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A Note About The Metrics Induced by Truncated Dodecahedron And Truncated Icosahedron

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Abstract. Polyhedrons have been studied by mathematicians and geometers during many years, because of their symmetries. There are only five regular convex polyhedra known as the Platonic solids. Semi-regular convex polyhedron which are composed of two or more types of regular polygons meeting in identical vertices are called Archimedean solids. There are some relations between metrics and polyhedra. For example, it has been shown that cube, octahedron, deltoidal icositetrahedron are maximum, taxicab, Chinese Checker's unit sphere, respectively. In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are truncated dodecahedron and truncated icosahedron. Also we give some properties about these metrics.

1. Introduction

What is a polyhedron? This question is interestingly hard to answer simply! A polyhedron is a three-dimensional figure made up of polygons. When discussing polyhedra one will use the terms faces, edges and vertices. Each polygonal part of the polyhedron is called a face. A line segment along which two faces come together is called an edge. A point where several edges and faces come together is called a vertex. Traditional polyhedra consist of flat faces, straight edges, and vertices. There are many thinkers that worked on polyhedra among the ancient Greeks. Early civilizations worked out mathematics as problems and their solutions. Polyhedrons have been studied by mathematicians, scientists during many years, because of their symmetries. A polyhedron is called regular if all its faces are equal and regular polygons. It is called semi-regular if all its faces are regular polygons and all its vertices are equal. An irregular polyhedron is defined by polygons that are composed of elements that are not all equal. A regular polyhedron is called Platonic solid, a semi-regular polyhedron is called Archimedean solid and an irregular polyhedron is called Catalan solid.

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As it is stated in [4] and [7], polyhedra have been used for explaining the world around us in philosophical and scientific way. There are only five regular convex polyhedra. These regular polyhedra were known by the Ancient Greeks. They are generally known as the "Platonic" or "cosmic" solids because Plato mentioned them in his dialogue Timeous, where each is associated with one of the five elements - the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron with universe (or with ether, the material of the heavens). The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a 'lost' classical text. Rather, it concerns the rediscovery of actual mathematics, and there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author's predecessors. In fact, as far as we know, there was no classical text recovered by Archimedes. The Archimedean solids have that name because in his Collection, Pappus stated that Archimedes had discovered thirteen solids whose faces were regular polygons of more than one kind. Pappus then listed the numbers and types of faces of each solid. Some of these polyhedra have been discovered many times. According to Heron, the third solid on Pappus' list, the cuboctahedron, was known to Plato. During the Renaissance, and especially after the introduction of perspective into art, painters and craftsmen made pictures of platonic solids. To vary their designs they sliced off the corners and edges of these solids, naturally producing some of the Archimedean solids as a result. For more detailed knowledge, see [4] and [7].

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [13]. Some mathematicians have been studied and improved metric space geometry. According to mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. In [2, 3, 5, 6, 8, 9, 10, 11, 12] the authors give some metrics which the spheres of the 3-dimensional analytical space furnished by these metrics are some of Platonic solids, Archimedian solids and Catalan solids. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solids surface. In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are truncated dodecahedron and truncated icosahedron. Also we give some properties about these metrics.

2. Truncated Dodecahedron Metric and Some Properties

One type of convex polyhedrons is the Archimedean solids. The fifth book of the “Synagoge” or “Collection” of the Greek mathematician Pappus of Alexandria, who lived in the beginning of the fourth century AD, gives the first known mention of the thirteen “Archimedean solids”. Although, Archimedes makes no mention of these solids in any of his extant works, Pappus lists this solids and attributes to Archimedes in his book [17].

An Archimedean solid is a symmetric, semiregular convex polyhedron composed of two or more types of regular polygons meeting in identical vertices. A polyhedron is called semiregular if its faces are all regular polygons and its corners are alike. And, identical vertices are usually means that for two taken vertices there must be an isometry of the entire solid that transforms one vertex to the other.

It has been stated in [1], [4] and [14], seven of the 13 Archimedean solids (the cuboctahedron, icosidodecahedron, truncated cube, truncated dodecahedron, truncated octahedron, truncated icosahedron, and truncated tetrahedron) can be obtained by truncation of a Platonic solid.

Two additional solids (the (small) rhombicosidodecahedron and (small) rhombicuboctahedron) can be obtained by expansion of a Platonic solid, and two further solids (the great rhombicosidodecahedron and great rhombicuboctahedron) can be obtained by expansion of one of the previous 9 Archimedean solids. It is sometimes stated that these four solids can be obtained by truncation of other solids. The confusion originated with Kepler himself, who used the terms ”truncated icosidodecahedron” and ”truncated cuboctahedron” for the great rhombicosidodecahedron and great rhombicuboctahedron, respectively. However, truncation alone is not capable of producing these solids, but must be combined with distorting to turn the resulting rectangles into squares .

