero [5][2] and from proposition 3.2 $I_2 = 0 = E$ □

Remark 3.7. It is easy to note that the sign of the total mechanical energy of the orbit is the opposite sign of I_2.

Corollary 3.8. *Kepler's third law* [5][2][1] *can be expressed as a function of metric invariants:*

$$T^2 = \frac{4\pi^2}{\mu} \left(\frac{\sqrt{-c_{ii} \cdot I_3}}{I_2} \right)^3 \tag{8}$$

Proof. Inserting the expression for a from equation (5) into the expression of Kepler's third law $T^2 = \frac{4\pi}{\mu} a^3$ gives the desired result. □

Corollary 3.9. *Newton vis-viva equation* (1) *can be expressed as a function of metric invariants as:*

$$v = \sqrt{\mu\left(\frac{2}{((x-x_0)^2+(y-y_0)^2)^{\frac{1}{2}}} - \frac{I_2}{\sqrt{-c_{ii}\cdot I_3}}\right)} \tag{9}$$

Proof. Inserting the expression for a from equation (5) into equation (1), gives the desired result. ☐

Theorem 3.10. *(Energy and eccentricity) If the conic matrix has been constructed from the focal equation, the total mechanical energy can be expressed as a function of I_2:*

$$E = -I_2\frac{\mu^2}{2L^2} \tag{10}$$

where L is the angular momentum and μ is the gravitational parameter [5][2][1].

Proof. The total mechanical energy of the orbit is related to its eccentricity e, [1] by

$$E = \frac{\mu^2}{2L^2}(e^2-1) \tag{11}$$

Following the definition of a conic as the locus of points whose distance to the focus and the directrix has a constant relation:

$$(x-x_0)^2+(y-y_0)^2-(ax+by+c)^2=0$$

where (x_0,y_0) are the focus affine coordinates and $ax+by+c=0$ is the directrix. Multiplying and dividing the directrix by a^2+b^2 and setting $e^2=a^2+b^2$

$$(x-x_0)^2+(y-y_0)^2-e^2\left(\frac{a}{\sqrt{a^2+b^2}}x+\frac{b}{\sqrt{a^2+b^2}}y+\frac{c}{\sqrt{a^2+b^2}}\right)^2=0$$

so the conic matrix is:

$$\begin{pmatrix} 1-e^2\frac{a^2}{a^2+b^2} & -e^2\frac{a}{\sqrt{a^2+b^2}}\frac{b}{\sqrt{a^2+b^2}} & x_0-e^2\frac{ac}{2(a^2+b^2)} \\ -e^2\frac{a}{\sqrt{a^2+b^2}}\frac{b}{\sqrt{a^2+b^2}} & 1-e^2\frac{b^2}{a^2+b^2} & y_0-e^2\frac{b}{2(a^2+b^2)} \\ x_0-e^2\frac{ac}{2(a^2+b^2)} & y_0-e^2\frac{b}{2(a^2+b^2} & x_0^2+y_0^2-e^2\frac{c}{a^2+b^2} \end{pmatrix}$$

and

$$I_2 = 1-e^2 \tag{12}$$

inserting (12) into equation (11) gives equation (10). ☐

4. Applications

Definition 4.1. A pencil of hyperquadrics [4] in $P_n(V_{n+1}(K))$ is a line of $P_{\frac{n(n+3)}{2}}(V_{n+1}(K))$. The set of all points of $P_n(V_{n+1}(K))$ which are in all of the hyperquadrics of the pencil are the pencil fixed points. For conics there are four fixed points which can be different or coincidental.

Definition 4.2. Conic foci [4] are real points $F(a, b)$ from which the tangent lines to the conic are the isotrope lines:

$$x_2 - b = \pm\imath(x_1 - a) \tag{13}$$

The polar of the foci are the directrixes of the conic.

