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AN IRRATIONAL REFINEMENT OF YUN'S INEQUALITY IN BICENTRIC QUADRILATERAL

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1. Introduction

In [1], Zhang Yun established the following inequality.

Theorem 1.1. *In every bicentric quadrilateral is true the the following inequality*

$$\frac{r\sqrt{2}}{R} \leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right) \leq 1.$$

In [2] Martin Josefsson gives another proof at Theorem 1.1.

An refinement of Yun's inequality is given by Vasile Jiglău in [3].

In [4] appear a Yun's inequality of type

$$f(R,r) \leq \frac{1}{2}\sum_{cyclic}\sin\frac{A}{2}\cos\frac{B}{2} \leq g(R,r)$$

where $f(R,r)$, $g(R,r)$ represent the best minimal and maximal homogenous functions for the sum $\frac{1}{2}\sum_{cyclic}\sin\frac{A}{2}\cos\frac{B}{2}$.

This theorem say that:

Theorem 1.2. *In every bicentric quadrilateral is true the inequality*

$$(1)\qquad \frac{1}{2} + \frac{1}{2}\sqrt{\frac{r(r+\sqrt{4R^2+r})}{2R^2}} \leq \frac{1}{2}\sum_{cyclic}\sin\frac{A}{2}\cos\frac{B}{2} \leq \frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R}.$$

The proof of this theorem is proved basing on the monotony of the function $F : [s_1, s_2] \to \mathbb{R}$

$$(2)\qquad F(s) = \frac{1}{2}\sum_{cyclic}\sin\frac{A}{2}\cos\frac{B}{2} = \frac{1}{2}\sqrt{1 + \frac{x_3}{4R^2} + \frac{x_3}{8R^2}\,s}$$

(see [4]).

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Since it is know that $\sin\frac{A}{2}=\cos\frac{C}{2}=\sqrt{\frac{bc}{ad+bd}}$ and $x_3=ac+bd=2r\left(r+\sqrt{4R^2+r^2}\right)$ (see [5]), and $s_1=\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$, $s_2=r+\sqrt{4R^2+r^2}$ represent the semiperimeter of bicentric quadrilaterals $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, which it makes up the minimal and maximal semiperimeter from Blundon Eddy inequality $s_1\le s\le s_2$ (see [6]), which is true in every bicentric quadrilateral.

2. Main results

In the following we find the best real constants α,β and $\gamma>-\sqrt{2}$ such that the inequality

(3)
$$\sqrt{\frac{\alpha R+\beta r}{R+\gamma r}}\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)$$

is true in every bicentric quadrilateral.

Theorem 2.1. *In every bicentric quadrilateral is true the inequality*

$$\sqrt{\frac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}\,r}}$$
$$\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right). \tag{4}$$

Proof. From (1) we have that

$$\frac{1}{2}+\frac{1}{2}\sqrt{\frac{r(r+\sqrt{4R^2+r^2})}{2R^2}}$$
$$\le\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)$$

In order to prove (4) it will be sufficient to prove that

$$\sqrt{\frac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}\,r}}\le\frac{1}{2}+\frac{1}{2}\sqrt{\frac{r(r+\sqrt{4R^2+r^2})}{2R^2}},$$

or after we denote $x=\frac{R}{r}$

$$\sqrt{\frac{2x+7\sqrt{2}}{8x+\sqrt{2}}}\le\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}. \tag{5}$$

We denote $t=\sqrt{\frac{2x+7\sqrt{2}}{8x+\sqrt{2}}}$ or $t^2=\frac{2x+7\sqrt{2}}{8x+\sqrt{2}}$,

$$8t^2x+\sqrt{2}t^2=2x+7\sqrt{2}\quad\text{or}\quad x=\frac{t^2-7}{\sqrt{2}(1-4t^2)}. \tag{6}$$

If we consider the function $g : [\sqrt{2}, +\infty)$, $g(x) = \dfrac{2x + 7\sqrt{2}}{8x + \sqrt{2}}$ with $g'(x) = \dfrac{-54\sqrt{2}}{(8x + \sqrt{2})^2} \leq 0$. It follows that g is an decreasing function. So $\lim\limits_{x\to\infty} g(x) \leq g(x) \leq g(\sqrt{2})$ for each $x \in [\sqrt{2}, +\infty)$ or $\dfrac{1}{4} \leq t^2 \leq 1$, or $t \in \left[\dfrac{1}{2}, 1\right]$.