The remaining two solids, the snub cube and snub dodecahedron, can be obtained by moving the faces of a cube and dodecahedron outward while giving each face a twist. The resulting spaces are then filled with ribbons of equilateral triangles.

One of the Archimedean solids is the truncated dodecahedron. A truncated dodecahedron is a polyhedron which has 12 regular decagonal faces, 20 regular triangular faces, 60 vertices and 90 edges. This polyhedron can be formed from a dodecahedron by truncating (cutting off) the corners so the pentagon faces become decagons and the corners become triangles. [15].

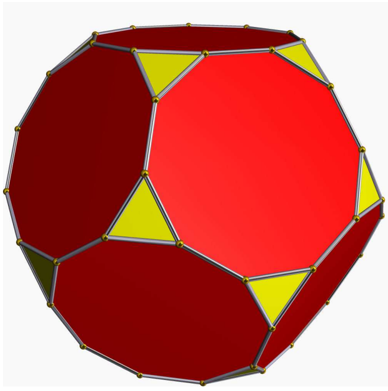

Figure 1(a) Truncated dodecahedron

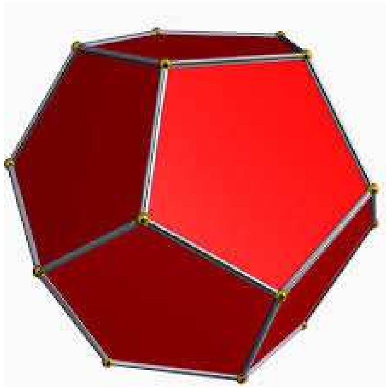

Figure 1(b) Dodecahedron

We describe the metric that unit sphere is truncated dodecahedron as following:

Definition 2.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{TD} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated dodecahedron distance between P_1 and P_2 is defined by*
$d_{TD}(P_1, P_2) =$

$$\max \left\{ \begin{array}{l} \frac{8-\varphi}{11} X_{12} + \frac{8\varphi-9}{11} \max \left\{ \varphi(Y_{12}+Z_{12}), \frac{3\varphi+1}{11} X_{12} + \frac{\varphi+7}{5} Y_{12}, X_{12}+Z_{12} \right\}, \\ \frac{8-\varphi}{11} Y_{12} + \frac{8\varphi-9}{11} \max \left\{ \varphi(X_{12}+Z_{12}), \frac{3\varphi+1}{11} Y_{12} + \frac{\varphi+7}{5} Z_{12}, X_{12}+Y_{12} \right\}, \\ \frac{8-\varphi}{11} Z_{12} + \frac{8\varphi-9}{11} \max \left\{ \varphi(X_{12}+Y_{12}), \frac{3\varphi+1}{11} Z_{12} + \frac{\varphi+7}{5} X_{12}, Y_{12}+Z_{12} \right\} \end{array} \right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$ and $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio.

According to truncated dodecahedron distance, there are three different paths from P_1 to P_2. These paths are

$i)$ union of three line segments each of which is parallel to a coordinate axis.

$ii)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{95030-20661\sqrt{5}}{219010})$ angle with another coordinate axis.

$iii)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{5\sqrt{5}-9}{8})$ angle with another coordinate axis.

Thus truncated dodecahedron distance between P_1 and P_2 is for (i) $\frac{15-\sqrt{5}}{22}$ times the sum of Euclidean lengths of three line segments, for (ii) $\frac{100-3\sqrt{5}}{121}$ times the sum of Euclidean lengths of mentioned two line segments, and for (iii) $\frac{5+7\sqrt{5}}{22}$ times the sum of Euclidean lengths of mentioned two line segments.

Figure 2 illustrates truncated dodecahedron way from P_1 to P_2 if maximum value is $(\frac{15-\sqrt{5}}{22})(|x_1-x_2|+|y_1-y_2|+|z_1-z_2|)$, $\frac{100-3\sqrt{5}}{121}\left(|y_1-y_2| + (\frac{1067\sqrt{5}-935}{1810})|z_1-z_2|\right)$ or $\frac{5+7\sqrt{5}}{22}\left(|y_1-y_2| + (3-\sqrt{5})|x_1-x_2|\right)$.

