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CHARACTERIZATIONS OF EXBICENTRIC QUADRILATERALS

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Abstract. We prove twelve necessary and sufficient conditions for when a convex quadrilateral has both an excircle and a circumcircle.

1. Introduction

This is the third and final paper in an extensive study of extangential quadrilaterals, preceded by [9] and [10]. An *extangential quadrilateral* is a convex quadrilateral with an *excircle*, that is, an external circle tangent to the extensions of all four sides. A related quadrilateral with an incircle is called a tangential quadrilateral. In contrast to a triangle, a quadrilateral can at most have one excircle [6, p.64].

A *cyclic* extangential quadrilateral, i.e. one that also has a circumcircle (a circle that touches each vertex) was called an *exbicentric quadrilateral* [1] in [11, p.44]. These have only been studied scarcely before. Five area formulas were deduced in [9, §7]. In this paper we shall prove twelve characterizations of exbicentric quadrilaterals that concerns angles or metric relations. We assume the quadrilateral is extangential and explore what additional properties it must have to be cyclic.

The following general notations and concepts will be used. If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of the sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ at W, X, Y, Z respectively, then the distances $e = AW$, $f = BX$, $g = CY$, $h = DZ$ are called the *tangent lengths*, and the distances WY and XZ are called the *tangency chords*, see Figure 1.

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[1]A tangential quadrilateral that is also cyclic is called a bicentric quadrilateral, see [4] and [5].

2. A concurrency in extangential quadrilaterals

It is well known that the two diagonals and the two tangency chords in a tangential quadrilateral are concurrent (see [13]). We start by proving that an extangential quadrilateral has a similar property. However, the two tangency chords in an extangential quadrilateral never intersect, but rather its their extensions that do so.

Theorem 2.1. *The two diagonals and the extensions of the two tangency chords in an extangential quadrilateral are concurrent.*

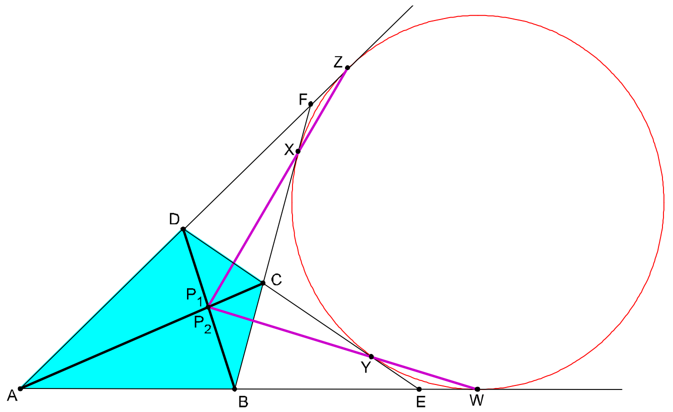

Figure 1. Extangential quadrilateral with intersecting tangency chords

Proof. In an extangential quadrilateral $ABCD$ where opposite sides AB and DC intersect at E and the other two sides intersect at F, let the extensions of WY and ZX intersect AC in P_1 and P_2 respectively (see Figure 1). We apply Menelaus' theorem (with non-directed distances) in triangles ACE and ACF using the transversals WYP_1 and ZXP_2 respectively. This yields

$$\frac{EW}{WA}\cdot\frac{AP_1}{P_1C}\cdot\frac{CY}{YE}=1 \quad\Rightarrow\quad \frac{AP_1}{P_1C}=\frac{WA}{CY},$$
$$\frac{FZ}{ZA}\cdot\frac{AP_2}{P_2C}\cdot\frac{CX}{XF}=1 \quad\Rightarrow\quad \frac{AP_2}{P_2C}=\frac{ZA}{CX}$$

since $EW = YE$ and $FZ = XF$ according to the two tangent theorem.[2] We further have $WA = ZA$ and $CY = CX$ by the same theorem, so we get

$$\frac{AP_1}{P_1C}=\frac{AP_2}{P_2C}.$$

This proves that $P_1 \equiv P_2$ since these two points divide the diagonal AC in the same ratio. Hence WY, ZX and AC intersect in a point on AC.