Inequality (5) may be written as $2(2t-1)^2x^2 \leq 1 + \sqrt{4x^2+1}$ or $4(2t-1)^4x^4 - 4x^2(2t-1)^2 + 1 \leq 4x^2 + 1$, or after dividing by $4x^2$

$$\begin{aligned}
&x^2(2t-1)^4 - (2t-1)^2 \leq 1, \quad \text{or}\\
&\frac{(t^2-7)^2}{2(2t-1)^2(2t+1)^2}(2t-1)^4 \leq 4t^2 - 4t + 2, \quad \text{or}\\
&(t^2-7)^2(2t-1)^2 \leq (2t+1)^2(8t^2-8t+4), \quad \text{or}\\
&(t^4 - 14t^2 + 49)(4t^2 - 4t + 1) \leq (4t^2 + 4t + 1)(8t^2 - 8t + 4), \quad \text{or}\\
&4t^6 - 4t^5 - 87t^4 + 56t^3 + 190t^2 - 204t + 45 \leq 0, \quad \text{or}\\
&(t-1)^2(4t^4 + 4t^3 - 83t^2 - 114t + 45 \leq 0 \quad \text{for each } t \in \left[\frac{1}{2}, 1\right].
\end{aligned}$$

It well be sufficient to prove that

$$4t^4 + 4t^3 - 83t^2 - 114t + 45 \leq 0. \tag{7}$$

But $4t^4 + 4t^3 + 45 \leq 4 + 4 + 45 = 53$ and $-83t^2 - 114t \leq -\dfrac{83}{4} - \dfrac{114}{2}$.
Adding we obtain

$$4t^4 + 4t^3 - 83t^2 - 114t + 45 \leq -4 - \frac{83}{4} < 0.$$

It follows that inequality (7) is true.
Next, we shall prove that the inequality (4) is the best of type (3).

We suppose that it exists $\alpha_0, \beta_0 \in \mathbb{R}$, $\gamma_0 \geq -\sqrt{2}$ such that

$$\begin{aligned}
&\sqrt{\frac{\alpha_0 R + \beta_0 r}{R + \gamma_0 r}}\\
&\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right)
\end{aligned} \tag{8}$$

is true in every bicentric quadrilateral and inequality (8) is the best of type (3).
It follows that

$$\begin{aligned}
&\sqrt{\frac{2R + 7\sqrt{2}\,r}{8R + \sqrt{2}\,r}} \leq \sqrt{\frac{\alpha_0 R + \beta_0 r}{R + \gamma_0 r}}\\
&\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{C}{2} + \sin\frac{C}{2}\cos\frac{D}{2} + \sin\frac{D}{2}\cos\frac{A}{2}\right)
\end{aligned} \tag{9}$$

is true in every bicentric quadrilateral.
If we consider the case of bicentric quadrilateral $A_1B_1C_1D_1$ which makes up

the semiperimeter minimal $s_1 = \sqrt{8r(\sqrt{4R^2+r^2}-r)}$.
From (9) it results that

$$\sqrt{\frac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}\,r}} \le \sqrt{\frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r}} \le \frac{1}{2}+\frac{1}{2}\sqrt{\frac{r\left(r+\sqrt{4R^2+r^2}\right)}{2R^2}}$$
$$= \frac{1}{2}\left(\sin\frac{A_1}{2}\cos\frac{B_1}{2}+\sin\frac{B_1}{2}\cos\frac{C_1}{2}+\sin\frac{C_1}{2}\cos\frac{D_1}{2}+\sin\frac{D_1}{2}\cos\frac{A_1}{2}\right)$$
$$\le \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \tag{10}$$

is true in every bicentric quadrilateral since from (2) we have that

$$\frac{1}{2}\left(\sin\frac{A_1}{2}\cos\frac{B_1}{2}+\sin\frac{B_1}{2}\cos\frac{C_1}{2}+\sin\frac{C_1}{2}\cos\frac{D_1}{2}+\sin\frac{D_1}{2}\cos\frac{A_1}{2}\right)$$
$$= \frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}s_1} \le \frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}s}$$
$$= \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right).$$