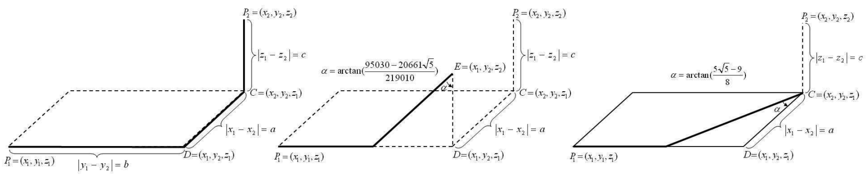

Figure 2: TD way from P_1 to P_2

Lemma 2.1. *Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ be distinct two points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$, respectively. Then*

$$d_{TD}(P_1,P_2)\geq \tfrac{8-\varphi}{11}X_{12}+\tfrac{8\varphi-9}{11}\max\left\{\varphi(Y_{12}+Z_{12}),\tfrac{3\varphi+1}{11}X_{12}+\tfrac{\varphi+7}{5}Y_{12},X_{12}+Z_{12}\right\},$$
$$d_{TD}(P_1,P_2)\geq \tfrac{8-\varphi}{11}Y_{12}+\tfrac{8\varphi-9}{11}\max\left\{\varphi(X_{12}+Z_{12}),\tfrac{3\varphi+1}{11}Y_{12}+\tfrac{\varphi+7}{5}Z_{12},X_{12}+Y_{12}\right\},$$
$$d_{TD}(P_1,P_2)\geq \tfrac{8-\varphi}{11}Z_{12}+\tfrac{8\varphi-9}{11}\max\left\{\varphi(X_{12}+Y_{12}),\tfrac{3\varphi+1}{11}Z_{12}+\tfrac{\varphi+7}{5}X_{12},Y_{12}+Z_{12}\right\}.$$

Proof. *Proof is trivial by the definition of maximum function.*

Theorem 2.1. *The distance function d_{TD} is a metric. Also according to d_{TD}, the unit sphere is an truncated dodecahedron in $\mathbb{R}^3$.*

Proof. Let $d_{TD}:\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty)$ be the truncated dodecahedron distance function and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ are distinct three points in $\mathbb{R}^3$. X_{12}, Y_{12}, Z_{12} denote $|x_1-x_2|$, $|y_1-y_2|$, $|z_1-z_2|$, respectively. To show that d_{TD} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3\in\mathbb{R}^3$.

M1) $d_{TD}(P_1,P_2)\geq 0$ and $d_{TD}(P_1,P_2)=0$ iff $P_1=P_2$
M2) $d_{TD}(P_1,P_2)=d_{TD}(P_2,P_1)$
M3) $d_{TD}(P_1,P_3)\leq d_{TD}(P_1,P_2)+d_{TD}(P_2,P_3)$.

Since absolute values is always nonnegative value $d_{TD}(P_1,P_2)\geq 0$. If $d_{TD}(P_1,P_2)=0$ then there are possible three cases. These cases are

1) $d_{TD}(P_1,P_2)=\frac{8-\varphi}{11}X_{12}+\frac{8\varphi-9}{11}\max\left\{\varphi(Y_{12}+Z_{12}),\frac{3\varphi+1}{11}X_{12}+\frac{\varphi+7}{5}Y_{12},X_{12}+Z_{12}\right\}$

2) $d_{TD}(P_1,P_2)=\frac{8-\varphi}{11}Y_{12}+\frac{8\varphi-9}{11}\max\left\{\varphi(X_{12}+Z_{12}),\frac{3\varphi+1}{11}Y_{12}+\frac{\varphi+7}{5}Z_{12},X_{12}+Y_{12}\right\}$

3) $d_{TD}(P_1,P_2)=\frac{8-\varphi}{11}Z_{12}+\frac{8\varphi-9}{11}\max\left\{\varphi(X_{12}+Y_{12}),\frac{3\varphi+1}{11}Z_{12}+\frac{\varphi+7}{5}X_{12},Y_{12}+Z_{12}\right\}$.

Case I: If

$$d_{TD}(P_1,P_2)=\frac{8-\varphi}{11}X_{12}+\frac{8\varphi-9}{11}\max\left\{\varphi(Y_{12}+Z_{12}),\frac{3\varphi+1}{11}X_{12}+\frac{\varphi+7}{5}Y_{12},X_{12}+Z_{12}\right\},$$

then

$\frac{8-\varphi}{11}X_{12}+\frac{8\varphi-9}{11}\max\left\{\varphi(Y_{12}+Z_{12}),\frac{3\varphi+1}{11}X_{12}+\frac{\varphi+7}{5}Y_{12},X_{12}+Z_{12}\right\}=0$
$\Leftrightarrow X_{12}=0$ and $\frac{8\varphi-9}{11}\max\left\{\varphi(Y_{12}+Z_{12}),\frac{3\varphi+1}{11}X_{12}+\frac{\varphi+7}{5}Y_{12},X_{12}+Z_{12}\right\}=0$
$\Leftrightarrow x_1=x_2,\ y_1=y_2,\ z_1=z_2$
$\Leftrightarrow (x_1,y_1,z_1)=(x_2,y_2,z_2)$
$\Leftrightarrow P_1=P_2$

The other cases can be shown by similar way in Case I. Thus we get $d_{TD}(P_1,P_2)=0$ iff $P_1=P_2$.