4.1. **Hohmann transfer ellipse.**

Proposition 4.3. *As shown in Figure 1, the Hohmann transfer ellipse* [5][2] *can be solved with the case of two pairs of coincidental points: A=B, C=D.*

Proof. For this case, the general equation of the pencil of conics is [4][3]:

$$\lambda \cdot r_{AA} \cdot r_{DD} + r_{AD} \cdot r_{AD} = 0 \tag{14}$$

where the pair of tangent lines (r_{AA}, r_{DD}) and the double line r_{AD} are the two degenerate conics which form the base of the pencil of conics. From Fig1 it is easy to realize that r_{AA} is the tangent line to the conic of the initial orbit at the point in which the first velocity change, Δv, takes place and r_{DD} is the tangent line to the conic of the destination orbit at the point in which the second Δv takes place. The double line r_{AD} is the line of apsides of the transfer ellipse. One of the foci (the planet center of mass) is known and from definition 4.2 the system formed by equations (13) and (14) gives the tangent point of the lines (13) from the focus.
Clearing x_2 from equation (13) and inserting it into equation (14), leads to an expression of the form:

$$Mx_1^2 + Nx_1 + P = 0$$

where M, N and P are functions of λ, a, b, a^2, b^2, $\imath$. Due to the fact that the tangent point must be a double point [4]

$$N^2 - 4MP = 0 \tag{15}$$

The parameters a and b are the focus affine coordinates, which are known, so from equation (15) the parameter λ can be determined, given the equation for the Hohmann transfer ellipse. □

4.2. **Hohmann braking ellipses.**

Proposition 4.4. *As shown in Figure 2 the Hohmann braking ellipses* [5] *can be solved with the case of four coincidental points; A=B=C=D.*

Proof. For this case, the general equation of the pencil of conics is [4][3]:

$$\lambda \cdot f(x, x) + r_{AA} = 0 \tag{16}$$

where the line r_{AA} is the degenerate conic and with the non-degenerate conic $C = \pi(f)$ form the base of the pencil of conics. r_{AA} is the tangent line to all the conics at the point of the orbits in which the brake takes place (the periastron) and $C = \pi(f)$ is the conic of one of the braking ellipses. For each ellipse, the parameter λ can be determinate because as in proposition 4.3, one of the foci (the planet center of mass) is fixed and the energy change,

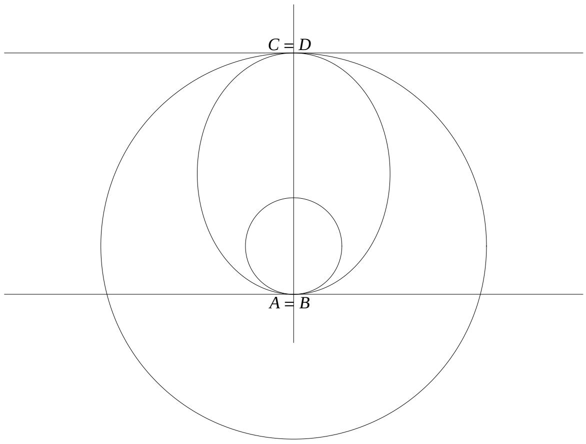

FIGURE 1. The pencil of conics case for a Hohmann transfer orbit

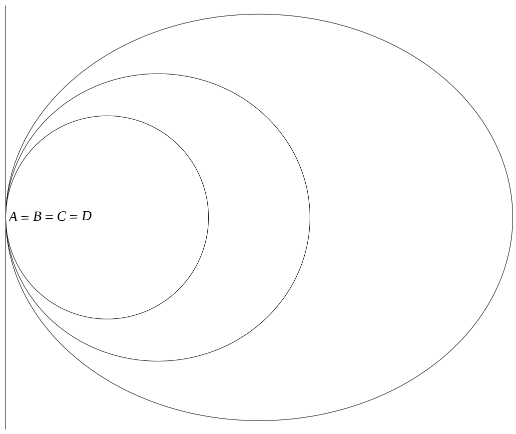

FIGURE 2. The pencil of conics case for Hohmann braking ellipses

ΔE, between orbits are known and from the equation (11) the eccentricity e of the orbit can be computed, so

$$d(A, focus) = e \cdot d(A, directrix)$$

and λ can be set for each Hohmann braking ellipse.
If the matrix of the non degenerate conic has been constructed from the focal equation as in theorem 3.10, the parameter λ can be set directly from equation (12) □

4.3. **Escape trajectories.**

Corollary 4.5. *As shown in Figure 3 the escape trajectories* [5][2] *can be solved with the case of four coincidental points; A=B=C=D.*

Proof. As in proposition 4.4 the general equation of the pencil of conics is given by equation (16). The degenerate conic r_{AA} is the tangent line to the conic at the point of the orbit in which the Δv takes place. The non degenerate conic $C = \pi(f)$ is the spacecraft current orbit. As in proposition 4.4, setting the focus and the energy condition for the escape parabola or the

