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MORE CHARACTERIZATIONS OF EXTANGENTIAL QUADRILATERALS

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Abstract. We prove ten necessary and sufficient conditions for a convex quadrilateral to have an excircle that concerns angles, areas, circles or concurrent lines.

1. Introduction

An *extangential quadrilateral* is a convex quadrilateral with an *excircle*, i.e. an external circle tangent to the extensions of all four sides, see Figure 1. A convex quadrilateral can at most have one excircle, and as with all classes of quadrilaterals, there are characterizations to determine when a quadrilateral has this property. In [7] we proved five metric characterizations of extangential quadrilaterals and compared them to similar conditions for tangential quadrilaterals (a quadrilateral with an incircle). In this paper we will prove ten more characterizations of extangential quadrilaterals that concerns angles, areas, circles or concurrent lines. A few corresponding theorems in tangential quadrilaterals were proved in [8].

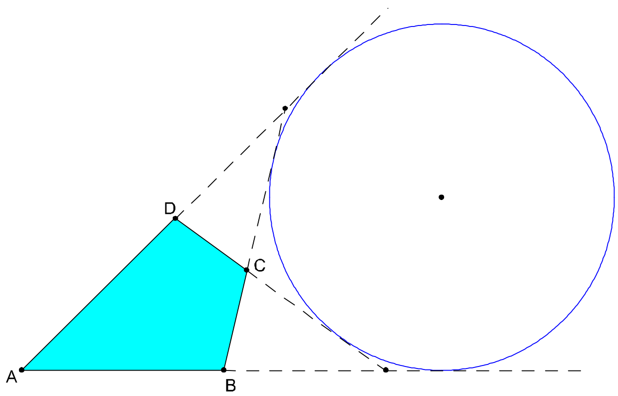

Figure 1. An extangential quadrilateral and its excircle

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We remind the reader that the Swiss mathematician Jakob Steiner proved in 1846 that a convex quadrilateral $ABCD$ has an excircle outside one of the vertices A and C if and only if (see [3, p.318])

$$AB + BC = CD + DA. \tag{1}$$

By symmetry there is an excircle outside one of the vertices B and D if and only if

$$DA + AB = BC + CD. \tag{2}$$

In these pairs of opposite vertices, the excircle is always outside the one with the biggest vertex angle.

2. Characterizations concerning angles or areas

We begin with a counterpart to Theorem 1 in [8] for an extangential quadrilateral.

Theorem 2.1. *Let the internal angle bisectors of two opposite angles in the convex quadrilateral $ABCD$ intersect at an exterior point J. Then it is an extangential quadrilateral if and only if $\angle AJD = \angle CJB$.*

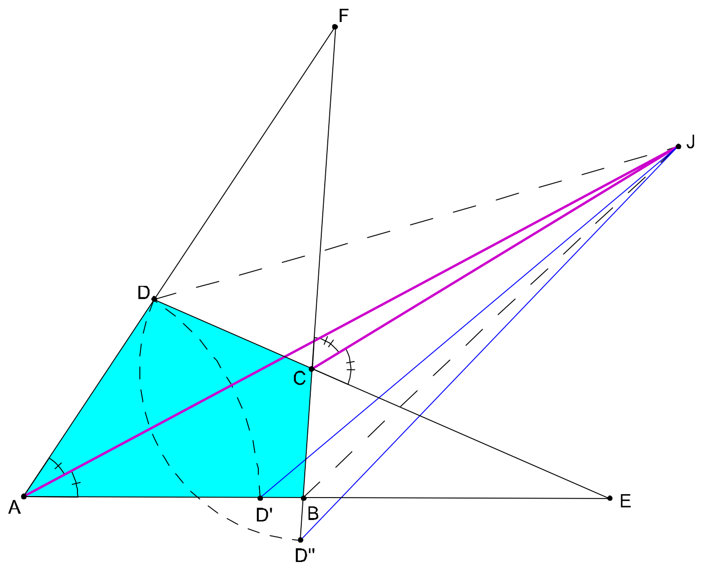

Figure 2. Intersecting angle bisectors

Proof. ($\Rightarrow$) Let $ABCD$ be an extangential quadrilateral where the extensions of opposite sides intersect at E and F (see Figure 2). Then the internal angle bisectors at two opposite vertex angles and the external angle bisectors at the other two vertex angles intersect at the excenter J (center of the excircle).[1] We assume that the excircle is outside of the vertex C (the proofs in the other three cases are the same). Then $\angle EDJ = \frac{\pi - D}{2}$, $\angle FBJ = \frac{\pi - B}{2}$ and $\angle BCJ = \pi - \frac{C}{2}$. Let us denote $\beta = \angle CJB$ and $\delta = \angle AJD$. Using the

[1] This is proved in the same way as the corresponding property in a triangle.

sum of angles in triangles BCJ and ADJ yields

$$\begin{aligned}\beta-\delta &= \left(\pi-\left(\pi-\frac{C}{2}+\frac{\pi-B}{2}\right)\right)-\left(\pi-\left(\frac{A}{2}+D+\frac{\pi-D}{2}\right)\right)\\ &= \frac{A+B+C+D}{2}-\pi=\frac{2\pi}{2}-\pi=0,\end{aligned}$$

where we also used the sum of angles in $ABCD$. Hence we get $\angle AJD = \angle CJB$.

