INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 6 (2017), No. 2, 109 - 111

A NEW TRIGONOMETRIC PROOF TO PTOLEMY THEOREMS IN CYCLIC QUADRILATERALS

MIHAI MICULIŢA

Abstract. We give a new trigonometric proof to both Ptolemey Theorems. The proof is elementary and it uses only the Law of Sines in a triangle.

In classical Euclidean geometry, Ptolemy Theorems are relations between the four sides and the two diagonals of a cyclic quadrilateral [6]. The theorems are named after the Greek astronomer and mathematician Ptolemy, on his Latin name Claudius Ptolemaeus (about 87-165 AC). He lived and worked in Alexandria and has used these results as an aid in creating his table of chords, a trigonometric table that he applied to Astronomy.

There are many known proofs, specially to the first Ptolemy Theorem: using some geometric constructions [6], [1] and [8], using vectors [7], using complex numbers [2] and [6], the connection with the Law of Cosines [4]. In fact, it is a special case of the Ptolemy inequality, a direct consequence of the Euler's Theorem on the area of the podar triangle of a point with respect to a given triangle (see [3], pp.375 or [2], Theorems 2 and 3, pp.143).

In the paper [5] it is proposed a proof based on areas to the first Ptolemy Theorem. Also, two characterizations of a bicentric quadrilateral are given. In this short note we give a new trigonometric proof to both Ptolemey theorems in cyclic quadrilaterals. Our proofs use only the basic Law of Sines.

Denoting by $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |AD|$ the side lengths and by $e = |AC|$, $f = |BD|$ the lengths of the diagonals of the cyclic quadrilateral $ABCD$ (see Figure 1), then the following two relations hold :

Keywords and phrases: cyclic quadrilaterals; Ptolemey Theorems; Law of Sines
(2010)Mathematics Subject Classification: 1Mxx, 51Axx
Received: 15.04.2017. In revised form: 10.09.2017. Accepted: 03.07.2017.

Theorem. 1) (The first Ptolemey Theorem) $ac + bd = ef$;

2) (The second Ptolemey Theorem) $\frac{e}{f} = \frac{ad+bc}{ab+cd}$.

Proof. Considering $\alpha = m(\sphericalangle ACB)$, $\beta = m(\sphericalangle BAC)$, $\gamma = m(\sphericalangle DAC)$ and $\delta = m(\sphericalangle ACD)$, we have that the property $ABCD$ is cyclic is equivalent to each of the following properties:

$$m(\sphericalangle ADB) = m(\sphericalangle ACB) = \alpha; m(\sphericalangle BDC) = m(\sphericalangle BAC)$$

$$\alpha + \beta + \gamma + \delta = m(\sphericalangle DAC) + m(\sphericalangle BCD) = 180^\circ.$$

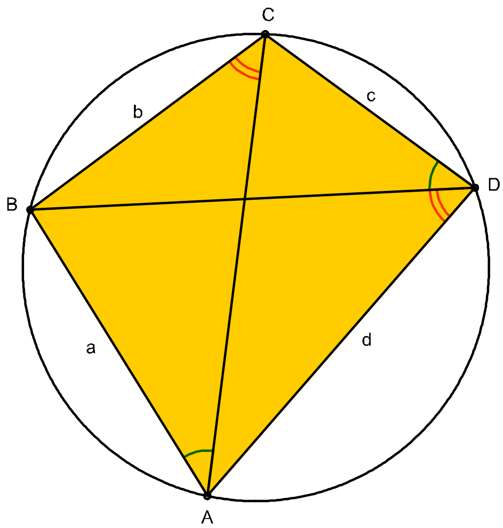

Figure 1

From the Law of Sines, we obtain the relations:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = \frac{d}{\sin\delta} = \frac{e}{\sin(\alpha+\beta)} = \frac{f}{\sin(\beta+\gamma)} = 2R.$$

This is equivalent to

$$a = 2R\cdot\sin\alpha, \quad b = 2R\cdot\sin\beta, \quad c = 2R\cdot\sin\gamma,$$

$$d = 2R\cdot\sin\delta = 2R\cdot\sin\left[180^\circ - (\alpha+\beta+\gamma)\right] = 2R\cdot\sin(\alpha+\beta+\gamma),$$

$$e = 2R\cdot\sin(\alpha+\beta), \quad f = 2R\cdot\sin(\beta+\gamma).$$

1) Using these relations it follows that $ac + bd = ef$ is equivalent to

$$\sin\alpha\cdot\sin\gamma + \sin\beta\cdot\sin(\alpha+\beta+\gamma) = \sin(\alpha+\beta)\cdot\sin(\beta+\gamma).$$

