CHARACTERIZATIONS OF EXBICENTRIC QUADRILATERALS

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Abstract. We prove twelve necessary and sufficient conditions for when a convex quadrilateral has both an excircle and a circumcircle.

1. Introduction

This is the third and final paper in an extensive study of extangential quadrilaterals, preceded by [9] and [10]. An extangential quadrilateral is a convex quadrilateral with an excircle, that is, an external circle tangent to the extensions of all four sides. A related quadrilateral with an incircle is called a tangential quadrilateral. In contrast to a triangle, a quadrilateral can at most have one excircle [6, p.64].

A cyclic extangential quadrilateral, i.e. one that also has a circumcircle (a circle that touches each vertex) was called an exbicentric quadrilateral in [11, p.44]. These have only been studied scarcely before. Five area formulas were deduced in [9, §7]. In this paper we shall prove twelve characterizations of exbicentric quadrilaterals that concerns angles or metric relations. We assume the quadrilateral is extangential and explore what additional properties it must have to be cyclic.

The following general notations and concepts will be used. If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of the sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ at $W$, $X$, $Y$, $Z$ respectively, then the distances $e = AW$, $f = BX$, $g = CY$, $h = DZ$ are called the tangent lengths, and the distances $WY$ and $XZ$ are called the tangency chords, see Figure 1.

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1 A tangential quadrilateral that is also cyclic is called a bicentric quadrilateral, see [4] and [5].
2. A CONCURRENCE IN EXTANGENTIAL QUADRILATERALS

It is well known that the two diagonals and the two tangency chords in a tangential quadrilateral are concurrent (see [13]). We start by proving that an extangential quadrilateral has a similar property. However, the two tangency chords in an extangential quadrilateral never intersect, but rather its their extensions that do so.

**Theorem 2.1.** The two diagonals and the extensions of the two tangency chords in an extangential quadrilateral are concurrent.

![Figure 1. Extangential quadrilateral with intersecting tangency chords](image)

**Proof.** In an extangential quadrilateral $ABCD$ where opposite sides $AB$ and $DC$ intersect at $E$ and the other two sides intersect at $F$, let the extensions of $WY$ and $ZX$ intersect $AC$ in $P_1$ and $P_2$ respectively (see Figure 1). We apply Menelaus’ theorem (with non-directed distances) in triangles $ACE$ and $ACF$ using the transversals $WYP_1$ and $ZXP_2$ respectively. This yields

$$\frac{EW}{WA} \cdot \frac{AP_1}{P_1C} \cdot \frac{CY}{YE} = 1 \Rightarrow \frac{AP_1}{P_1C} = \frac{WA}{CY}.$$  

$$\frac{FZ}{ZA} \cdot \frac{AP_2}{P_2C} \cdot \frac{CX}{XF} = 1 \Rightarrow \frac{AP_2}{P_2C} = \frac{ZA}{CX}$$

since $EW = YE$ and $FZ = XF$ according to the two tangent theorem.\(^2\)

We further have $WA = ZA$ and $CY = CX$ by the same theorem, so we get

$$\frac{AP_1}{P_1C} = \frac{AP_2}{P_2C}.$$  

This proves that $P_1 \equiv P_2$ since these two points divide the diagonal $AC$ in the same ratio. Hence $WY$, $ZX$ and $AC$ intersect in a point on $AC$.

In the second half of the proof, consider triangles $BDE$ and $BDF$. We apply Menelaus’ theorem in these triangles using the same two transversals as before. If they intersect $BD$ in $Q_1$ and $Q_2$, then

$$\frac{EW}{WB} \cdot \frac{BQ_1}{Q_1D} \cdot \frac{DY}{YE} = 1 \Rightarrow \frac{BQ_1}{Q_1D} = \frac{WB}{DY},$$  

$$\frac{FZ}{ZD} \cdot \frac{DQ_2}{Q_2B} \cdot \frac{BX}{XF} = 1 \Rightarrow \frac{BQ_2}{Q_2D} = \frac{XB}{ZD}.$$  

\(^2\)The two tangents to a circle through an external point have the same lengths.
But $WB = XB$ and $DY = ZD$, so it follows that

$$\frac{BQ_1}{Q_1D} = \frac{BQ_2}{Q_2D}.$$ 

This proves that $Q_1 \equiv Q_2$. Hence $WY$, $ZX$, and $BD$ intersect in a point on $BD$. But the lines $WY$ and $ZX$ can only have one point of intersection, which proves that all four of $WY$, $ZX$, $AC$ and $BD$ intersect in the same point.