In the second half of the proof, consider triangles BDE and BDF. We apply Menelaus' theorem in these triangles using the same two transversals as before. If they intersect BD in Q_1 and Q_2, then

$$\frac{EW}{WB}\cdot\frac{BQ_1}{Q_1D}\cdot\frac{DY}{YE}=1 \quad\Rightarrow\quad \frac{BQ_1}{Q_1D}=\frac{WB}{DY},$$
$$\frac{FZ}{ZD}\cdot\frac{DQ_2}{Q_2B}\cdot\frac{BX}{XF}=1 \quad\Rightarrow\quad \frac{BQ_2}{Q_2D}=\frac{XB}{ZD}.$$

[2]The two tangents to a circle through an external point have the same lengths.

But $WB = XB$ and $DY = ZD$, so it follows that

$$\frac{BQ_1}{Q_1D} = \frac{BQ_2}{Q_2D}.$$

This proves that $Q_1 \equiv Q_2$. Hence WY, ZX, and BD intersect in a point on BD. But the lines WY and ZX can only have one point of intersection, which proves that all four of WY, ZX, AC and BD intersect in the same point. □

3. Characterizations concerning angle bisectors

The first two characterizations of exbicentric quadrilaterals are about different angle bisectors. The following necessary and sufficient condition is related to the theorem we just proved.

Theorem 3.1. *An extangential quadrilateral is also cyclic if and only if the extension of a tangency chords is an angle bisector to one of the angles between the diagonals.*

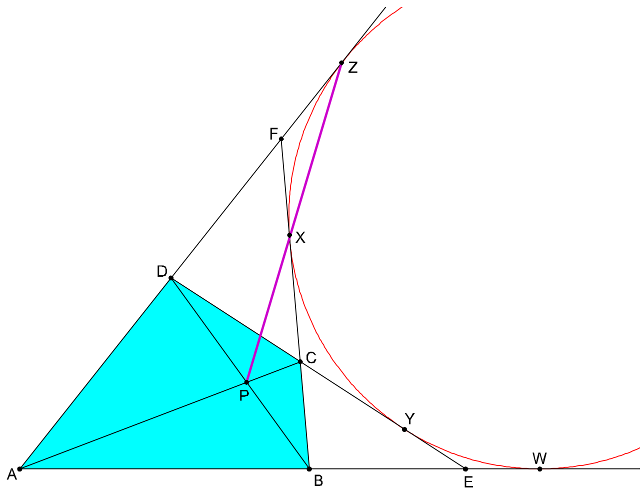

Figure 2. Extangential quadrilateral with one tangency chord

Proof. The extension of the tangency chord XZ and the two diagonals are concurrent at a point P according to Theorem 2.1. If the extensions of the sides AD and BC intersect at F, then triangle FXZ is isosceles according to the two tangent theorem. Thus we have that $\angle DZP = \angle FXZ = \angle CXP$, see Figure 2. It is well known that a convex quadrilateral $ABCD$ is cyclic if and only if $\angle ADB = \angle BCA$. This is equivalent to $\angle ZDP = \angle XCP$, which in turn is equivalent to $\angle DPZ = \angle CPX$ since $\angle DZP = \angle CXP$. Hence we have proved that the extangential quadrilateral $ABCD$ is cyclic if and only if ZP is an angle bisector to the angle CPD between the diagonals. By symmetry, the result also holds for the other tangency chord. □

In [4, Thm 5] we proved an angle characterization of bicentric quadrilaterals, where the considered angle was between the two angle bisectors to the angles created at the intersections of the extensions of opposite sides. That proof was elementary, but here we will give an even simpler proof of the corresponding characterization of an exbicentric quadrilateral.[3]

Theorem 3.2. *Let the extensions of opposite sides in an extangential quadrilateral intersect at E and F. If the external angle bisectors at E and F intersect at J, then the quadrilateral is also cyclic if and only if EJF is a right angle.*