Since $|x_1 - x_2| = |x_2 - x_1|$, $|y_1 - y_2|=|y_2 - y_1|$ and $|z_1 - z_2| = |z_2 - z_1|$, obviously $d_{TD}(P_1, P_2) = d_{TD}(P_2, P_1)$. That is, d_{TD} is symmetric.

$X_{13}, Y_{13}, Z_{13}, X_{23}, Y_{23}, Z_{23}$ denote $|x_1 - x_3|, |y_1 - y_3|, |z_1 - z_3|, |x_2 - x_3|, |y_2 - y_3|, |z_2 - z_3|$, respectively. Then by using the property $|a - b + b - c| \leq |a - b| + |b - c|$ for $a, b, c \in \mathbb{R}$.

$$
\begin{aligned}
&d_{TD}(P_1, P_3) \\
&= \max \left\{ \begin{array}{l} \frac{8-\varphi}{11} X_{13} + \frac{8\varphi-9}{11} \max\left\{ \varphi(Y_{13}+Z_{13}), \frac{3\varphi+1}{11} X_{13} + \frac{\varphi+7}{5} Y_{13}, X_{13}+Z_{13} \right\}, \\ \frac{8-\varphi}{11} Y_{13} + \frac{8\varphi-9}{11} \max\left\{ \varphi(X_{13}+Z_{13}), \frac{3\varphi+1}{11} Y_{13} + \frac{\varphi+7}{5} Z_{13}, X_{13}+Y_{13} \right\}, \\ \frac{8-\varphi}{11} Z_{13} + \frac{8\varphi-9}{11} \max\left\{ \varphi(X_{13}+Y_{13}), \frac{3\varphi+1}{11} Z_{13} + \frac{\varphi+7}{5} X_{13}, Y_{13}+Z_{13} \right\} \end{array} \right\} \\
&\leq \max \left\{ \begin{array}{l} \frac{8-\varphi}{11}(X_{12}+X_{23}) + \frac{8\varphi-9}{11} \max \left\{ \begin{array}{r} \frac{3\varphi+1}{11}(X_{12}+X_{23}) + \frac{\varphi+7}{5}(Y_{12}+Y_{23}), \\ \varphi(Y_{12}+Y_{23}+Z_{12}+Z_{13}), \\ (X_{12}+X_{23}+Z_{12}+Z_{23}) \end{array} \right\}, \\ \frac{8-\varphi}{11}(Y_{12}+Y_{23}) + \frac{8\varphi-9}{11} \max \left\{ \begin{array}{r} \frac{3\varphi+1}{11}(Y_{12}+Y_{23}) + \frac{\varphi+7}{5}(Z_{12}+Z_{23}), \\ \varphi(X_{12}+X_{23}+Z_{12}+Z_{13}), \\ (X_{12}+X_{23}+Y_{12}+Y_{23}) \end{array} \right\}, \\ \frac{8-\varphi}{11}(Z_{12}+Z_{23}) + \frac{8\varphi-9}{11} \max \left\{ \begin{array}{r} \frac{3\varphi+1}{11}(Z_{12}+Z_{23}) + \frac{\varphi+7}{5}(X_{12}+X_{23}), \\ \varphi(X_{12}+X_{23}+Y_{12}+Y_{23}), \\ (Y_{12}+Y_{23}+Z_{12}+Z_{23}) \end{array} \right\} \end{array} \right\} \\
&= I.
\end{aligned}
$$

Therefore one can easily find that $I \leq d_{TD}(P_1, P_2) + d_{TD}(P_2, P_3)$ from Lemma 2.2. So $d_{TD}(P_1, P_3) \leq d_{TD}(P_1, P_2) + d_{TD}(P_2, P_3)$. Consequently, truncated dodecahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that truncated dodecahedron distance is 1 from $O = (0, 0, 0)$ is $S_{TD} =$

$$
\left\{ (x,y,z) \colon \max \left\{ \begin{array}{l} \frac{8-\varphi}{11}|x| + \frac{8\varphi-9}{11} \max\left\{ \varphi(|y|+|z|), \frac{3\varphi+1}{11}|x| + \frac{\varphi+7}{5}|y|, |x|+|z| \right\}, \\ \frac{8-\varphi}{11}|y| + \frac{8\varphi-9}{11} \max\left\{ \varphi(|x|+|z|), \frac{3\varphi+1}{11}|y| + \frac{\varphi+7}{5}|z|, |x|+|y| \right\}, \\ \frac{8-\varphi}{11}|z| + \frac{8\varphi-9}{11} \max\left\{ \varphi(|x|+|y|), \frac{3\varphi+1}{11}|z| + \frac{\varphi+7}{5}|x|, |y|+|z| \right\} \end{array} \right\} = 1 \right\}.
$$