If we consider the case of square with sides $a=b=c=d=1$, $R=\frac{1}{\sqrt{2}}$, $r=\frac{1}{2}$, if we replace in (10) we obtain

$$1 \le \sqrt{\frac{\frac{\alpha_0}{\sqrt{2}}+\frac{\beta_0}{2}}{\frac{1}{\sqrt{2}}+\frac{\gamma_0}{2}}} \le 1 \quad \text{or} \quad \sqrt{2}\alpha_0+\beta_0=\sqrt{2}+\gamma_0. \tag{11}$$

Also if we denote with $x=\frac{R}{r}$ from (10) we obtain

$$\sqrt{\frac{2x+7\sqrt{2}}{8x+\sqrt{2}}} \le \sqrt{\frac{\alpha_0 x+\beta_0}{x+\gamma_0}} \le \frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}. \tag{12}$$

If we take in (12) $x\to\infty$ we obtain $\frac{1}{2}\le\sqrt{\alpha_0}\le\frac{1}{2}$ or $\alpha_0=\frac{1}{4}$. From (11) we obtain

$$\beta_0=\frac{3\sqrt{2}}{4}+\gamma_0 \tag{13}$$

So inequality (13) may be written as

$$\sqrt{\frac{2x+7\sqrt{2}}{8x+\sqrt{2}}} \le \sqrt{\frac{\frac{1}{4}\,x+\frac{3\sqrt{2}}{4}+\gamma_0}{x+\gamma_0}} \le \frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}} \tag{14}$$

The left side of inequality (14) may be written as

$$\left(6\gamma_0-\frac{3\sqrt{2}}{4}\right)(x-\sqrt{2})\ge 0 \text{ for each } x\ge\sqrt{2} \quad \text{or} \quad \gamma_0\ge\frac{\sqrt{2}}{8}. \tag{15}$$

We denote $\alpha=\sqrt{4x^2+1}$.

The right side of inequality (14) may be written, after squaring, as

$$\frac{\frac{1}{4}x+\frac{3\sqrt{2}}{4}+\gamma_0}{x+\gamma_0}\le\frac{1}{4}+\frac{1+\alpha}{8x^2}+\frac{1}{2}\sqrt{\frac{1+\alpha}{2x^2}}\quad\text{or}$$

$$\frac{x}{4}+\frac{1+\alpha}{8x}+\frac{1}{2}\sqrt{\frac{1+\alpha}{2}}+\gamma_0\left(\frac{1}{4}+\frac{1+\alpha}{8x^2}+\frac{1}{2}\sqrt{\frac{1+\alpha}{2x^2}}\right)$$

$$\ge\frac{x}{4}+\frac{3\sqrt{2}}{4}+\gamma_0,\quad\text{or}$$

(16)
$$\left(\frac{3}{2}-\frac{1+\alpha}{4x^2}-\sqrt{\frac{1+\alpha}{2x^2}}\right)\gamma_0\le\frac{1+\alpha}{4x}+\sqrt{\frac{1+\alpha}{2}-\frac{3\sqrt{2}}{2}}.$$

We prove that

(17)
$$\frac{3}{2}-\frac{1+\alpha}{4x^2}-\sqrt{\frac{1+\alpha}{2x^2}}\ge 0$$

We denote $u=\sqrt{\frac{1+\alpha}{2x^2}}$.
Inequality (17) may be written as

$$\frac{3}{2}-\frac{u^2}{2}-u\ge 0\quad\text{or}\quad u^2+2u-3\le 0\quad\text{or}\quad (u-1)(u+3)\le 0$$

We prove that $u\le 1$ or $1+\sqrt{4x^3+1}\le 2x^2$ or, after squaring, $4x^2(x^2-2)\ge 0$ which is true for each $x\ge\sqrt{2}$.
Inequality (16) may be written as

(18)
$$\gamma_0\le\frac{\frac{1+\alpha}{4x}+\sqrt{\frac{1+\alpha}{2}}-\frac{3\sqrt{2}}{2}}{\frac{3}{2}-\frac{1+\alpha}{4x^2}-\sqrt{\frac{1+\alpha}{2x^2}}}.$$

We consider the function

$$f:(\sqrt{2},+\infty)\to\mathbb{R},\ f(x)=\frac{\frac{1+\sqrt{4x^2+1}}{4x}+\sqrt{\frac{1+\sqrt{4x^2+1}}{2}}-\frac{3\sqrt{2}}{2}}{\frac{3}{2}-\frac{1+\sqrt{4x^2+1}}{4x^2}-\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}}$$

Inequality (18) may be written as $\gamma_0\le f(x)$ for each $x\ge\sqrt{2}$.
It follows that $\gamma_0\le\min\limits_{x\ge\sqrt{2}}f(x)$.