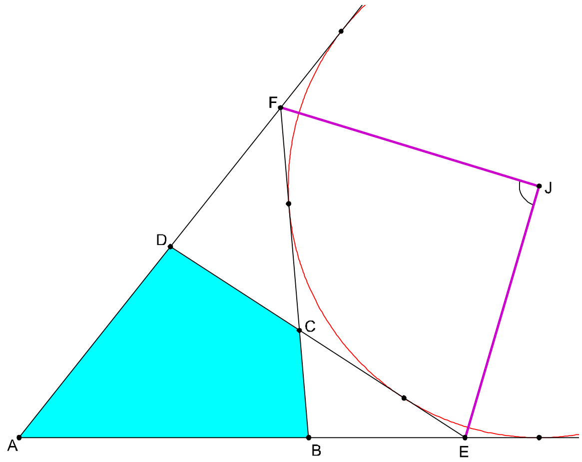

Figure 3. Extangential quadrilateral with two external angle bisectors

Proof. Let $ABCD$ be the extangential quadrilateral. The external angle to triangle ABF at F is $A+B$, and the external angle to triangle ADE at E is $A+D$. Thus using the sum of angles in quadrilateral $CEJF$, we get (see Figure 3)

$$\angle EJF = 2\pi - C - \frac{A+B}{2} - \frac{A+D}{2} = 2\pi - \frac{A+B+C+D}{2} - \frac{A+C}{2}.$$

By the sum of angles in quadrilateral $ABCD$, this is simplified to

$$\angle EJF = \pi - \frac{A+C}{2}.$$

Hence we have that $\angle EJF = \frac{\pi}{2}$ if and only if $A+C=\pi$, which is a well-known characterization for $ABCD$ to be a cyclic quadrilateral.[4] □

4. Characterizations concerning perpendicular lines

There are several more characterizations of exbicentric quadrilaterals that concern perpendicular lines (besides the one in Theorem 3.2). Now we prove two and in the next section we study an additional two where rectangles are involved. The following has a well-known counterpart for bicentric quadrilaterals, see [3, p.124].

Theorem 4.1. *If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of AB, BC, CD, DA at W, X, Y, Z respectively, then the quadrilateral is also cyclic if and only if the extensions of the tangency chords WY and XZ are perpendicular.*

[3]The same method can be used to prove when a tangential quadrilateral is cyclic.

[4]We note that the point J is the *excenter* (the center of the excircle), see [6, pp.63–64].

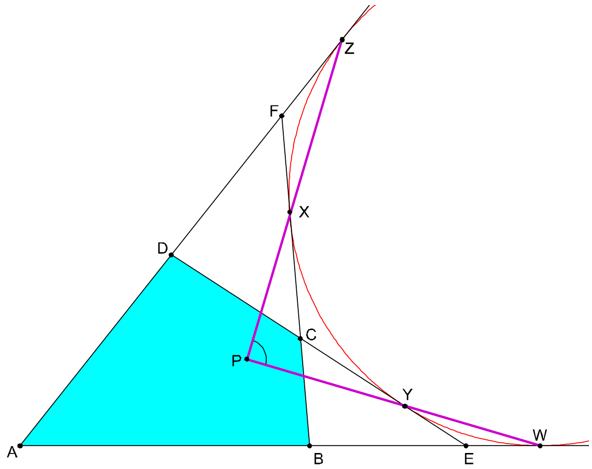

Figure 4. Extangential quadrilateral with two tangency chord

Proof. Let the extensions of opposite sides AB, DC and AD, BC in an extangential quadrilateral intersect at E and F respectively. The external angle to triangle ABF at F is $A+B$. Triangle XFZ is isosceles according to the two tangent theorem. If the extensions of the tangency chords intersect at P, then $\angle PXC = \frac{1}{2}(\pi - A - B)$. In the same way, $\angle PYC = \frac{1}{2}(\pi - A - D)$. According to Lemma 4.1 in [10] it holds that[5] (see Figure 4)

$$\angle C = \angle A + \angle E + \angle F \tag{1}$$

in the *concave* quadrilateral $AECF$ if $\angle C > \angle A$. Applying this equality in the concave quadrilateral $PYCX$ yields

$$\angle XPY = C - \frac{\pi - A - B}{2} - \frac{\pi - A - D}{2} = -\pi + \frac{A+B+C+D}{2} + \frac{A+C}{2}.$$