($\Leftarrow$) Let $\angle AJD = \angle CJB$ in a convex quadrilateral where the angle bisectors of A and C intersect at a point J outside of C. We assume without loss of generality that $AB > AD$. (If instead there is equality, then the assumption $\angle AJD = \angle CJB$ can only be fulfilled in a kite. But then the two angle bisectors don't intersect since they coincide.) First we prove that $CD > CB$. We construct a point D' on AB and D'' on CB or its extension such that $AD' = AD$ and $CD'' = CD$, see Figure 2. Then

$$\angle CJD - \angle BJC = \angle AJC + \angle AJD - \angle CJB = \angle AJC > 0.$$

Thus

$$\angle CJD > \angle BJC \quad \Rightarrow \quad \angle CJD'' > \angle BJC$$

which in turn imply that $CD'' > CB$ and therefore $CD > CB$.

Now triangles ADJ and $AD'J$ are congruent, so $DJ = D'J$, and triangles CDJ and $CD''J$ are congruent, so $DJ = D''J$. Thus $D'J = D''J$. We also have that $\angle D''JC = \angle DJC$ and $\angle AJD = \angle AJD'$. Whence

$$(3) \qquad \angle D''JC - \angle CJB = \angle DJC - \angle AJD \quad \Rightarrow \quad \angle D''JB = \angle AJC.$$

In addition, $\angle CJB = \angle AJD'\,(= \angle AJD)$, so

$$(4) \qquad \angle CJD' + \angle D'JB = \angle AJC + \angle CJD' \quad \Rightarrow \quad \angle D'JB = \angle AJC.$$

From (3) and (4) we conclude that $\angle D''JB = \angle D'JB$. Thus, since $D'J = D''J$ and BJ is a common side, triangles $D'JB$ and $D''JB$ are congruent, so $D'B = D''B$. Hence

$$AB + BC - CD - DA = AD' + D'B + BC - D''B - BC - AD' = 0$$

which proves that $ABCD$ is an extangential quadrilateral with an excircle outside of A or C according to (1). □

An alternative and equivalent formulation of this angle characterization exists. Since

$$\angle AJD = \angle CJB \quad \Leftrightarrow \quad \angle CJD = \angle AJB$$

the theorem could as well have had the same formulation except for the angle equality, that instead would have been that $\angle AJB = \angle CJD$. The rewrite between these two equalities is simply a matter of adding or subtracting the common angle AJC.

At first glance it may seem remarkable that the angle equality is the same in all four cases of extangential quadrilaterals outside any of the vertices. But transforming the vertices (in several steps) according to $A \to B \to C \to D \to A$ we see that there are only two cases of angle equalities (since for instance $\angle AJD = \angle DJA$). These are $\angle AJD = \angle CJB$ and $\angle AJB = \angle CJD$, which we just noted to be equivalent.

There is the following related characterization to the one in Theorem 2.1, where the equality is between four triangle areas instead. We reviewed the corresponding characterization for a tangential quadrilateral in [8, p.3]. That theorem was proved in [10] and [11, pp.134–135], where the latter attributes it to V. Pop and I. Gavrea. Note that by symmetry, there is a similar necessary and sufficient condition for the other pair of opposite vertex angles.

Theorem 2.2. *A convex quadrilateral $ABCD$ has an excircle outside one of the vertices A or C if and only if*

$$S_{AJB} + S_{BJC} = S_{CJD} + S_{DJA}$$

where J is the intersection of the angle bisectors at A and C, and S_{AJB} stands for the area of triangle AJB.

Proof. ($\Rightarrow$) The direct part of the theorem is a trivial corollary to (1). Simply multiply both sides of that equation by $\frac{1}{2}\rho$, where ρ is the exradius (the radius in the excircle), and the equality follows.

($\Leftarrow$) Conversely, if the equality between the four areas holds in a convex quadrilateral, we construct points D' and D'' as in the proof of Theorem 2.1. Then

$$S_{AJD'} + S_{D'JB} + S_{D''JC} - S_{D''JB} = S_{CJD} + S_{DJA}.$$

But triangles AJD' and DJA as well as triangles $D''JC$ and CJD are congruent. Thus we get that $S_{D'JB} = S_{D''JB}$. Then

$$\frac{BJ \cdot JD' \sin\angle D'JB}{2} = \frac{BJ \cdot JD'' \sin\angle D''JB}{2}.$$

From the two pairs of congruent triangles, we also have that $JD' = JD = JD''$. Thus $\sin\angle D'JB = \sin\angle D''JB$ and it follows that $\angle D'JB = \angle D''JB$ since these are both acute angles. This proves that triangles $D'JB$ and $D''JB$ are congruent, so $D'B = D''B$. Hence we finally have

$$AB + BC - CD - DA = AD' + D'B + BC - D''B - BC - AD' = 0$$

which proves that $ABCD$ is an extangential quadrilateral with an excircle outside of A or C according to (1). □

3. Characterizations concerning circles

The first characterization regarding circles is about incircles in the two subtriangles created by a diagonal. The direct part of (i) in this theorem was a problem solved in [2, p.116].