We shall prove that $\min\limits_{x\geq\sqrt{2}} f(x)=\lim\limits_{x\to\sqrt{2}} f(x)$. We have

$$\lim_{x\to\sqrt{2}}\frac{\frac{1+\sqrt{4x^2+1}}{4x}+\sqrt{\frac{1+\sqrt{4x^2+1}}{2}}-\frac{3\sqrt{2}}{2}}{-\frac{1}{2}\left(\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}-1\right)\left(\sqrt{\frac{1+\sqrt{4x^2+1}}{2x^2}}+3\right)}$$

$$-\frac{\sqrt{2}}{8}\lim_{x\to\sqrt{2}}\frac{1+\sqrt{4x^2+1}+2\sqrt{2}\,x\sqrt{1+\sqrt{4x^2+1}}-6\sqrt{2}\,x}{\sqrt{1+\sqrt{4x^2+1}}-x\sqrt{2}}$$

$$=-\frac{\sqrt{2}}{8}\lim_{x\to\sqrt{2}}\frac{2\sqrt{2}\,x\left(\sqrt{1+\sqrt{4x^2+1}}-x\sqrt{2}\right)+4x^2-6\sqrt{2}\,x+1+\sqrt{4x^2+1}}{\sqrt{1+\sqrt{4x^2+1}}-x\sqrt{2}}$$

$$=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{8}\lim_{x\to\sqrt{2}}\frac{8x-6\sqrt{2}+\frac{4x}{\sqrt{4x^2+1}}}{\frac{1}{2\sqrt{1+\sqrt{4x^2+1}}}\frac{4x}{\sqrt{4x^2+1}}-\sqrt{2}}=\frac{\sqrt{2}}{8}\,.$$

We prove that

$$f(x)\geq\frac{\sqrt{2}}{8} \tag{19}$$

and as $\lim\limits_{x\to\sqrt{2}} f(x)=\dfrac{\sqrt{2}}{8}$ it follows that

$$\min_{x\geq\sqrt{2}} f(x)=\frac{\sqrt{2}}{8}\,. \tag{20}$$

Inequality (19) is equivalent with

$$\frac{\frac{1+\alpha}{4x}+\sqrt{\frac{1+\alpha}{2}}-\frac{3\sqrt{2}}{2}}{\frac{3}{2}-\frac{1+\alpha}{4x^2}-\sqrt{\frac{1+\alpha}{2x^2}}}\geq\frac{\sqrt{2}}{8}\quad\text{or}$$

$$\frac{2(1+\alpha)}{x}+8\sqrt{\frac{1+\alpha}{2}}-12\sqrt{2}\geq\frac{3\sqrt{2}}{2}-\frac{\sqrt{2}(1+\alpha)}{4x^2}-\sqrt{2}\sqrt{\frac{1+\alpha}{2x}}\,,\quad\text{or}$$

$$\sqrt{2}\sqrt{\frac{1+\alpha}{2x^2}}+8x\sqrt{\frac{1+\alpha}{2x^2}}+\frac{2(1+\alpha)}{x^2}\,x+\frac{\sqrt{2}(1+\alpha)}{4x^2}\geq\frac{27\sqrt{2}}{2}\,,\quad\text{or}$$

$$\frac{1}{2}\sqrt{\frac{1+\alpha}{2x^2}}(2\sqrt{2}+16x)+\frac{1}{8}\,\frac{1+\alpha}{x^2}\,(16x+2\sqrt{2})+\frac{1}{4}(16x+2\sqrt{2})\geq$$

$$\geq\frac{27\sqrt{2}}{2}+\frac{1}{4}(16x+2\sqrt{2})\,,\quad\text{or}$$

$$\left(\frac{1}{2}\sqrt{\frac{1+\alpha}{2x^2}}+\frac{1}{8}\,\frac{1+\alpha}{x^2}+\frac{1}{4}\right)(8x+\sqrt{2})\geq\frac{28\sqrt{2}+8x}{4}=7\sqrt{2}+2x.$$

So we obtain that

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1+\alpha}{2x^2}}\right)^2\geq\frac{7\sqrt{2}+2x}{8x+\sqrt{2}}\,,$$

inequality which is prove in the context of the Theorem 2.1.
So we obtain that $\gamma_0\leq\dfrac{\sqrt{2}}{8}$ and using (15) we have $\gamma_0=\dfrac{\sqrt{2}}{8}$.