This is simplified to

$$\angle XPY = \frac{A+C}{2}$$

and we have that $\angle XPY = \frac{\pi}{2}$ if and only if $A + C = \pi$, which is true if and only if $ABCD$ is a cyclic quadrilateral. □

Theorem 9 in [7] gave a characterization of bicentric quadrilaterals, where two incircles were involved. A similar necessary and sufficient condition holds for exbicentric quadrilaterals according to the next theorem.

Theorem 4.2. *In an extangential quadrilateral $ABCD$ with an excircle outside of C, let the extensions of opposite sides intersect at E and F. Also, let the incircle in triangle CEF be tangent to the extensions of BC and DC at M and N respectively, and the excircle to side EF in triangle AEF be tangent to the extensions of AB and AD at K and L respectively. Then $ABCD$ is also cyclic if and only if the extensions of LM and NK are perpendicular.*

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[5] This formula is easy to prove using the sum of angles in a triangle and a quadrilateral.

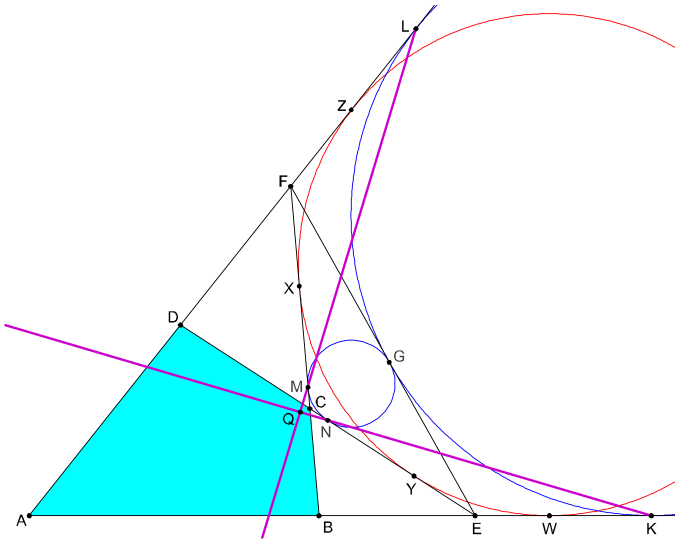

Figure 5. Extangential quadrilateral with a related incircle and excircle

Proof. The incircle in triangle CEF and the excircle to triangle AEF are tangent to EF at the same point G (see Figure 4) according to Theorem 3.3 (ii) in [9]. Thus $EN = EG = EK$ according to the two tangent theorem, so triangle ENK is isosceles. Let the extensions of LM and KN intersect at Q. The external angle at E in triangle ADE is $A + D$, so we have that $\angle QNC = \angle ENK = \frac{1}{2}(\pi - A - D)$. In the same way we get $\angle QMC = \frac{1}{2}(\pi - A - B)$. Applying the angle equality (1) in the concave quadrilateral $CMQN$ yields

$$\angle MQN = C - \angle QMC - \angle QNC = -\pi + \frac{A+B+C+D}{2} + \frac{A+C}{2}$$

which simplifies to

$$\angle MQN = \frac{A+C}{2}.$$

Hence the angle MQN between the extensions of LM and NK are perpendicular if and only if $ABCD$ is a cyclic quadrilateral. □

5. Characterizations concerning rectangles

The characterizations of exbicentric quadrilaterals in this section are corollaries to Theorems 3.2 and 4.1, but we state them here as theorems anyway. They are both about rectangles.