3. Characterizations concerning angle bisectors

The first two characterizations of exbicentric quadrilaterals are about different angle bisectors. The following necessary and sufficient condition is related to the theorem we just proved.

**Theorem 3.1.** An extangential quadrilateral is also cyclic if and only if the extension of a tangency chords is an angle bisector to one of the angles between the diagonals.

**Proof.** The extension of the tangency chord $XZ$ and the two diagonals are concurrent at a point $P$ according to Theorem 2.1. If the extensions of the sides $AD$ and $BC$ intersect at $F$, then triangle $FXZ$ is isosceles according to the two tangent theorem. Thus we have that $\angle DZP = \angle FXZ = \angle CXP$, see Figure 2. It is well known that a convex quadrilateral $ABCD$ is cyclic if and only if $\angle ADB = \angle BCA$. This is equivalent to $\angle ZDP = \angle XCP$, which in turn is equivalent to $\angle DPZ = \angle CPX$ since $\angle DZP = \angle CXP$. Hence we have proved that the extangential quadrilateral $ABCD$ is cyclic if and only if $ZP$ is an angle bisector to the angle $CPD$ between the diagonals. By symmetry, the result also holds for the other tangency chord. \qed

In [4, Thm 5] we proved an angle characterization of bicentric quadrilaterals, where the considered angle was between the two angle bisectors to the angles created at the intersections of the extensions of opposite sides. That
proof was elementary, but here we will give an even simpler proof of the corresponding characterization of an exbicentric quadrilateral.

**Theorem 3.2.** Let the extensions of opposite sides in an extangential quadrilateral intersect at $E$ and $F$. If the external angle bisectors at $E$ and $F$ intersect at $J$, then the quadrilateral is also cyclic if and only if $EJF$ is a right angle.

![Figure 3. Extangential quadrilateral with two external angle bisectors](image_url)

**Proof.** Let $ABCD$ be the extangential quadrilateral. The external angle to triangle $ABF$ at $F$ is $A + B$, and the external angle to triangle $ADE$ at $E$ is $A + D$. Thus using the sum of angles in quadrilateral $CEJF$, we get (see Figure 3)

$$\angle EJF = 2\pi - C - \frac{A + B}{2} - \frac{A + D}{2} = 2\pi - \frac{A + B + C + D}{2} - \frac{A + C}{2}.$$

By the sum of angles in quadrilateral $ABCD$, this is simplified to

$$\angle EJF = \pi - \frac{A + C}{2}.$$

Hence we have that $\angle EJF = \frac{\pi}{2}$ if and only if $A + C = \pi$, which is a well-known characterization for $ABCD$ to be a cyclic quadrilateral.\(^4\) \(\Box\)

4. **Characterizations concerning perpendicular lines**

There are several more characterizations of exbicentric quadrilaterals that concern perpendicular lines (besides the one in Theorem 3.2). Now we prove two and in the next section we study an additional two where rectangles are involved. The following has a well-known counterpart for bicentric quadrilaterals, see [3, p.124].

**Theorem 4.1.** If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of $AB$, $BC$, $CD$, $DA$ at $W$, $X$, $Y$, $Z$ respectively, then the quadrilateral is also cyclic if and only if the extensions of the tangency chords $WY$ and $XZ$ are perpendicular.

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\(^3\)The same method can be used to prove when a tangential quadrilateral is cyclic.

\(^4\)We note that the point $J$ is the excenter (the center of the excircle), see [6, pp.63–64].
Figure 4. Extangential quadrilateral with two tangency chord

Proof. Let the extensions of opposite sides $AB$, $DC$ and $AD$, $BC$ in an extangential quadrilateral intersect at $E$ and $F$ respectively. The external angle to triangle $ABF$ at $F$ is $A + B$. Triangle $XFZ$ is isosceles according to the two tangent theorem. If the extensions of the tangency chords intersect at $P$, then $\angle PXC = \frac{1}{2}(\pi - A - B)$. In the same way, $\angle PYC = \frac{1}{2}(\pi - A - D)$. According to Lemma 4.1 in [10] it holds that $\angle C = \angle A + \angle E + \angle F$ (see Figure 4)

$$\angle C = \angle A + \angle E + \angle F$$

in the concave quadrilateral $AECF$ if $\angle C > \angle A$. Applying this equality in the concave quadrilateral $PYCX$ yields

$$\angle XPY = C - \frac{\pi - A - B}{2} - \frac{\pi - A - D}{2} = -\pi + \frac{A + B + C + D}{2} + \frac{A + C}{2}.$$ 

This is simplified to

$$\angle XPY = \frac{A + C}{2}$$

and we have that $\angle XPY = \frac{\pi}{2}$ if and only if $A + C = \pi$, which is true if and only if $ABCD$ is a cyclic quadrilateral.