From (14) we obtain $\beta_0 = \dfrac{7\sqrt{2}}{8}$. We obtain a contradiction.
It results that inequality (4) is the best of type (3).

Next we shall find the best real constants α, β and $\gamma > -\sqrt{2}$ such that the inequality

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \le \sqrt{\frac{\alpha R+\beta r}{R+\gamma r}} \tag{21}$$

is true in every bicentric quadrilateral.

Theorem 2.2. *In every bicentric quadrilateral is true the inequality*

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \le \sqrt{\frac{2R+\sqrt{2}\,r}{4R-\sqrt{2}\,r}}\,. \tag{22}$$

Proof. From (1) we have that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \le \frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}\,R}\,.$$

In order to prove (22) it will be sufficient to prove that

$$\frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}\,R} \le \sqrt{\frac{2R+\sqrt{2}\,r}{4R-\sqrt{2}\,r}} \quad \text{or}$$

$$\frac{\sqrt{4x^2+1}+1}{2\sqrt{2}\,x} \le \sqrt{\frac{2x+\sqrt{2}}{4x-\sqrt{2}}}, \quad \text{or after squaring}$$

$$\frac{2x^2+1+\sqrt{4x^2+1}}{4x^2} \le \frac{2x+\sqrt{2}}{4x-\sqrt{2}}, \quad \text{or}$$

$$8x^3+4x-2\sqrt{2}x^2-\sqrt{2}+(4x-\sqrt{2})\sqrt{4x^2+1} \le 8x^3+4\sqrt{2}x^2, \quad \text{or}$$

$$(4x-\sqrt{2})\sqrt{4x^2+1} \le 6\sqrt{2}x^2-4x+\sqrt{2}\,, \quad \text{or after squaring}$$

$$(4x^2+1)(16x^2-8\sqrt{2}x+2) \le (6\sqrt{2}x^2-4x+\sqrt{2})^2, \quad \text{or}$$

$$64x^4-32\sqrt{2}x^3+8x^2+16x^2-8\sqrt{2}\,x+2 \le$$
$$\le 72x^4+16x^2+2-48\sqrt{2}\,x^3-24x^2-8\sqrt{2}\,x \quad \text{or}$$

$8x^4-16\sqrt{2}\,x^3+16x^2 \ge 0$ or $8x^2(x-\sqrt{2})^2 \ge 0$, which is true for each $x \ge \sqrt{2}$.

In the next we shall prove that the inequality (22) is the best of type (21). Ve suppose that it exists $\alpha_0, \beta_0 \in \mathbb{R}$, $\gamma_0 \geq -\sqrt{2}$ such that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \leq \sqrt{\frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r}}$$

is the best inequality of type (21).
It follows that

$$\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right) \leq \sqrt{\frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r}} \leq \sqrt{\frac{R+2\sqrt{2}\,r}{2R+\sqrt{2}\,r}} \tag{23}$$

is true in every bicentric quadrilateral.
If we consider the bicentric quadrilateral $A_2B_2C_2D_2$ which makes up the maximal semiperimeter $S_2=\sqrt{4R^2+r^2}+r$ from (23) it follows that

$$\begin{aligned}&\frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)\\ &\leq \frac{1}{2}\left(\sin\frac{A_2}{2}\cos\frac{B_2}{2}+\sin\frac{B_2}{2}\cos\frac{C_2}{2}+\sin\frac{C_2}{2}\cos\frac{D_2}{2}+\sin\frac{D_2}{2}\cos\frac{A_2}{2}\right)\\ &= \frac{\sqrt{4R^2+r^2}+r}{2\sqrt{2}R} \leq \sqrt{\frac{\alpha_0 R+\beta_0 r}{R+\gamma_0 r}} \leq \sqrt{\frac{2R+\sqrt{2}\,r}{4R-\sqrt{2}\,r}}\end{aligned} \tag{24}$$

is true in every bicentric quadrilateral since we have

$$\begin{aligned}&\frac{1}{2}\left(\sin\frac{A_2}{2}\cos\frac{B_2}{2}+\sin\frac{B_2}{2}\cos\frac{C_2}{2}+\sin\frac{C_2}{2}\cos\frac{D_2}{2}+\sin\frac{D_2}{2}\cos\frac{A_2}{2}\right)\\ &= \frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}S_2} \geq \frac{1}{2}\sqrt{1+\frac{x_3}{4R^2}+\frac{x_3}{8R^2r}S}\\ &= \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right).\end{aligned}$$