Theorem 5.1. *Let the extensions of opposite sides in an extangential quadrilateral intersect at E and F. If the internal and external angle bisectors at these points intersect at I and J respectively, then the extangential quadrilateral is also cyclic if and only if quadrilateral $EIFJ$ is a rectangle.*

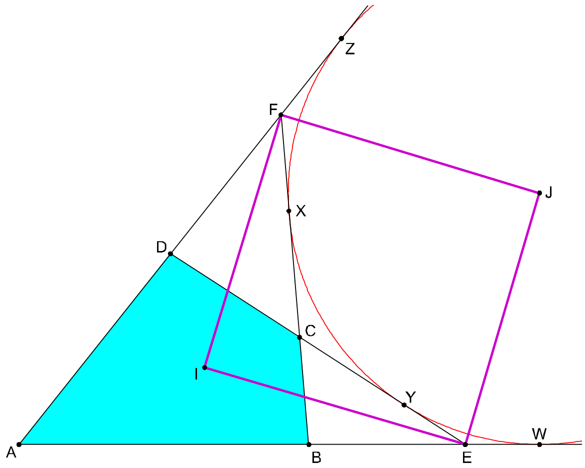

Figure 6. Extangential quadrilateral with the quadrilateral $EIFJ$

Proof. The angle between an internal and an external angle bisector in a triangle is always a right angle, so $\angle IEJ = \angle IFJ = \frac{\pi}{2}$ in all convex quadrilaterals. A convex quadrilateral is a rectangle if and only if three of its angles are right angles, so this characterization is a direct consequence of Theorem 3.2. □

The Varignon parallelogram is the parallelogram formed in any quadrilateral by joining the midpoints of the sides. The *contact quadrilateral* in a tangential or extangential quadrilateral is obtained by joining the tangency points between the sides and the incircle or excircle. In [4, p.166] we reviewed a characterization of bicentric quadrilaterals which can be stated that the Varignon parallelogram in the contact quadrilateral belonging to a tangential quadrilateral is a rectangle if and only if the tangential quadrilateral is also cyclic. To prove that theorem was an Olympiad problem in China in 2003. There is the similar necessary and sufficient condition for extangential quadrilaterals to be cyclic, as we will prove now.

Theorem 5.2. *The Varignon parallelogram in the contact quadrilateral belonging to an extangential quadrilateral is a rectangle if and only if the extangential quadrilateral is also cyclic.*

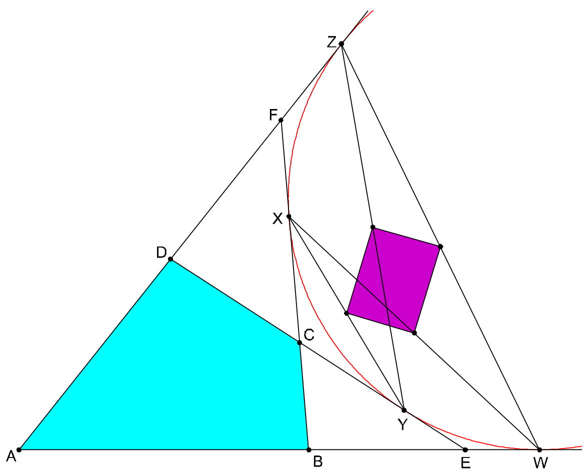

Figure 7. The Varignon parallelogram in the contact quadrilateral

Proof. It is well known that the Varignon parallelogram belonging to a quadrilateral is a rectangle if and only if the diagonals in the quadrilateral are perpendicular, see [8, p.137]. The diagonals in the contact quadrilateral $WXYZ$ (which is crossed) are the tangency chords WY and XZ, see Figure 7. According to Theorem 4.1, they are perpendicular if and only if the extangential quadrilateral is also cyclic. □

6. Characterizations concerning metric relations

Next we prove three characterizations of exbicentric quadrilaterals that are corollaries to formulas in our paper [10]. The last has already been proved in Theorem 4.1 in the present paper, so here we get a second proof of it.

Theorem 6.1. *In an extangential quadrilateral with consecutive sides a, b, c, d and tangent lengths e, f, g, h, the following statements are equivalent:*
(i) *The quadrilateral is cyclic.*
(ii) *The quadrilateral has area $K = \sqrt{abcd}$*
(iii) $eg = fh$
(iv) *The tangency chords are perpendicular.*

Proof. The area of an extangential quadrilateral is given by the formula

$$K = \sqrt{abcd}\sin\frac{A+C}{2}$$

according to Theorem 3.2 (i) in [10]. Thus the area is given by $K = \sqrt{abcd}$ if and only if opposite angles are supplementary angles, which is a well-known characterization of a cyclic quadrilateral. This proves that statements (i) and (ii) are equivalent.