Thus the graph of S_{TD} is as in the figure 3:

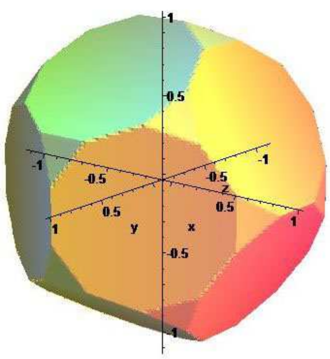

Figure 3 The unit sphere in terms of d_{TD}: Truncated dodecahedron

Corrolary 2.1. *The equation of the truncated dodecahedron with center (x_0, y_0, z_0) and radius r is*

$$\max\left\{\begin{array}{l} \frac{8-\varphi}{11}|x-x_0| + \frac{8\varphi-9}{11}\max\left\{\begin{array}{r} \varphi(|y-y_0|+|z-z_0|), |x-x_0|+|z-z_0|, \\ \frac{3\varphi+1}{11}|x-x_0| + \frac{\varphi+7}{5}|y-y_0| \end{array}\right\}, \\ \frac{8-\varphi}{11}|y-y_0| + \frac{8\varphi-9}{11}\max\left\{\begin{array}{r} \varphi(|x-x_0|+|z-z_0|), |x-x_0|+|y-y_0|, \\ \frac{3\varphi+1}{11}|y-y_0| + \frac{\varphi+7}{5}|z-z_0| \end{array}\right\}, \\ \frac{8-\varphi}{11}|z-z_0| + \frac{8\varphi-9}{11}\max\left\{\begin{array}{r} \varphi(|x-x_0|+|y-y_0|), |y-y_0|+|z-z_0|, \\ \frac{3\varphi+1}{11}|z-z_0| + \frac{\varphi+7}{5}|x-x_0| \end{array}\right\} \end{array}\right\} = r$$

which is a polyhedron which has 32 faces and 60 vertices. Coordinates of the vertices are translation to (x_0, y_0, z_0) all posible +/- sign components of the points $(0, \frac{3\alpha-1}{2}r, r)$, $(r, 0, \frac{3\alpha-1}{2}r)$, $(\frac{3\alpha-1}{2}r, r, 0)$, $(\frac{3\alpha-1}{2}r, \alpha r, 2\alpha r)$, $(2\alpha r, \frac{3\alpha-1}{2}r, \alpha r)$, $(\alpha r, 2\alpha r, \frac{3\alpha-1}{2}r)$, $(\alpha r, (1-\alpha)r, \frac{\alpha+1}{2}r)$, $(\frac{\alpha+1}{2}r, \alpha r, (1-\alpha)r)$ and $((1-\alpha)r, \frac{\alpha+1}{2}r, \alpha r)$ where $\alpha = \frac{\sqrt{5}}{5}$.

Lemma 2.2. *Let l be the line through the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p, q, r), then*

$$d_{TD}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where $\mu(P_1P_2) =$

$$\frac{\max\left\{\begin{array}{l} \frac{8-\varphi}{11}|p| + \frac{8\varphi-9}{11}\max\left\{\varphi(|q|+|r|), \frac{3\varphi+1}{11}|p| + \frac{\varphi+7}{5}|q|, |p|+|r|\right\}, \\ \frac{8-\varphi}{11}|q| + \frac{8\varphi-9}{11}\max\left\{\varphi(|p|+|r|), \frac{3\varphi+1}{11}|q| + \frac{\varphi+7}{5}|r|, |p|+|q|\right\}, \\ \frac{8-\varphi}{11}|r| + \frac{8\varphi-9}{11}\max\left\{\varphi(|p|+|q|), \frac{3\varphi+1}{11}|r| + \frac{\varphi+7}{5}|p|, |q|+|r|\right\} \end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1 - x_2 = \lambda p$, $y_1 - y_2 = \lambda q$, $z_1 - z_2 = \lambda r$, $r \in \mathbb{R}$. Thus,$d_{TD}(P_1, P_2)$ is equal to

$$|\lambda|\left(\max\left\{\begin{array}{l} \frac{8-\varphi}{11}|p| + \frac{8\varphi-9}{11}\max\left\{\varphi(|q|+|r|), \frac{3\varphi+1}{11}|p| + \frac{\varphi+7}{5}|q|, |p|+|r|\right\}, \\ \frac{8-\varphi}{11}|q| + \frac{8\varphi-9}{11}\max\left\{\varphi(|p|+|r|), \frac{3\varphi+1}{11}|q| + \frac{\varphi+7}{5}|r|, |p|+|q|\right\}, \\ \frac{8-\varphi}{11}|r| + \frac{8\varphi-9}{11}\max\left\{\varphi(|p|+|q|), \frac{3\varphi+1}{11}|r| + \frac{\varphi+7}{5}|p|, |q|+|r|\right\} \end{array}\right\}\right)$$

and $d_E(A, B) = |\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result.