In the case of square with the sides $a=b=c=d=1$, $R=\frac{1}{\sqrt{2}}$, $r=\frac{1}{2}$, from (24) we obtain

$$1 \leq \sqrt{\frac{\frac{\alpha_0}{\sqrt{2}}+\frac{\beta_0}{2}}{\frac{1}{\sqrt{2}}+\frac{\gamma_0}{2}}} \leq 1 \qquad \text{or} \qquad \alpha_0\sqrt{2}+\beta_0=\gamma_0+\sqrt{2}\,. \tag{25}$$

From (24) we have

$$\frac{\sqrt{4x^2+1}+1}{2\sqrt{2}\,x} \leq \sqrt{\frac{\alpha_0 x+\beta_0}{x+\gamma_0}} \leq \sqrt{\frac{2x+\sqrt{2}}{4x-\sqrt{2}}}\,. \tag{26}$$

If we take in (26), $x \to \infty$ we obtain $\dfrac{1}{\sqrt{2}} \le \sqrt{\alpha_0} \le \dfrac{1}{\sqrt{2}}$ or $\alpha_0 = \dfrac{1}{2}$.
From (25) we obtain that

$$\beta_0 = \gamma_0 + \frac{\sqrt{2}}{2}\,. \tag{27}$$

So from (26) we have after squaring

$$\frac{2x^2+1+\sqrt{4x^2+1}}{4x^2} \le \frac{\frac{1}{2}\,x+\gamma_0+\frac{\sqrt{2}}{2}}{x+\gamma_0} \qquad \text{or}$$

$$2x^3+x+x\sqrt{4x^2+1}+\gamma_0(2x^2+1+\sqrt{4x^2+1}) \le 2x^3+2\sqrt{2}\,x^2+4x^2\gamma_0 \text{ or}$$

$$x\sqrt{4x^2+1}+x-2\sqrt{2}\,x^2 \le (2x^2-1-\sqrt{4x^2+1})\gamma_0\,. \tag{28}$$

We have $2x^2-1 \ge \sqrt{4x^2+1}$ or $4x^4-4x^2+1 \ge 4x^2+1$ or $x^2 \ge 2$ which is true. From (28) we have

$$\gamma_0 \ge \frac{x\sqrt{4x^2+1}+x-2\sqrt{2}\,x^2}{2x^2-1-\sqrt{4x^2+1}}\,. \tag{29}$$

We consider the function

$$f : [\sqrt{2}, +\infty) \to \mathbb{R},\ f(x) = \frac{x\sqrt{4x^2+1}+x-2\sqrt{2}\,x^2}{2x^2-1-\sqrt{4x^2+1}}\,.$$

From (29) we have $\gamma_0 \ge f(x)$ for each $x \ge \sqrt{2}$. It follows that

$$\gamma_0 \ge \max_{x\ge\sqrt{2}} f(x) \tag{30}$$

We compute

$$\lim_{x\to\sqrt{2}} f(x) = \lim_{x\to\sqrt{2}} \frac{\sqrt{4x^2+1}+\frac{4x^2}{\sqrt{4x^2+1}}+1-4\sqrt{2}\,x}{4x-\frac{4x}{\sqrt{4x^2+1}}} = -\frac{\sqrt{2}}{4}\,.$$

We shall prove that $f(x) \le -\dfrac{\sqrt{2}}{2} = \lim\limits_{x\to\sqrt{2}} f(x)$ for each $x \ge \sqrt{2}$ which imply that $\max\limits_{x\ge\sqrt{2}} f(x) = -\dfrac{\sqrt{2}}{4}$.

Inequality $f(x) \le -\dfrac{\sqrt{2}}{4}$ is proved in the context of Theorem 2.2.
From (30) it results that

$$\gamma_0 \ge -\frac{\sqrt{2}}{4}\,. \tag{31}$$

From the right side of inequality (16) we have that

$$\frac{\frac{1}{2}\,x+\gamma_0+\frac{\sqrt{2}}{2}}{x+\gamma_0} \le \frac{2x+\sqrt{2}}{4x-\sqrt{2}} \quad \text{or} \quad (x-\sqrt{2})\left(-2\gamma_0-\frac{\sqrt{2}}{2}\right) \ge 0,$$

for each $x \geq \sqrt{2}$, which imply $\gamma_0 \leq -\dfrac{\sqrt{2}}{4}$ and from (31) we obtain $\gamma_0 = -\dfrac{\sqrt{2}}{4}$. Also from (26) it follows that $\beta_0 = \dfrac{\sqrt{2}}{4}$.