The area of an extangential quadrilateral is also given by the formula

$$K = \sqrt{abcd - (eg - fh)^2}$$

according to Theorem 3.2 (iv) in [10]. Hence we directly get that statements (ii) and (iii) are equivalent, since

$$K = \sqrt{abcd - (eg - fh)^2} = \sqrt{abcd} \quad \Leftrightarrow \quad eg = fh.$$

The acute angle φ between the tangency chords in an extangential quadrilateral is given by

$$\cos\varphi = \frac{|eg - fh|}{\sqrt{(e-f)(f-g)(g-h)(h-e)}}$$

according to Theorem 4.2 in [10]. The equivalence of (iii) and (iv) is a direct consequence of this formula, since $\cos\varphi = 0$ if and only if $\varphi = \frac{\pi}{2}$. □

It is interesting that the same three necessary and sufficient conditions (ii), (iii) and (iv) hold for when a tangential quadrilateral is cyclic (a bicentric quadrilateral). That was proved in [5, pp.156–157], [2, p.104] and [3, p.124] respectively, but those characterizations of bicentric quadrilaterals were probably known long before those publications.

The radius in the excircle to an extangential quadrilateral is called the *exradius* and it is a part of the following characterization. The corresponding relation in a bicentric quadrilateral was the subject of Juan Carlos Salazar's question at [12].

Theorem 6.2. *An extangential quadrilateral $ABCD$ with excenter J is also cyclic if and only if*

$$\frac{1}{AJ^2}+\frac{1}{CJ^2}=\frac{1}{\rho^2}=\frac{1}{BJ^2}+\frac{1}{DJ^2}$$

where ρ is the exradius.

Proof. ($\Rightarrow$) In an extangential quadrilateral, we have $\frac{1}{AJ}=\frac{1}{\rho}\sin\frac{A}{2}$ since AJ is an angle bisector to vertex angle A. A similar equality holds for CJ. Thus

$$\frac{1}{AJ^2}+\frac{1}{CJ^2}=\frac{1}{\rho^2}\left(\sin^2\frac{A}{2}+\sin^2\frac{C}{2}\right). \tag{2}$$

Cyclic quadrilaterals have the property that opposite angles are supplementary angles, so $\sin\frac{C}{2}=\cos\frac{A}{2}$. The first equality now follows from the trigonometric Pythagorean identity $\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1$. The second equality is proved in the same way.

($\Leftarrow$) If $\frac{1}{AJ^2}+\frac{1}{CJ^2}=\frac{1}{\rho^2}$ holds in an extangential quadrilateral, then

$$\sin^2\frac{A}{2}+\sin^2\frac{C}{2}=1 \tag{3}$$

by (2). For all angles C we have $\sin^2\frac{C}{2}+\cos^2\frac{C}{2}=1$, so (3) yields that

$$\sin^2\frac{A}{2}=\cos^2\frac{C}{2}\quad\Rightarrow\quad\pm\sin\frac{A}{2}=\sin\left(\frac{\pi}{2}-\frac{C}{2}\right).$$

The four solutions to this equation satisfy $\pm\frac{A}{2}=\frac{\pi}{2}-\frac{C}{2}$ and $\pm\frac{A}{2}=\pi-\left(\frac{\pi}{2}-\frac{C}{2}\right)$. These can be summarized into $\pm A\pm C=\pi$. Here the solution with double minus signs is obviously false, but so are the two with one minus sign. This is because a vertex angle in an extangential quadrilateral must be less than π (it is a convex quadrilateral). Hence $A+C=\pi$ is the only valid solution, implying it's a cyclic quadrilateral. The converse for the second formula is proved in the same way. $\square$