The above lemma says that d_{TD}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corrolary 2.2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{TD}(P_1, X) = d_{TD}(P_2, X)$.*

Corrolary 2.3. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TD}(X, P_1) \ / \ d_{TD}(X, P_2) = d_E(X, P_1) \ / \ d_E(X, P_2) \ .$$

That is, the ratios of the Euclidean and $d_{TD}-$distances along a line are the same.

3. Truncated Icosahedron Metric and Some Properties

The truncated icosahedron is an Archimedean solid, one of 13 convex isogonal nonprismatic solids whose faces are two or more types of regular polygons. It has 12 regular pentagonal faces, 20 regular hexagonal faces, 60 vertices and 90 edges. It is the Goldberg polyhedron GV(1,1), containing pentagonal and hexagonal faces. This geometry is associated with footballs (soccer balls) typically patterned with white hexagons and black pentagons. Geodesic domes such as those whose architecture Richard Buckminister Fuller pioneered are often based on this structure. It also corresponds to the geometry of the fullerene C60 ("buckyball") molecule [16] (See Figure 4(a)-(d)).

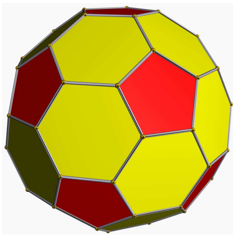

Figure 4(a)truncated icosahedron

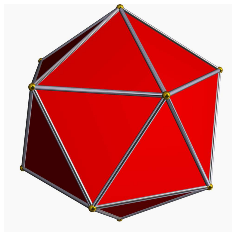

Figure 4(b) Icosahedron

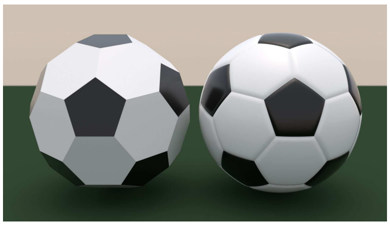

Figure 4(c)tru. icosahedron and soccer balls

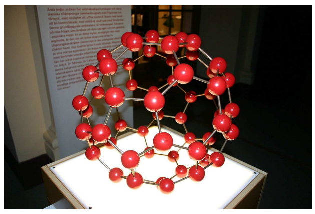

Figure 4(d) C60 molecule

We describe the metric that unit sphere is truncated dodecahedron as following:

Definition 3.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{TI} : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated icosahedron distance between P_1 and P_2 is defined by*
$d_{TI}(P_1, P_2) =$

$$(\varphi - 2)\max \left\{ \begin{array}{l} X_{12} + \max \left\{ \begin{array}{r} (2-\varphi)(X_{12}+Z_{12}), (\varphi-2)(Y_{12}+Z_{12}), \\ (\frac{31-16\varphi}{19})X_{12} + \frac{12\varphi-9}{19}Y_{12} \end{array} \right\}, \\ Y_{12} + \max \left\{ \begin{array}{r} (2-\varphi)(X_{12}+Y_{12}), (\varphi-2)(X_{12}+Z_{12}), \\ (\frac{31-16\varphi}{19})Y_{12} + \frac{12\varphi-9}{19}Z_{12} \end{array} \right\}, \\ Z_{12} + \max \left\{ \begin{array}{r} (2-\varphi)(Y_{12}+Z_{12}), (\varphi-2)(X_{12}+Y_{12}), \\ (\frac{31-16\varphi}{19})Z_{12} + \frac{12\varphi-9}{19}X_{12} \end{array} \right\} \end{array} \right\}$$

where $X_{12} = |x_1 - x_2|$, $Y_{12} = |y_1 - y_2|$, $Z_{12} = |z_1 - z_2|$ and $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio.

According to truncated icosahedron distance, there are three different paths from P_1 to P_2. These paths are

$i)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{11126\sqrt{5}-8155}{25992})$ angle with another coordinate axis.

$ii)$ union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan(\frac{33\sqrt{5}+50}{10})$ angle with another coordinate axis.

$iii)$ union of three line segments which one is parallel to a coordinate axis and other line segments makes $\arctan(\frac{1}{2})$ angle with another coordinate axes.