Theorem 3.1. *Consider a convex quadrilateral $ABCD$.*
(i) *Let the incircles in triangles ABD and CBD be tangent to the diagonal BD at S and T respectively. Then the quadrilateral has an excircle outside one of the vertices A or C if and only if $BT = DS$.*
(ii) *Let the incircles in triangles BAC and DAC be tangent to the diagonal AC at U and V respectively. Then the quadrilateral has an excircle outside one of the vertices B or D if and only if $AV = CU$.*

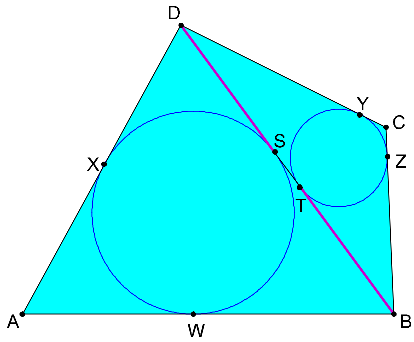

Figure 3. Incircles in two subtriangles

Proof. We prove the first statement, the second is proved in the same way. Let the incircles be tangent to the sides BA, AD, DC and CB at W, X, Y and Z respectively. We assume without loss of generality that $DS < DT$. Then $AW = XA$, $WB = BS$, $BZ = BT$, $ZC = CY$, $YD = DT$, $DX = DS$ (see Figure 3) according to the two tangent theorem (the two tangents to a circle through an external point have the same lengths). Thus

$$\begin{aligned} AB + BC - CD - DA &= AW + WB + BZ + ZC - CY - YD - DX - AX \\ &= BS + BT - DT - DS \\ &= BT + ST + BT - DS - ST - DS \\ &= 2(BT - DS). \end{aligned}$$

Hence we have that

$$AB + BC = CD + DA \quad \Leftrightarrow \quad BT = DS$$

which proves that the quadrilateral has an excircle outside one of the vertices A or C if and only if $BT = DS$ according to (1). □

Since the line segments ST and UV are common to the considered distances in pairs, alternative equivalent statements would have been that the excircle is outside one of the vertices A or C if and only if $BS = DT$, and that it is outside one of B or D if and only if $AU = CV$.

The next characterization is a counterpart to Theorem 1 in [6] for an extangential quadrilateral.

Theorem 3.2. *A convex quadrilateral $ABCD$ has an excircle outside one of the vertices A or C if and only if it holds that*
(i) *the incircle in triangle ABD and the excircle to triangle CBD are tangent to the diagonal BD at the same point, or*
(ii) *the incircle in triangle CBD and the excircle to triangle ABD are tangent to the diagonal BD at the same point.*
A convex quadrilateral $ABCD$ has an excircle outside one of the vertices B or D if and only if it holds that
(iii) *the incircle in triangle BAC and the excircle to triangle DAC are tangent to the diagonal AC at the same point, or*
(iv) *the incircle in triangle DAC and the excircle to triangle BAC are tangent to the diagonal AC at the same point.*

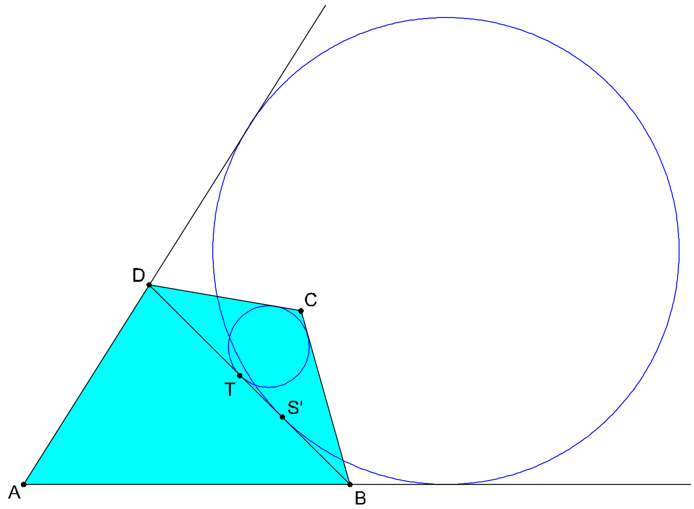

Figure 4. Tangency points on diagonal BD

Proof. Since the four proofs are so similar, we only do one of them. Let's prove (ii). The length of the segments on the sides of a triangle determined by the points of tangency of the incircle and the excircle have well-known formulas (see [5, p.184]). If the incircle and excircle are tangent to BD at T and S' respectively (see Figure 4), then

$$2(BT-BS') = (BD+BC-CD)-(BD+AD-AB) = AB+BC-CD-DA.$$

Thus

$$T \equiv S' \quad \Leftrightarrow \quad BT = BS' \quad \Leftrightarrow \quad AB+BC = CD+DA$$

which proves that the two circles are tangent at the same point on BD if and only if the quadrilateral has an excircle outside one of the vertices A or C according to (1). □

Now we prove the corresponding characterization to Theorem 5 in [8] for an extangential quadrilateral.