The next two characterizations concern the quotient of the diagonals $p=AC$ and $q=BD$ in an extangential quadrilateral $ABCD$. First we need to prove a lemma. The following formulas are the counterparts to the ones derived by Mowaffaq Hajja in [2] for the tangential quadrilateral.[6]

Lemma 6.1. *The lengths of the diagonals AC and BD in an extangential quadrilateral $ABCD$ with tangent lengths e, f, g, h are respectively*

$$p=\sqrt{\frac{e+g}{f+h}\big((e+g)(f+h)-4fh\big)},$$

$$q=\sqrt{\frac{f+h}{e+g}\big((e+g)(f+h)-4eg\big)}.$$

[6] He called them circumscriptible quadrilaterals, which is a synonym. Note that he used a, b, c, d for the tangent lengths.

Proof. In [10] we concluded that the only difference between formulas in terms of the tangent lengths in tangential and extangential quadrilaterals are the signs of f and h. We use that symmetry here to quickly get the formulas in this lemma. Starting with Hajja's formula for the length of diagonal p in a tangential quadrilateral (see [2, p.104]),

$$p = \sqrt{\frac{e+g}{f+h}\big((e+g)(f+h)+4fh\big)},$$

and making the changes $f \to -f$ and $h \to -h$ yields the formula that holds in an extangential quadrilateral

$$p = \sqrt{\frac{e+g}{-f-h}\big((e+g)(-f-h)+4(-f)(-h)\big)}.$$

This is equivalent to the first formula in the theorem. The second follows in the same way from

$$q = \sqrt{\frac{f+h}{e+g}\big((e+g)(f+h)+4eg\big)}$$

which holds in a tangential quadrilateral according to [2, p.104]. $\square$

Another (longer) way of deriving the two formulas for an extangential quadrilaterals is to mimic the method used by Hajja in [2].

We can use the formulas in Lemma 6.1 to get a second proof of the characterization in Theorem 6.1 (iii). A convex quadrilateral is cyclic if and only if its consecutive sides a, b, c, d and diagonals p, q satisfy Ptolemy's theorem $pq = ac + bd$. When the excircle to an extangential quadrilateral is outside C, this is equivalent to

$$\big((e+g)(f+h)-4fh\big)\big((e+g)(f+h)-4eg\big) = \big((e-f)(h-g)+(f-g)(e-h)\big)^2.$$

Expanding, simplifying and factoring this equality yields $4(eg - fh)^2 = 0$, which is equivalet to $eg = fh$.

Now we use the formulas in Lemma 6.1 to prove a characterization of exbicentric quadrilaterals that is the counterpart to one for bicentric quadrilaterals which was proved by Hajja in [2]. That characterization as well as the one in the following theorem was previously derived in [11] using rather long arguments.

Theorem 6.3. *An extangential quadrilateral with tangent lengths e, f, g, h is also cyclic if and only if the quotient of the diagonals satisfies*

$$\frac{p}{q} = \frac{e+g}{f+h}.$$

Proof. An extangential quadrilateral is also cyclic if and only if the tangent lengths satisfy $eg = fh$ according to Theorem 6.1 (iii). This condition is equivalent to

$$\begin{aligned} &(e+g)(f+h)-4fh=(e+g)(f+h)-4eg \\ \Leftrightarrow\quad &\frac{p^2(f+h)}{(e+g)}=\frac{q^2(e+g)}{(f+h)} \\ \Leftrightarrow\quad &\frac{p^2}{q^2}=\frac{(e+g)^2}{(f+h)^2} \end{aligned}$$

which is equivalent to the equality we want to prove. In the first equivalence we used the formulas in Lemma 6.1. □

Corollary 6.1. *An extangential quadrilateral $ABCD$ with excenter J is also cyclic if and only if the quotient of the diagonals satisfies*

$$\frac{p}{q}=\frac{AJ\cdot CJ}{BJ\cdot DJ}.$$

Proof. Corollary 3.1 in [10] states that the tangent lengths in all extangential quadrilaterals (except parallelograms) satisfy

$$\frac{e+g}{f+h}=\frac{AJ\cdot CJ}{BJ\cdot DJ}.$$

Combining this equality with Theorem 6.3 directly yields this second characterization in terms of the quotient of the diagonals. □