But, we have

$$\sin\alpha\cdot\sin\gamma + \sin\beta\cdot\sin(\alpha+\beta+\gamma) - \sin(\alpha+\beta)\cdot\sin(\beta+\gamma) =$$

$$\frac{1}{2}\cdot[\cos(\alpha-\gamma) - \cos(\alpha+\gamma)] + \frac{1}{2}\cdot[\cos(\alpha+\gamma) - \cos(\alpha+2\beta+\gamma)] -$$

$$-\frac{1}{2}\cdot[\cos(\alpha-\gamma) - \cos(\alpha+2\beta+\gamma)] = 0,$$

that is

$$\sin\alpha\cdot\sin\gamma + \sin\beta\cdot\sin(\alpha+\beta+\gamma) = \sin(\alpha+\beta)\cdot\sin(\beta+\gamma),$$

and the first relation is proved.

2) To prove the second relation, observe that the relation $\frac{e}{f}=\frac{ad+bc}{ab+bc}$ is equivalent to

$$\frac{\sin(\alpha+\beta)}{\sin(\beta+\gamma)}=\frac{\sin\alpha\cdot\sin(\alpha+\beta+\gamma)+\sin\beta\cdot\sin\gamma}{\sin\alpha\cdot\sin\beta+\sin\gamma\cdot\sin(\beta+\gamma)}.$$

But we have

$$\frac{\sin\alpha\cdot\sin(\alpha+\beta+\gamma)+\sin\beta\cdot\sin\gamma}{\sin\alpha\cdot\sin\beta+\sin\gamma\cdot\sin(\beta+\gamma)}=$$

$$\frac{\frac{1}{2}\cdot[\cos(\beta+\gamma)-\cos(2\alpha+\beta+\gamma)]+\frac{1}{2}\cdot[\cos(\beta-\gamma)-\cos(\beta+\gamma)]}{\frac{1}{2}\cdot[\cos(\alpha-\beta)-\cos(\alpha+\beta)]+\frac{1}{2}\cdot[\cos(\alpha+\beta)-\cos(\alpha+\beta+2\gamma)]}=$$

$$\frac{\frac{1}{2}\cdot[\cos(\beta-\gamma)-\cos(2\alpha+\beta+\gamma)]}{\frac{1}{2}\cdot[\cos(\alpha-\beta)-\cos(\alpha+\beta+2\gamma)]}=$$

$$\frac{\sin(\alpha+\beta)\cdot\sin(\alpha+\gamma)}{\sin(\alpha+\gamma)\cdot\sin(\beta+\gamma)}=\frac{\sin(\alpha+\beta)}{\sin(\beta+\gamma)},$$

and we are done.

References

[1] Amarasinghe, G.I.S., *A concise elementary proof for the Ptolemy's theorem*, Global Journal of Advanced Research on Classical and Modern Geometries, **2(1) (2013)**, 20-25.

[2] Andreescu, T. and Andrica, D., *Complex Numbers from A to...Z*, Second Edition, Birkhäuser, 2014.

[3] Barbu, C., *Fundamental Theorems in the Triangle Geometry* (Romanian), UNIQUE Publishing House, Bacău, 2008.

[4] Kung, S.H., *Proof without words: the law of cosines via Ptolemy's theorem*, Mathematics Magazine, 1992.

[5] Krishna, D.N.V., *A new proof of Ptolemey theorem*, Journal of Classical Geometry, **4 (2017)**.

[6] Pop, O.T., Minculete, N. and Bencze, M., *An introduction to quadrilateral geometry*, Editura Didactică şi Pedagogică, R.A., Bucureşti, 2013.

[7] Schaumberger, E.J.N., *A vector approach to Ptolemy's theorem*, Mathematics Magazin, **77(5)(1) (2004)**, 386-388.

[8] Shisha, O., *On Ptolemy's theorem*, International Journal of Mathematics and Mathematical Sciences, **14(2) (1991)**, 410.

PARCUL TRAIAN 4, ORADEA, ROMANIA
E-mail address: `miculitam@yahoo.com`