Theorem 3.3. *In a convex quadrilateral $ABCD$ that is not a trapezoid,*[2] *let the extensions of opposite sides intersect at E and F.*
(i) *The quadrilateral has an excircle outside of A if and only if the incircle in triangle AEF and the excircle to triangle CEF are tangent to EF at the same point.*
(ii) *The quadrilateral has an excircle outside of C if and only if the incircle in triangle CEF and the excircle to triangle AEF are tangent to EF at the same point.*
(iii) *The quadrilateral has an excircle outside of B if and only if the incircle in triangle BEF and the excircle to triangle DEF are tangent to EF at the same point.*
(iv) *The quadrilateral has an excircle outside of D if and only if the incircle in triangle DEF and the excircle to triangle BEF are tangent to EF at the same point.*

[2] And thus neither of the special cases parallelogram, rhombus, rectangle nor a square.

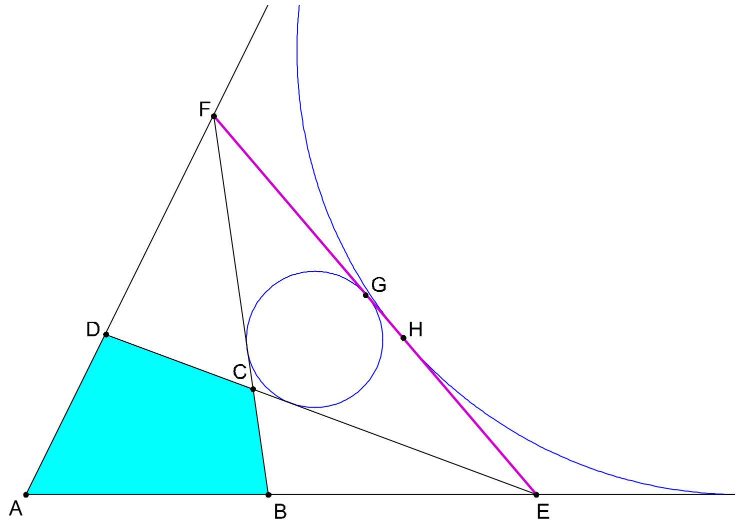

Figure 5. Tangency points on EF

Proof. We prove (ii), the other proofs are similar. Let the incircle and excircle be tangent to EF at G and H respectively (see Figure 5). In the same way as in the proof of Theorem 3.2, we have

$$2(FH-FG) = (EF+AE-AF)-(EF+CF-CE) = AE-AF-CF+CE.$$

Hence

$$G \equiv H \quad \Leftrightarrow \quad FG = FH \quad \Leftrightarrow \quad AE + CE = AF + CF.$$

According to equation (5) in [7], this proves that the two circles are tangent to EF at the same point if and only if the quadrilateral $ABCD$ has an excircle outside of A or C. It is evident that the excircle to the quadrilateral must be outside of the vertex where the incircle to triangle CEF is, since that is the only place where it can be tangent to the extensions of all four sides of the quadrilateral (the opposite sides diverge outside of the other vertex A). □

The next characterization is the counterpart to Theorem 6 in [8] for an extangential quadrilateral. There are four different versions of this theorem depending on outside which vertex the excircle can be located, but we just formulate one of them and trust the reader can make the appropriate changes of letters for the other three cases.

Theorem 3.4. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at E and F. Suppose C is a vertex such that no parts except one point of the sides of triangle CEF coincides with the sides of $ABCD$. Let the excircle outside of EF to triangle AEF and the incircle in triangle CEF be tangent to the extensions of AD, AB, DC, BC at K, L, M, N respectively. Then $ABCD$ is an extangential quadrilateral with an excircle outside of C if and only if $KLMN$ is a cyclic quadrilateral.*

Proof. (⇒) In an extangential quadrilateral $ABCD$, let AB and DC intersect at E, and AD and BC intersect at F. We have that $AK = AL$ according to the two tangent theorem, so $\angle AKL = \frac{\pi - A}{2}$ in the isosceles triangle AKL (see Figure 6). It further holds that $\angle AFB = \pi - A - B$ and if the incircle in CEF is tangent to EF at G, then $FK = FG = FN$. Thus $\angle AKN = \angle FKN = \frac{\pi - A - B}{2}$ by the exterior angle theorem. This yields

$$\angle NKL = \angle AKL - \angle AKN = \frac{\pi - A}{2} - \frac{\pi - A - B}{2} = \frac{B}{2}.$$

In the same way $CM = CN$, so $\angle CMN = \frac{\pi - C}{2}$, and $\angle AED = \pi - A - D$. Then, since $EL = EG = EM$, it follows that $\angle EML = \frac{\pi - A - D}{2}$. Thus

$$\angle NML = \pi - \angle CMN - \angle EML = \pi - \frac{\pi - C}{2} - \frac{\pi - A - D}{2} = \frac{A + C + D}{2}.$$

Hence two opposite angles in $KLMN$ have the sum

$$\angle NKL + \angle NML = \frac{B}{2} + \frac{A + C + D}{2} = \frac{2\pi}{2} = \pi.$$

This proves that $KLMN$ is a cyclic quadrilateral according to a well-known characterization.