Theorem 9 in [7] gave a characterization of bicentric quadrilaterals, where two incircles were involved. A similar necessary and sufficient condition holds for exbicentric quadrilaterals according to the next theorem.

Theorem 4.2. In an extangential quadrilateral $ABCD$ with an excircle outside of $C$, let the extensions of opposite sides intersect at $E$ and $F$. Also, let the incircle in triangle $CEF$ be tangent to the extensions of $BC$ and $DC$ at $M$ and $N$ respectively, and the excircle to side $EF$ in triangle $AEF$ be tangent to the extensions of $AB$ and $AD$ at $K$ and $L$ respectively. Then $ABCD$ is also cyclic if and only if the extensions of $LM$ and $NK$ are perpendicular.
Figure 5. Extangential quadrilateral with a related incircle and excircle

**Proof.** The incircle in triangle $CEF$ and the excircle to triangle $AEF$ are tangent to $EF$ at the same point $G$ (see Figure 4) according to Theorem 3.3 (ii) in [9]. Thus $EN = EG = EK$ according to the two tangent theorem, so triangle $ENK$ is isosceles. Let the extensions of $LM$ and $KN$ intersect at $Q$. The external angle at $E$ in triangle $ADE$ is $A + D$, so we have that $\angle QNC = \angle ENK = \frac{1}{2}(\pi - A - D)$. In the same way we get $\angle QMC = \frac{1}{2}(\pi - A - B)$. Applying the angle equality (1) in the concave quadrilateral $CMQN$ yields

$$\angle MQN = C - \angle QMC - \angle QNC = -\pi + \frac{A + B + C + D}{2} + \frac{A + C}{2}$$

which simplifies to

$$\angle MQN = \frac{A + C}{2}.$$

Hence the angle $MQN$ between the extensions of $LM$ and $NK$ are perpendicular if and only if $ABCD$ is a cyclic quadrilateral.  \[\square\]

5. **Characterizations concerning rectangles**

The characterizations of exbicentric quadrilaterals in this section are corollaries to Theorems 3.2 and 4.1, but we state them here as theorems anyway. They are both about rectangles.

**Theorem 5.1.** Let the extensions of opposite sides in an extangential quadrilateral intersect at $E$ and $F$. If the internal and external angle bisectors at these points intersect at $I$ and $J$ respectively, then the extangential quadrilateral is also cyclic if and only if quadrilateral $EIFJ$ is a rectangle.
Figure 6. Extangential quadrilateral with the quadrilateral $EIFJ$

**Proof.** The angle between an internal and an external angle bisector in a triangle is always a right angle, so $\angle IEJ = \angle IFJ = \frac{\pi}{2}$ in all convex quadrilaterals. A convex quadrilateral is a rectangle if and only if three of its angles are right angles, so this characterization is a direct consequence of Theorem 3.2. $\square$

The Varignon parallelogram is the parallelogram formed in any quadrilateral by joining the midpoints of the sides. The *contact quadrilateral* in a tangential or extangential quadrilateral is obtained by joining the tangency points between the sides and the incircle or excircle. In [4, p.166] we reviewed a characterization of bicentric quadrilaterals which can be stated that the Varignon parallelogram in the contact quadrilateral belonging to a tangential quadrilateral is a rectangle if and only if the tangential quadrilateral is also cyclic. To prove that theorem was an Olympiad problem in China in 2003. There is the similar necessary and sufficient condition for extangential quadrilaterals to be cyclic, as we will prove now.

**Theorem 5.2.** The Varignon parallelogram in the contact quadrilateral belonging to an extangential quadrilateral is a rectangle if and only if the extangential quadrilateral is also cyclic.