7. Perpendicular opposite sides in a cyclic quadrilateral

In Theorem 4.2 we proved a characterization for exbicentric quadrilaterals with perpendicular opposite sides in an associated cyclic quadrilateral ($KLMN$ is cyclic according to Theorem 3.4 in [9]). Here we shall study another such configuration where the condition is expressed as a metric relation. We start by proving a necessary and sufficient condition in a cyclic quadrilateral for the extensions of opposite sides to be perpendicular. The direct part of the theorem was a problem in [1, p.31].

Theorem 7.1. *The extensions of opposite sides b and d in a cyclic quadrilateral with consecutive sides a, b, c, d are perpendicular if and only if*

$$(ab+cd)^2+(ad+bc)^2=(a^2-c^2)^2$$

as long as the other two sides does not have the same lengths.

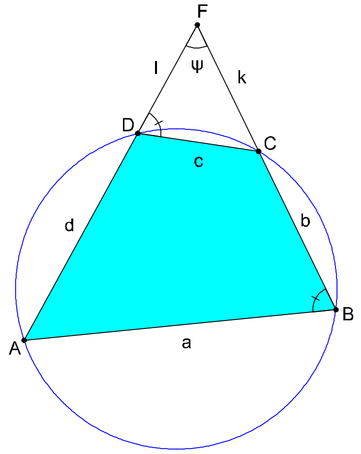

Figure 8. Cyclic quadrilateral with angle between two extended sides

Proof. Let the extensions of the sides b and d intersect at F, and $k = CF$ and $l = DF$. The triangles CDF and ABF are similar (by AAA) since the angle at B and the external angle at D are equal, see Figure 8. Thus

$$\frac{k}{l+d} = \frac{c}{a}, \qquad \frac{l}{k+b} = \frac{c}{a}.$$

We get $ak = cl + cd$ and $al = ck + bc$. Multiply the first equation by c and the second by a to get $ack = c^2l + c^2d$ and $a^2l = ack + abc$. These yields

$$a^2l = c^2l + c^2d + abc \quad \Leftrightarrow \quad l = \frac{c(ab+cd)}{a^2-c^2}, \ a \neq c. \tag{4}$$

By symmetry we get a formula for k by making the change $b \leftrightarrow d$ in (4). Whence

$$k = \frac{c(ad+bc)}{a^2-c^2}, \ a \neq c. \tag{5}$$

Now we apply the Law of cosines in triangle CDF to get

$$c^2 = k^2 + l^2 - 2kl\cos\psi$$

where ψ is the angle between the extensions of b and d. Inserting (5) and (4), dividing both sides by c^2 and multiplying them by $(a^2-c^2)^2$ yields

$$(a^2-c^2)^2 = (ad+bc)^2 + (ab+cd)^2 - 2(ad+bc)(ab+cd)\cos\psi.$$

Here we see that the extensions of b and d are perpendicular if and only if the equality in the theorem holds, since $\cos\psi = 0$ if and only if $\psi = \frac{\pi}{2}$. □

We conclude this paper by returning to the configuration we studied in Theorem 4.1 and stating a characterization of exbicentric quadrilaterals that concerns the quadrilateral $WYXZ$. Note that this is the convex version of the crossed quadrilateral $WXYZ$, see Figure 7. Since $WYXZ$ is convex, the tangency chords (diagonals in the contact quadrilateral) are now two opposite sides.

Corollary 7.1. *If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of AB, BC, CD, DA at W, X, Y, Z respectively, then the quadrilateral is also cyclic if and only if*

$$(XY\cdot YW+WZ\cdot ZX)^2+(YX\cdot XZ+ZW\cdot WY)^2=(WY^2-XZ^2)^2.$$

Proof. The extangential quadrilateral is also cyclic if and only if the tangency chords are perpendicular according to Theorem 4.1. They are perpendicular if and only if the equation in this corollary is satisfied according to Theorem 7.1. □

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