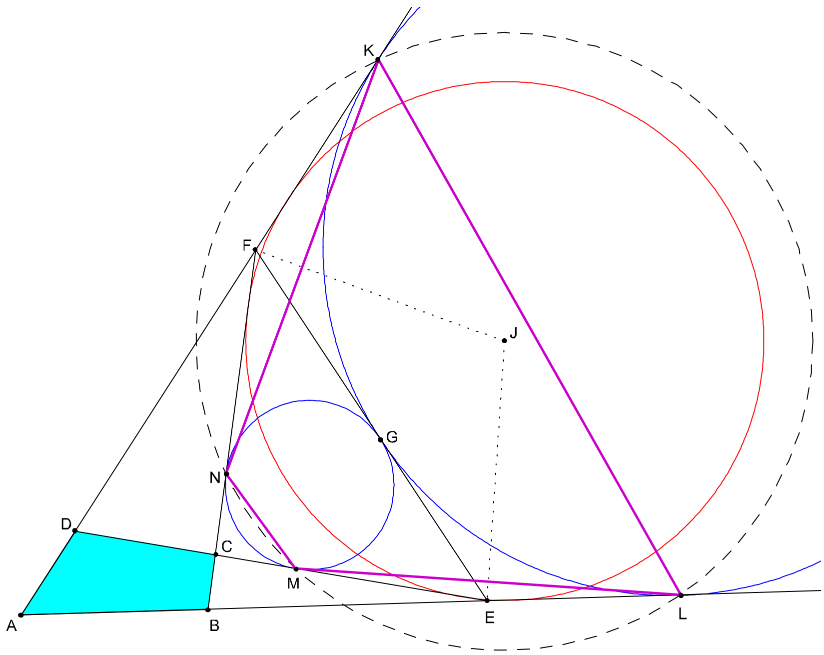

Figure 6. Here $KLMN$ is a cyclic quadrilateral

($\Leftarrow$) We do an indirect proof of the converse. If $ABCD$ is not an extangential quadrilateral, assume that the incircle in CEF and the excircle to AEF are tangent to EF at G and H respectively (these are different points by Theorem 3.3, see Figure 7). We assume without loss of generality that $FH < FG$. Then $FK = FH < FG = FN$, so $\angle AKN = \angle FKN > \frac{\pi - A - B}{2}$ since a longer side in a triangle is opposite a larger angle. Thus

$$\angle NKL = \angle AKL - \angle AKN < \frac{\pi - A}{2} - \frac{\pi - A - B}{2} = \frac{B}{2}.$$

We also have $EL = EH > EG = EM$, so $\angle EML > \frac{\pi - A - D}{2}$, and we deduce that

$$\angle NML = \pi - \angle CMN - \angle EML < \pi - \frac{\pi - C}{2} - \frac{\pi - A - D}{2} = \frac{A + C + D}{2}.$$

Hence two opposite angles in $KLMN$ have the sum

$$\angle NKL + \angle NML < \frac{A+B+C+D}{2} = \pi$$

which proves that $KLMN$ is not a cyclic quadrilateral. □

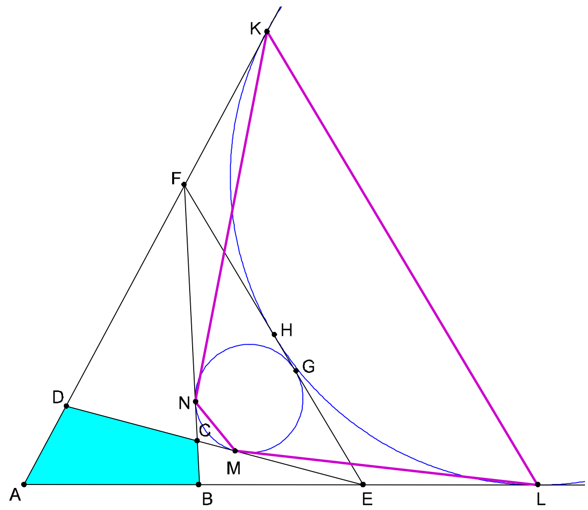

Figure 7. Here $KLMN$ is not a cyclic quadrilateral

Corollary 3.1. *If $ABCD$ is an extangential quadrilateral, then its excircle and the circumcircle to quadrilateral $KLMN$ in Theorem 3.4 are concentric.*

Proof. Triangles KFN and LEM are isosceles, so their perpendicular bisectors to the sides KN and ML and the angle bisectors to the angles KFN and LEM are identical in pairs. Hence they have the same point of intersection J, see Figure 6, so the two circles are concentric. □

What happens if we in Theorem 3.4 instead consider the incircle in triangle AEF and the excircle tangent to EF in triangle CEF? It will probably not come as a big surprise that the result is the same; this too gives a characterization of extangential quadrilaterals, see Figure 8. Since the method of proof is the same, we only state the theorem here, and let the reader record the proof.