Figure 7. The Varignon parallelogram in the contact quadrilateral
**Proof.** It is well known that the Varignon parallelogram belonging to a quadrilateral is a rectangle if and only if the diagonals in the quadrilateral are perpendicular, see [8, p.137]. The diagonals in the contact quadrilateral \( WXYZ \) (which is crossed) are the tangency chords \( WY \) and \( XZ \), see Figure 7. According to Theorem 4.1, they are perpendicular if and only if the extangential quadrilateral is also cyclic. □

### 6. CHARACTERIZATIONS CONCERNING METRIC RELATIONS

Next we prove three characterizations of exbicentric quadrilaterals that are corollaries to formulas in our paper [10]. The last has already been proved in Theorem 4.1 in the present paper, so here we get a second proof of it.

**Theorem 6.1.** In an extangential quadrilateral with consecutive sides \( a, b, c, d \) and tangent lengths \( e, f, g, h \), the following statements are equivalent:

(i) The quadrilateral is cyclic.

(ii) The quadrilateral has area \( K = \sqrt{abcd} \)

(iii) \( eg = fh \)

(iv) The tangency chords are perpendicular.

**Proof.** The area of an extangential quadrilateral is given by the formula

\[
K = \sqrt{abcd} \sin \frac{A + C}{2}
\]

according to Theorem 3.2 (i) in [10]. Thus the area is given by \( K = \sqrt{abcd} \) if and only if opposite angles are supplementary angles, which is a well-known characterization of a cyclic quadrilateral. This proves that statements (i) and (ii) are equivalent.

The area of an extangential quadrilateral is also given by the formula

\[
K = \sqrt{abcd - (eg - fh)^2}
\]

according to Theorem 3.2 (iv) in [10]. Hence we directly get that statements (ii) and (iii) are equivalent, since

\[
K = \sqrt{abcd - (eg - fh)^2} = \sqrt{abcd} \iff eg = fh.
\]

The acute angle \( \varphi \) between the tangency chords in an extangential quadrilateral is given by

\[
\cos \varphi = \frac{|eg - fh|}{\sqrt{(e - f)(f - g)(g - h)(h - e)}}
\]

according to Theorem 4.2 in [10]. The equivalence of (iii) and (iv) is a direct consequence of this formula, since \( \cos \varphi = 0 \) if and only if \( \varphi = \frac{\pi}{2} \). □

It is interesting that the same three necessary and sufficient conditions (ii), (iii) and (iv) hold for when a tangential quadrilateral is cyclic (a bicentric quadrilateral). That was proved in [5, pp.156–157], [2, p.104] and [3, p.124] respectively, but those characterizations of bicentric quadrilaterals were probably known long before those publications.

The radius in the excircle to an extangential quadrilateral is called the **exradius** and it is a part of the following characterization. The corresponding
relation in a bicentric quadrilateral was the subject of Juan Carlos Salazar’s question at [12].

**Theorem 6.2.** An extangential quadrilateral $ABCD$ with excenter $J$ is also cyclic if and only if

$$\frac{1}{AJ^2} + \frac{1}{CJ^2} = \frac{1}{\rho^2} = \frac{1}{BJ^2} + \frac{1}{DJ^2}$$

where $\rho$ is the exradius.

**Proof.** ($\Rightarrow$) In an extangential quadrilateral, we have $\frac{1}{AJ} = \frac{1}{\rho} \sin \frac{A}{2}$ since $AJ$ is an angle bisector to vertex angle $A$. A similar equality holds for $CJ$. Thus

$$\frac{1}{AJ^2} + \frac{1}{CJ^2} = \frac{1}{\rho^2} \left( \sin^2 \frac{A}{2} + \sin^2 \frac{C}{2} \right).$$

Cyclic quadrilaterals have the property that opposite angles are supplementary angles, so $\sin \frac{C}{2} = \cos \frac{A}{2}$. The first equality now follows from the trigonometric Pythagorean identity $\sin^2 \frac{A}{2} + \cos^2 \frac{A}{2} = 1$. The second equality is proved in the same way.

