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# AN EQUILATERAL TRIANGLE IN THE ARBELOS

HIROSHI OKUMURA

**Abstract.** An equilateral triangle is derived from the incircle of the arbelos.

Let us consider an arbelos with two inner semicircles with diameters $AO$ and $BO$ for a point $O$ on the segment $AB$ in the plane. Let $\delta$ be the incircle of the arbelos of radius $d$. The distance between the center of $\delta$ and the line $AB$ equals $2d$ by Pappus chain theorem. Therefore the segment $AB$, the diameter of $\delta$ parallel to $AB$ and the tangents of $\delta$ perpendicular to $AB$ form a square (see Figure 1) [1]. In this note we show that the same circle also yields an equilateral triangle. Let $\gamma$ be the outer semicircle of the arbelos. The circle with center $P$ passing through $Q$ is denoted by $P(Q)$ for points $P$ and $Q$. We use the following theorem (see Figure 2).

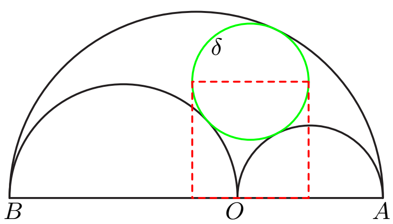

Figure 1.

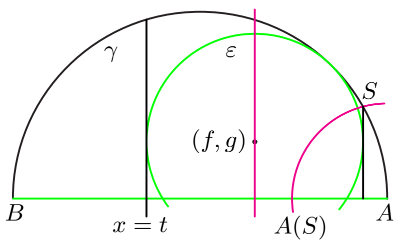

Figure 2.

**Theorem 1.** *If a circle $\varepsilon$ touches the semicircle $\gamma$ from the inside and $S$ is the point on $\gamma$ such that $\varepsilon$ touches the perpendicular from $S$ to $AB$ from the side opposite to $A$, then the pencil of circles determined by $\varepsilon$ and $AB$ is orthogonal to the pencil of circles determined by the circle $A(S)$ and the perpendicular from the center of $\varepsilon$ to $AB$.*

**Proof.** It is sufficient to show that the circles $\varepsilon$ and $A(S)$ are orthogonal. Let $c$ be the radius of $\gamma$. We use a rectangular coordinate system with origin at the center of $\gamma$ such that $A$ has coordinates $(c, 0)$. We assume that $s$ is the $x$-coordinate of the points $S$ and the circle $\varepsilon$ touches the line $x = t$ from

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the side opposite to $B$ and has radius $e$ and center with coordinates $(f, g)$. Then we have

$$2e = s - t, \quad 2f = s + t, \tag{1}$$

and from $f^2 + g^2 = (c - e)^2$,

$$f^2 + g^2 - e^2 = c^2 - 2ce. \tag{2}$$

While

$$|AS|^2 = (c - s)^2 + c^2 - s^2 = 2c(c - s). \tag{3}$$

The square of the distance between the centers of $\varepsilon$ and $A(S)$ is $(c-f)^2+g^2$. Using (3), (2), (1) in this order, we have

$$\begin{aligned} & (c - f)^2 + g^2 - e^2 - |AS|^2 = c^2 - 2cf + (f^2 + g^2 - e^2) - 2c(c - s) \\ = \ & c^2 - 2cf + (c^2 - 2ce) - 2c(c - s) = -2cf - 2ce + 2cs \\ = \ & -c(s + t) - c(s - t) + 2cs = 0. \end{aligned}$$

Therefore $A(S)$ and $\varepsilon$ are orthogonal.

**Corollary 1.** *Let us assume that a circle $\varepsilon$ touches the semicircle $\gamma$ from the inside and has no point in common with the line $AB$ and $S$ is the point on $\gamma$ such that $\varepsilon$ touches the perpendicular from $S$ to $AB$ from the side opposite to $A$. Then a point $L$ is one of the limiting points of the pencil of circles determined by $\varepsilon$ and $AB$ if and only if $L$ lies on the circle $A(S)$ and the perpendicular from the center of $\varepsilon$ to $AB$.*

The if part of the corollary can be found in [2, Theorem 3]. We now consider the arbelos (see Figure 3).

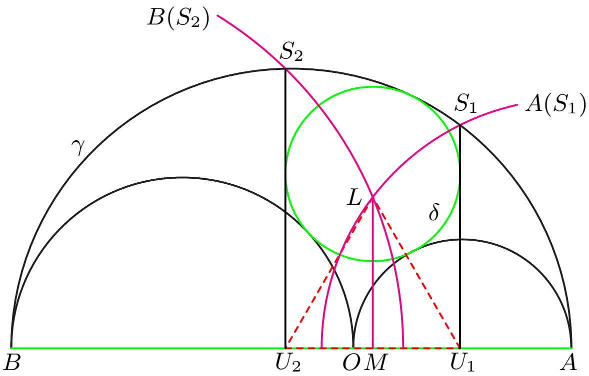


Figure 3.

**Theorem 2.** *Let $S_i$ $(i = 1, 2)$ be the points on $\gamma$ such that $U_i$ is the foot of perpendicular from $S_i$ to $AB$ and $S_iU_i$ touches the circle $\delta$ and the points $A$, $S_1$, $S_2$ lie on $\gamma$ in this order. Then the circles $A(S_1)$ and $B(S_2)$ intersect, and if $L$ is one of the points of intersection, the triangle $U_1LU_2$ is equilateral.*

**Proof.** The circles $A(S_1)$ and $B(S_2)$ meet in the limiting points of the pencil determined by $\varepsilon$ and $AB$ by Corollary 1. Let $L$ be one of points of intersection, and let $M$ be the foot of perpendicular from $L$ to $AB$. Since the power of $M$ with respect to $\delta$ equals $|LM|^2$ and the distance between the center of $\delta$ and $AB$ equals $2d$, where recall $d$ being the radius of $\delta$,

$|LM| = \sqrt{(2d+d)(2d-d)} = \sqrt{3}d$. While $|U_1U_2| = 2d$. Therefore the triangle $U_1LU_2$ is equilateral.

## References

[1] Gutierrez A., *Archimedes' Arbelos and Square 2*, Go Geometry web site, http://gogeometry.blogspot.jp/2008/12/archimedes-arbelos-and-square-2.html

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DEPARTMENT OF MATHEMATICS
YAMATO UNIVERSITY
2-5-1 Katayama Suita Osaka 564-0082, JAPAN
*E-mail address*: okumura.hiroshi@yamato-u.ac.jp