Theorem 3.5. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at E and F. Suppose C is a vertex such that no parts except one point of the sides of triangle CEF coincides with the sides of $ABCD$. Let the excircle outside of EF to triangle AEF and the incircle in triangle CEF be tangent to the extensions of AD, AB, DC, BC at K', L', M', N' respectively. Then $ABCD$ is an extangential quadrilateral with an excircle outside of C if and only if $K'L'M'N'$ is a cyclic quadrilateral.*

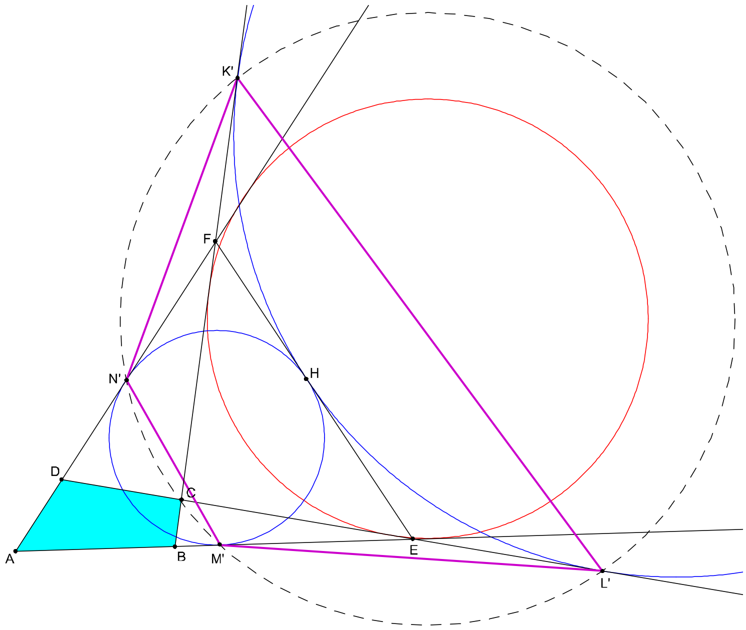

Figure 8. The cyclic quadrilateral $K'L'M'N'$

Again it is easy to deduce that there are two concentric circles (see Figure 8):

Corollary 3.2. *If $ABCD$ is an extangential quadrilateral, then its excircle and the circumcircle to quadrilateral $K'L'M'N'$ in Theorem 3.5 are concentric.*

4. Characterizations concerning concurrent lines

The first characterization regarding concurrent lines is about the same configuration as the one in Theorem 3.4.

Theorem 4.1. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at E and F. Suppose C is a vertex such that no parts except one point of the sides of triangle CEF coincides with the sides of $ABCD$. Let the excircle outside of EF to triangle AEF and the incircle in triangle CEF be tangent to the extensions of AD, AB, DC, BC at K, L, M, N respectively. Then $ABCD$ is an extangential quadrilateral with an excircle outside of C if and only if KN, LM and AC are concurrent.*

Proof. ($\Rightarrow$) Let KN and LM intersect AC at Q_1 and Q_2 respectively in an extangential quadrilateral, see Figure 9. We apply Menelaus' theorem in triangle ACF with the transversal KNQ_1 to get[3]

$$\frac{FK}{KA} \cdot \frac{AQ_1}{Q_1C} \cdot \frac{CN}{NF} = 1 \quad \Rightarrow \quad \frac{AQ_1}{Q_1C} = \frac{KA}{CN} \tag{5}$$

where $FK = FG = NF$ according to the two tangent theorem and the fact that the excircle to triangle AEF and the incircle in triangle CEF are

[3]We use non-directed distances, in which case one of the sides in Menelaus' theorem is a $+1$ instead of a -1.

tangent to EF at the same point G by Theorem 3.3. Using the transversal LMQ_2 in triangle ACE yields in the same way

$$\frac{EL}{LA}\cdot\frac{AQ_2}{Q_2C}\cdot\frac{CM}{ME}=1 \quad\Rightarrow\quad \frac{AQ_2}{Q_2C}=\frac{LA}{CM} \tag{6}$$

since $EL = EG = ME$. But we also have that $KA = LA$ and $CM = CN$ according to the two tangent theorem. Thus

$$\frac{AQ_1}{Q_1C}=\frac{AQ_2}{Q_2C}$$

which means that the two points Q_1 and Q_2 divide the line segment AC in the same ratio. Hence they must coincide, so we have proved that KN, LM and AC are concurrent at $Q_1 \equiv Q_2$.