($\Leftarrow$) If $\frac{1}{AJ^2} + \frac{1}{CJ^2} = \frac{1}{\rho^2}$ holds in an extangential quadrilateral, then

$$\sin^2 \frac{A}{2} + \sin^2 \frac{C}{2} = 1$$

by (2). For all angles $C$ we have $\sin^2 \frac{C}{2} + \cos^2 \frac{C}{2} = 1$, so (3) yields that

$$\sin^2 \frac{A}{2} = \cos^2 \frac{C}{2} \Rightarrow \pm \sin \frac{A}{2} = \sin \left( \frac{\pi}{2} - \frac{C}{2} \right).$$

The four solutions to this equation satisfy $\pm \frac{A}{2} = \frac{\pi}{2} - \frac{C}{2}$ and $\pm \frac{A}{2} = \pi - \left( \frac{\pi}{2} - \frac{C}{2} \right)$. These can be summarized into $\pm A \pm C = \pi$. Here the solution with double minus signs is obviously false, but so are the two with one minus sign. This is because a vertex angle in an extangential quadrilateral must be less than $\pi$ (it is a convex quadrilateral). Hence $A + C = \pi$ is the only valid solution, implying it’s a cyclic quadrilateral. The converse for the second formula is proved in the same way. \qed

The next two characterizations concern the quotient of the diagonals $p = AC$ and $q = BD$ in an extangential quadrilateral $ABCD$. First we need to prove a lemma. The following formulas are the counterparts to the ones derived by Mowaffaq Hajja in [2] for the tangential quadrilateral.$^6$

**Lemma 6.1.** The lengths of the diagonals $AC$ and $BD$ in an extangential quadrilateral $ABCD$ with tangent lengths $e, f, g, h$ are respectively

$$p = \sqrt{\frac{e + g}{f + h} ((e + g)(f + h) - 4fh)},$$

$$q = \sqrt{\frac{f + h}{e + g} ((e + g)(f + h) - 4eg)}.$$
Proof. In [10] we concluded that the only difference between formulas in
terms of the tangent lengths in tangential and extangential quadrilaterals
are the signs of \( f \) and \( h \). We use that symmetry here to quickly get the
formulas in this lemma. Starting with Hajja’s formula for the length of
diagonal \( p \) in a tangential quadrilateral (see [2, p.104]),

\[
p = \sqrt{\frac{e + g}{f + h} (e + g)(f + h) + 4fh},
\]

and making the changes \( f \to -f \) and \( h \to -h \) yields the formula that holds
in an extangential quadrilateral

\[
p = \sqrt{\frac{e + g}{-f - h} ((e + g)(-f - h) + 4(-f)(-h))}.
\]

This is equivalent to the first formula in the theorem. The second follows in
the same way from

\[
q = \sqrt{\frac{f + h}{e + g} ((e + g)(f + h) + 4eh)}
\]

which holds in a tangential quadrilateral according to [2, p.104].

Another (longer) way of deriving the two formulas for an extangential
quadrilaterals is to mimic the method used by Hajja in [2].

We can use the formulas in Lemma 6.1 to get a second proof of the
characterization in Theorem 6.1 (iii). A convex quadrilateral is cyclic if and
only if its consecutive sides \( a, b, c, d \) and diagonals \( p, q \) satisfy Ptolemy’s
theorem \( pq = ac + bd \). When the excircle to an extangential quadrilateral is
outside \( C \), this is equivalent to

\[
((e+g)(f+h) - 4fh)((e+g)(f+h) - 4eg) = ((e-f)(h-g) + (f-g)(e-h))^2.
\]

Expanding, simplifying and factoring this equality yields \( 4(eg - fh)^2 = 0 \),
which is equivalent to \( eg = fh \).

Now we use the formulas in Lemma 6.1 to prove a characterization of
exbicentric quadrilaterals that is the counterpart to one for bicentric quadri-
laterals which was proved by Hajja in [2]. That characterization as well as
the one in the following theorem was previously derived in [11] using rather
long arguments.

**Theorem 6.3.** An extangential quadrilateral with tangent lengths \( e, f, g, h \)
is also cyclic if and only if the quotient of the diagonals satisfies

\[
\frac{p}{q} = \frac{e + g}{f + h}.
\]

**Proof.** An extangential quadrilateral is also cyclic if and only if the tangent
lengths satisfy \( eg = fh \) according to Theorem 6.1 (iii). This condition is
equivalent to

\[(e + g)(f + h) - 4fh = (e + g)(f + h) - 4eg\]

\[\Leftrightarrow \frac{p^2(f + h)}{(e + g)} = \frac{q^2(e + g)}{(f + h)}\]

\[\Leftrightarrow \frac{p^2}{q^2} = \left(\frac{e + g}{f + h}\right)^2\]

which is equivalent to the equality we want to prove. In the first equivalence we used the formulas in Lemma 6.1.