Thus truncated dodecahedron distance between P_1 and P_2 is for (i) $\frac{50\sqrt{5}-82}{38}$ times the sum of Euclidean lengths of mentioned two line segments, for (ii) $\frac{3\sqrt{5}-5}{2}$ times the sum of Euclidean lengths of mentioned two line segments, and for (iii) $\frac{\sqrt{5}-1}{2}$ times the sum of Euclidean lengths of three line segments. Figure 5 shows that the path between P_1 and P_2 in case of the maximum is $\frac{50\sqrt{5}-82}{38}\left(|y_1-y_2|+\frac{72\sqrt{5}+36}{361}|z_1-z_2|\right)$, $\frac{3\sqrt{5}-5}{2}\left(|y_1-y_2|+\frac{25-11\sqrt{5}}{10}|x_1-x_2|\right)$ or $\frac{\sqrt{5}-1}{2}\left(|y_1-y_2|+\frac{\sqrt{5}-1}{2}(|x_1-x_2|+|z_1-z_2|)\right)$.

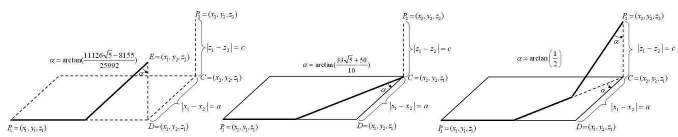

Figure 5: TI way from P_1 to P_2

Lemma 3.1. *Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. Then*

$$d_{TI}(P_1,P_2) \geq (\varphi-2)\left(X_{12}+\max\left\{\begin{array}{r}(2-\varphi)(X_{12}+Z_{12}),(\varphi-2)(Y_{12}+Z_{12}),\\ (\frac{31-16\varphi}{19})X_{12}+\frac{12\varphi-9}{19}Y_{12}\end{array}\right\}\right)$$
$$d_{TI}(P_1,P_2) \geq (\varphi-2)\left(Y_{12}+\max\left\{\begin{array}{r}(2-\varphi)(X_{12}+Y_{12}),(\varphi-2)(X_{12}+Z_{12}),\\ (\frac{31-16\varphi}{19})Y_{12}+\frac{12\varphi-9}{19}Z_{12}\end{array}\right\}\right)$$
$$d_{TI}(P_1,P_2) \geq (\varphi-2)\left(Z_{12}+\max\left\{\begin{array}{r}(2-\varphi)(Y_{12}+Z_{12}),(\varphi-2)(X_{12}+Y_{12}),\\ (\frac{31-16\varphi}{19})Z_{12}+\frac{12\varphi-9}{19}X_{12}\end{array}\right\}\right).$$

where $X_{12}=|x_1-x_2|$, $Y_{12}=|y_1-y_2|$, $Z_{12}=|z_1-z_2|$ and $\varphi=\frac{1+\sqrt{5}}{2}$ the golden ratio.

Proof. Proof is trivial by the definition of maximum function.

Theorem 3.1. *The distance function d_{TI} is a metric. Also according to d_{TI}, unit sphere is a truncated icosahedron in $\mathbb{R}^3$.*

Proof. One can easily show that the truncated icosahedron distance function satisfies the metric axioms by similar way in Theorem 2.3.

Consequently, the set of all points $X = (x, y, z) \in \mathbb{R}^3$ that truncated icosahedron distance is 1 from $O = (0,0,0)$ is $S_{TI} =$

$$\left\{ (x,y,z)\colon (\varphi-2)\max \left\{ \begin{array}{l} |x| + \max\left\{ \begin{array}{r} (2-\varphi)(|x|+|z|), (\varphi-2)(|y|+|z|), \\ (\frac{31-16\varphi}{19})|x| + \frac{12\varphi-9}{19}|y| \end{array} \right\}, \\ |y| + \max\left\{ \begin{array}{r} (2-\varphi)(|x|+|y|), (\varphi-2)(|x|+|z|), \\ (\frac{31-16\varphi}{19})|y| + \frac{12\varphi-9}{19}|z| \end{array} \right\}, \\ |z| + \max\left\{ \begin{array}{r} (2-\varphi)(|y|+|z|), (\varphi-2)(|x|+|y|), \\ (\frac{31-16\varphi}{19})|z| + \frac{12\varphi-9}{19}|x| \end{array} \right\} \end{array} \right\} = 1 \right\}.$$

Thus the graph of S_{TI} is as in the figure 6:

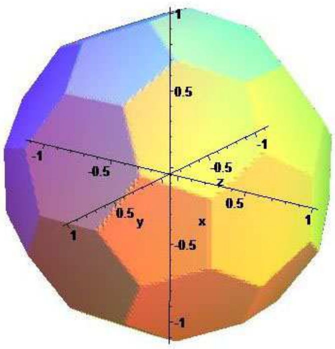

Figure 6 The unit sphere in terms of d_{TI}: Truncated icosahedron

Corrolary 3.1. *The equation of the truncated icosahedron with center (x_0, y_0, z_0) and radius r is*