We obtain $\alpha_0 = \dfrac{1}{2}$, $\beta_0 = \dfrac{\sqrt{2}}{4}$, $\gamma_0 = -\dfrac{\sqrt{2}}{4}$ which represents a contradiction.

Theorem 2.3 (An irrational refinement of Yun's inequality)**.** *In every bicentric quadrilateral is true the inequality*

$$\frac{r\sqrt{2}}{R} \leq \sqrt{\frac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}\,r}}$$
$$\leq \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)$$
$$\leq \sqrt{\frac{2R+\sqrt{2}\,r}{4R-\sqrt{2}\,r}} \leq 1. \tag{32}$$

Proof. It follows from Theorem 2.1 and Theorem 2.2.

Inequality $\dfrac{r\sqrt{2}}{R} \leq \sqrt{\dfrac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}}}$ is equivalent after squaring and perform some calculation with $(x-\sqrt{2})(2x^2+9\sqrt{2}\,x+2) \geq 0$ for each $x \geq 2$.

Inequality $\sqrt{\dfrac{2R+\sqrt{2}\,r}{4R-\sqrt{2}}} \leq 1$ is equivalent with $R \geq \sqrt{2}\,r$.

Next we consider the function

$$F : \left(-\sqrt{2}, \frac{\sqrt{2}}{8}\right] \to \mathbb{R}$$
$$F(\gamma_1) = \frac{R+(3\sqrt{2}+4\gamma_1)r}{4R+4\gamma_1 r} \quad \text{with} \quad F'(\gamma_1) = \frac{12r(R-\sqrt{2}\,r)}{(4R+4\gamma_1\,r)^2} \geq 0.$$

It results that F is an increasing function.

We have $\sqrt{F(-\sqrt{2})} < \sqrt{F(\gamma_1)} \leq \sqrt{F\left(\dfrac{\sqrt{2}}{8}\right)}$ for each $\gamma_1 \in \left(-\sqrt{2}, \dfrac{\sqrt{2}}{8}\right)$

$$\frac{1}{2} < \sqrt{\frac{R+(3\sqrt{2}+4\gamma_1)r}{4R+4\gamma_1 r}} \leq \sqrt{\frac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}\,r}}. \tag{33}$$

Also we consider the functions

$$G : \left(-\frac{\sqrt{2}}{4}, +\infty\right) \to \mathbb{R}$$
$$G(\gamma_2) = \frac{R+(2\gamma_2+\sqrt{2})r}{2R+2\gamma_2 r} \quad \text{with} \quad G'(\gamma_2) = \frac{2r(R-\sqrt{2}\,r)}{(2R+2\gamma_2 r)^2} \geq 0.$$

It results that G is an increasing function and $G\left(-\frac{\sqrt{2}}{4}\right) \le G(\gamma_2) \le G(\infty)$ for each $\gamma_2 \in \left[-\frac{\sqrt{2}}{4}, +\infty\right)$ or

$$\sqrt{\frac{2R+\sqrt{2}\,r}{4R-\sqrt{2}\,r}} \le \sqrt{\frac{R+2(\gamma_2+\sqrt{2})r}{2R+2\gamma_2\,r}} \le 1. \tag{34}$$

From (32), (33) and (34) we obtain the following theorem

Theorem 2.4. *In every bicentric quadrilateral is true the following inequality*

$$\frac{1}{2} < \sqrt{\frac{R+(3\sqrt{2}+4\gamma_1)r}{4R+4\gamma_1\,r}} \le \sqrt{\frac{2R+7\sqrt{2}\,r}{8R+\sqrt{2}\,r}}$$
$$\le \frac{1}{2}\left(\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{D}{2}+\sin\frac{D}{2}\cos\frac{A}{2}\right)$$
$$\le \sqrt{\frac{2R+\sqrt{2}\,r}{4R-\sqrt{2}\,r}} \le \sqrt{\frac{R+2(\gamma_2+\sqrt{2})r}{2R+2\gamma_2\,r}} \le 1.$$

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