\[\square\]

**Corollary 6.1.** An extangential quadrilateral \(ABCD\) with excenter \(J\) is also cyclic if and only if the quotient of the diagonals satisfies

\[\frac{p}{q} = \frac{AJ \cdot CJ}{BJ \cdot DJ}\]

**Proof.** Corollary 3.1 in [10] states that the tangent lengths in all extangential quadrilaterals (except parallelograms) satisfy

\[\frac{e + g}{f + h} = \frac{AJ \cdot CJ}{BJ \cdot DJ}\]

Combining this equality with Theorem 6.3 directly yields this second characterization in terms of the quotient of the diagonals.

\[\square\]

7. **Perpendicular opposite sides in a cyclic quadrilateral**

In Theorem 4.2 we proved a characterization for exbicentric quadrilaterals with perpendicular opposite sides in an associated cyclic quadrilateral (\(KLMN\) is cyclic according to Theorem 3.4 in [9]). Here we shall study another such configuration where the condition is expressed as a metric relation. We start by proving a necessary and sufficient condition in a cyclic quadrilateral for the extensions of opposite sides to be perpendicular. The direct part of the theorem was a problem in [1, p.31].

**Theorem 7.1.** The extensions of opposite sides \(b\) and \(d\) in a cyclic quadrilateral with consecutive sides \(a, b, c, d\) are perpendicular if and only if

\[(ab + cd)^2 + (ad + bc)^2 = (a^2 - c^2)^2\]

as long as the other two sides does not have the same lengths.
Proof. Let the extensions of the sides $b$ and $d$ intersect at $F$, and $k = CF$ and $l = DF$. The triangles $CDF$ and $ABF$ are similar (by AAA) since the angle at $B$ and the external angle at $D$ are equal, see Figure 8. Thus

$$\frac{k}{l + d} = \frac{c}{a}, \quad \frac{l}{k + b} = \frac{c}{a}.$$  

We get $ak = cl + cd$ and $al = ck + bc$. Multiply the first equation by $c$ and the second by $a$ to get $ack = c^2l + c^2d$ and $a^2l = ack + abc$. These yields

$$a^2l = c^2l + c^2d + abc \quad \Leftrightarrow \quad l = \frac{c(ab + cd)}{a^2 - c^2}, \quad a \neq c. \quad (4)$$

By symmetry we get a formula for $k$ by making the change $b \leftrightarrow d$ in (4). Whence

$$k = \frac{c(ad + bc)}{a^2 - c^2}, \quad a \neq c. \quad (5)$$

Now we apply the Law of cosines in triangle $CDF$ to get

$$c^2 = k^2 + l^2 - 2kl \cos \psi$$

where $\psi$ is the angle between the extensions of $b$ and $d$. Inserting (5) and (4), dividing both sides by $c^2$ and multiplying them by $(a^2 - c^2)^2$ yields

$$(a^2 - c^2)^2 = (ad + bc)^2 + (ab + cd)^2 - 2(ad + bc)(ab + cd) \cos \psi.$$  

Here we see that the extensions of $b$ and $d$ are perpendicular if and only if the equality in the theorem holds, since $\cos \psi = 0$ if and only if $\psi = \frac{\pi}{2}$. □

We conclude this paper by returning to the configuration we studied in Theorem 4.1 and stating a characterization of exbicentric quadrilaterals that concerns the quadrilateral $WYXZ$. Note that this is the convex version of the crossed quadrilateral $WXYZ$, see Figure 7. Since $WXYZ$ is convex, the tangency chords (diagonals in the contact quadrilateral) are now two opposite sides.
Corollary 7.1. If the excircle to an extangential quadrilateral $ABCD$ is tangent to the extensions of $AB$, $BC$, $CD$, $DA$ at $W$, $X$, $Y$, $Z$ respectively, then the quadrilateral is also cyclic if and only if

$$(XY \cdot YW + WZ \cdot ZX)^2 + (YX \cdot XZ + ZW \cdot WY)^2 = (WY^2 - XZ^2)^2.$$ \[\square\]

Proof. The extangential quadrilateral is also cyclic if and only if the tangency chords are perpendicular according to Theorem 4.1. They are perpendicular if and only if the equation in this corollary is satisfied according to Theorem 7.1.

References


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