$$(\varphi-2)\max \left\{ \begin{array}{l} |x-x_0| + \max\left\{ \begin{array}{r} (2-\varphi)(|x-x_0|+|z-z_0|), (\varphi-2)(|y-y_0|+|z-z_0|), \\ (\frac{31-16\varphi}{19})|x-x_0| + \frac{12\varphi-9}{19}|y-y_0| \end{array} \right\}, \\ |y-y_0| + \max\left\{ \begin{array}{r} (2-\varphi)(|x-x_0|+|y-y_0|), (\varphi-2)(|x-x_0|+|z-z_0|), \\ (\frac{31-16\varphi}{19})|y-y_0| + \frac{12\varphi-9}{19}|z-z_0| \end{array} \right\}, \\ |z-z_0| + \max\left\{ \begin{array}{r} (2-\varphi)(|y-y_0|+|z-z_0|), (\varphi-2)(|x-x_0|+|y-y_0|), \\ (\frac{31-16\varphi}{19})|z-z_0| + \frac{12\varphi-9}{19}|x-x_0| \end{array} \right\} \end{array} \right\} = r$$

which is a polyhedron which has 30 faces and 60 vertices. Coordinates of the vertices are translation to (x_0, y_0, z_0) all posible +/- sign components of the points $\left(\frac{\beta}{2}r, 0, r\right)$, $\left(r, \frac{\beta}{2}r, 0\right)$, $\left(0, r, \frac{\beta}{2}r\right)$, $\left(\beta r, \frac{1}{3}r, \frac{3\beta+4}{6}r\right)$, $\left(\frac{3\beta+4}{6}r, \beta r, \frac{1}{3}r\right)$, $\left(\frac{1}{3}r, \frac{3\beta+4}{6}r, \beta r\right)$, $\left(\frac{\beta}{2}r, \frac{2}{3}r, \frac{\sqrt{5}}{3}r\right)$, $\left(\frac{\sqrt{5}}{3}r, \frac{\beta}{2}r, \frac{2}{3}r\right)$ and $\left(\frac{2}{3}r, \frac{\sqrt{5}}{3}r, \frac{\beta}{2}r\right)$, where $\beta = \frac{\sqrt{5}-1}{3}$.

Lemma 3.2. *Let l be the line through the points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p, q, r), then*

$$d_{TI}(P_1, P_2) = \mu(P_1P_2)d_E(P_1, P_2)$$

where

$$\mu(P_1P_2)=\frac{(\varphi-2)\max\left\{\begin{array}{c}|p|+\max\left\{\begin{array}{r}(2-\varphi)(|p|+|r|),(\varphi-2)(|q|+|r|),\\(\frac{31-16\varphi}{19})|p|+\frac{12\varphi-9}{19}|q|\end{array}\right\},\\|q|+\max\left\{\begin{array}{r}(2-\varphi)(|p|+|q|),(\varphi-2)(|p|+|r|),\\(\frac{31-16\varphi}{19})|q|+\frac{12\varphi-9}{19}|r|\end{array}\right\},\\|r|+\max\left\{\begin{array}{r}(2-\varphi)(|q|+|r|),(\varphi-2)(|p|+|q|),\\(\frac{31-16\varphi}{19})|r|+\frac{12\varphi-9}{19}|p|\end{array}\right\}\end{array}\right\}}{\sqrt{p^2+q^2+r^2}}.$$

Proof. Equation of l gives us $x_1-x_2=\lambda p$, $y_1-y_2=\lambda q$, $z_1-z_2=\lambda r$, $r\in\mathbb{R}$. Thus,

$$d_{TI}(P_1,P_2)=|\lambda|\left((\varphi-2)\max\left\{\begin{array}{c}|p|+\max\left\{\begin{array}{r}(2-\varphi)(|p|+|r|),(\varphi-2)(|q|+|r|),\\(\frac{31-16\varphi}{19})|p|+\frac{12\varphi-9}{19}|q|\end{array}\right\},\\|q|+\max\left\{\begin{array}{r}(2-\varphi)(|p|+|q|),(\varphi-2)(|p|+|r|),\\(\frac{31-16\varphi}{19})|q|+\frac{12\varphi-9}{19}|r|\end{array}\right\},\\|r|+\max\left\{\begin{array}{r}(2-\varphi)(|q|+|r|),(\varphi-2)(|p|+|q|),\\(\frac{31-16\varphi}{19})|r|+\frac{12\varphi-9}{19}|p|\end{array}\right\}\end{array}\right\}\right)$$

and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the required result.

The above lemma says that d_{RT}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corrolary 3.2. *If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then $d_E(P_1,X)=d_E(P_2,X)$ if and only if $d_{TI}(P_1,X)=d_{TI}(P_2,X)$.*

Corrolary 3.3. *If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then*

$$d_{TI}(X,P_1)\ /\ d_{TI}(X,P_2)=d_E(X,P_1)\ /\ d_E(X,P_2)\ .$$

That is, the ratios of the Euclidean and $d_{TI}-$distances along a line are the same.

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