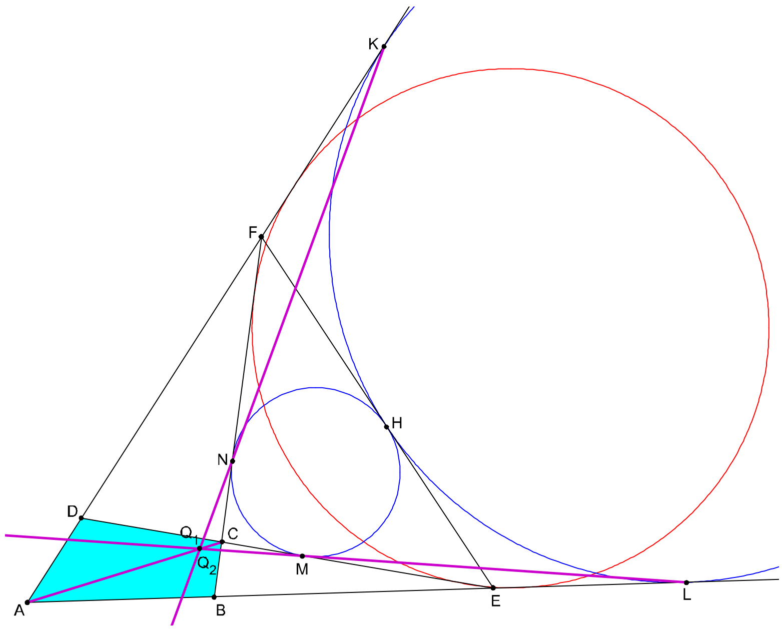

Figure 9. Points of intersection on AC

($\Leftarrow$) If $ABCD$ is not an extangential quadrilateral, then the incircle in triangle CEF and the excircle to triangle AEF are tangent to EF at different points G and H respectively (Theorem 3.3). Assume without loss of generality that $EG < EH$ (see Figure 7). The first equality in (5) still holds, but since we now have that $FK = FH < FG = NF$, it yields

$$FK\cdot\frac{AQ_1}{Q_1C}=NF\cdot\frac{KA}{CN}>FK\cdot\frac{KA}{CN}$$

so we have

$$\frac{AQ_1}{Q_1C}>\frac{KA}{CN}. \tag{7}$$

The first equality in (6) also still holds, and applying $EL = EH > EG = EM$, we get

$$\frac{AQ_2}{Q_2C}<\frac{LA}{CM}=\frac{KA}{CN}<\frac{AQ_1}{Q_1C}.$$

We used that $KA = LA$ and $CM = CN$ still holds, and applied (7) to get the last inequality. Thus Q_2 and Q_1 divide AC in different ratios, so

$Q_1 \neq Q_2$. This proves that the line segments KN, LM and AC are not concurrent. □

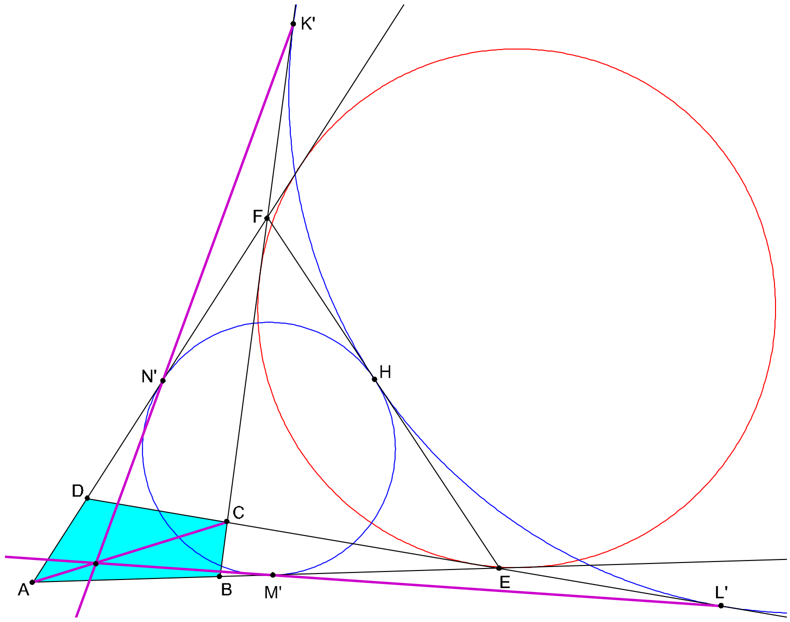

Figure 10. The lines $K'N'$ and $L'M'$ intersect on AC

In the same way that we got a similar characterization when we exchanged the roles of the incircle and excircle in Theorem 3.4, which gave Theorem 3.5, we have a similar characterization to Theorem 4.1 when making that change (see Figure 10). The following theorem can be proved using the same method we used to prove Theorem 4.1, so the proof is omitted.

Theorem 4.2. *In a convex quadrilateral $ABCD$ that is not a trapezoid, let the extensions of opposite sides intersect at E and F. Suppose C is a vertex such that no parts except one point of the sides of triangle CEF coincides with the sides of $ABCD$. Let the excircle outside of EF to triangle AEF and the incircle in triangle CEF be tangent to the extensions of AD, AB, DC, BC at K', L', M', N' respectively. Then $ABCD$ is an extangential quadrilateral with an excircle outside of C if and only if $K'N'$, $L'M'$ and AC are concurrent.*

In the beginning of May in 2010, a problem was posted at *Art of Problem Solving* [9] that is the direct part of the following necessary and sufficient condition for when a convex quadrilateral has an excircle. Three days later, a short solution using insimilicenter, exsimilicenter and the Monge-d'Alembert theorem was given by Luis González. Here we give a more elementary proof of the direct part and prove that the converse is true as well.