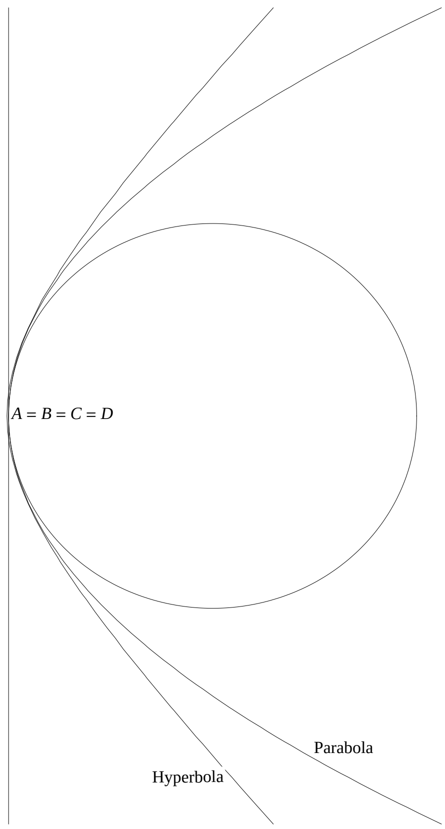

FIGURE 3. The pencil of conics case for escape trajectories

hyperbola (11), the eccentricity of the escape trajectory can be computed and from there the parameter λ can be set for the escape trajectory as in proposition 4.4.
If the matrix of the non degenerate conic has been constructed from the focal equation as in theorem 3.10, the parameter λ can be set directly from equation (12) □

4.4. **Advance of the line of apsides.** The planet's equator bulge induce to the orbit a rotation around the planet center of mass called the advance of the line of apsides [5][2][1].

Proposition 4.6. *From a geometrical point of view, the advance of the line of apsides is an isometry.*

Proof. Given a metric $f \in S_2(V_3(K))$ and an isometry σ, the following diagram commutes[4][3]:

$$\begin{tikzcd} V_3(K) \arrow[r, "f(\sigma)"] \arrow[d, "\pi"] & V_3(K) \arrow[d, "\pi"] \\ P_2(V_3(K)) \arrow[r, "\pi(f(\sigma))"] & P_2(V_3(K)) \end{tikzcd}$$

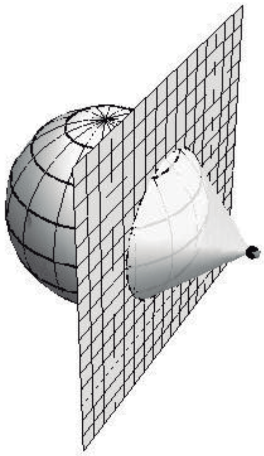

FIGURE 4. represents the sighting cone for a satellite and the polar plane of the satellite position for a spheroid model of the Earth.

so, the geometrical properties of a conic $C = \pi(f)$ are conserved under the isometry σ. If A is the matrix of the conic C and B is the new matrix of the conic under the isometry, they are similar matrices i.e.:

$$B = T \cdot A \cdot T^t \qquad T = \begin{pmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ x_0 & y_0 & 1 \end{pmatrix}$$

where (x_0, y_0) are the affine coordinates of the planet center of mass and ω is the rotation angle of the line of apsides [5][2][1]. $\square$

4.5. **Satellite sighting cone.** Taking the spheroid (or ellipsoid) model of a planet $C = \pi(f)$ with $f \in S_2(V_4(K))$ and a given position of a satellite $M = \pi(v)$ with $v \in V_4(K)$ and $M \notin ImC$, it is possible to make all the tangent lines from the satellite position to the planet, which form a cone with equation [4]:

$$f(v,v)f(x,x) - (f(v,x))^2 = 0$$

The polar of a point with respect a quadric is a plane and in this case, the intersection of the conjugate plane with respect the spheroid (or ellipsoid) of the satellite position with the spheroid (or ellipsoid), gives a conic which represents the horizon of the sighting cone. The plane equation in homogeneous coordinates is given by:

$$\begin{pmatrix} m_1 & m_2 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

where m_i are the homogeneous coordinates of M.

5. Conclusions

This paper has shown how projective hyperquadrics properties can be used to describe orbits, the relation between their energy and the conic metric invariants and the application to some basic problems of orbital mechanics in a different way, focusing in the geometrical aspects of the orbit and using basic results of linear algebra and geometry instead of angular parameterizations and differential calculus.

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