Theorem 4.3. *In a convex quadrilateral $ABCD$, let I_1 and I_2 be the incenters in triangles BCD and DAB respectively. Then the quadrilateral has an excircle outside of A or C if and only if AC, BD and I_1I_2 are concurrent.*

Proof. (⇒) In an extangential quadrilateral where the sides satisfy $AB + BC = CD + DA$, let P' be the intersection between BD and I_1I_2, and let J be the center of the excircle (which we assume without loss of generality is outside of the vertex C). Also, let P_1, P_2, P_3, P_4, P_5, P_6 be points on

BD, AB, CD or their extensions where the two incircles and the excircle are tangent to these lines (see Figure 11). Then we have three pairs of similar triangles, $JAP_3 \sim I_2AP_4$, $I_2P'P_1 \sim I_1P'P_2$ and $I_1CP_5 \sim JCP_6$. Thus

$$\frac{JA}{I_2A} = \frac{JP_3}{I_2P_4}, \qquad \frac{I_2P'}{I_1P'} = \frac{I_2P_1}{I_1P_2}, \qquad \frac{I_1C}{JC} = \frac{I_1P_5}{JP_6}.$$

Forming the product of these yields

$$\frac{JA}{AI_2} \cdot \frac{I_2P'}{P'I_1} \cdot \frac{I_1C}{CJ} = \frac{JP_3}{I_2P_4} \cdot \frac{I_2P_1}{I_1P_2} \cdot \frac{I_1P_5}{JP_6} = \frac{JP_3}{I_2P_1} \cdot \frac{I_2P_1}{I_1P_2} \cdot \frac{I_1P_2}{JP_3} = 1,$$

where we used that $I_2P_1 = I_2P_4$, $I_1P_5 = I_1P_2$ and $JP_3 = JP_6$ (these are radii in the three circles). According to the converse of Menelaus' theorem applied in triangle I_1I_2J with the transversal AC, the points C, P' and A are collinear. Since we already know that BD and I_1I_2 intersect at P', this proves that AC, BD and I_1I_2 are concurrent at P'.

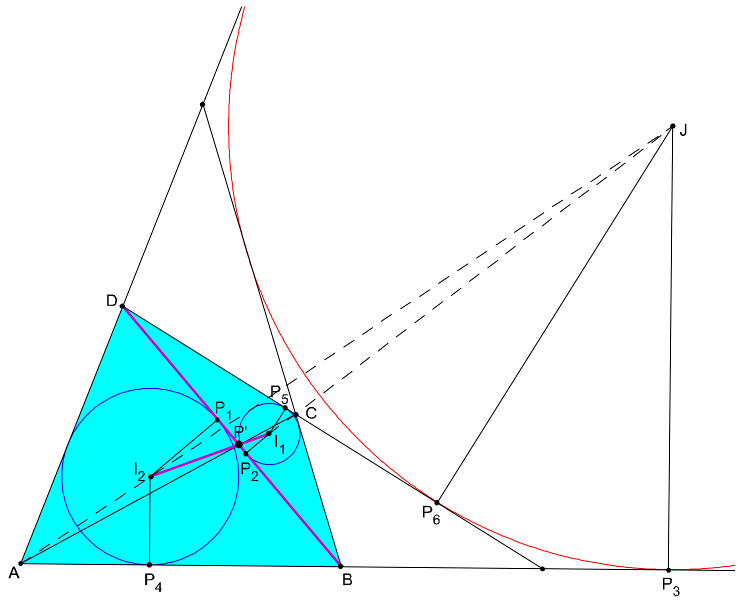

Figure 11. P' is the intersection of BD and I_1I_2

($\Leftarrow$) In a convex quadrilateral where AC, BD and I_1I_2 are concurrent at a point P, let the lines AI_2 and I_1C intersect at a point J'. We use the notation $d(J', AB)$ for the distance between the point J' and the line AB. Also, let r_{ABD} be the inradius in triangle ABD. The similarities used in the first part of the proof still hold if we exchange the exradius for the appropriate distances between J' and a side or its extension. Applying the direct part of Menelaus' theorem yields

$$1 = \frac{J'A}{AI_2} \cdot \frac{I_2P}{PI_1} \cdot \frac{I_1C}{CJ'} = \frac{d(J', AB)}{r_{ABD}} \cdot \frac{r_{ABD}}{r_{BCD}} \cdot \frac{r_{BCD}}{d(J', CD)} = \frac{d(J', AB)}{d(J', CD)}.$$

Thus we conclude that $d(J', AB) = d(J', CD)$.

But we might as well consider similar triangles where the normals from J' are drawn to AD and BC or their extensions instead (see Figure 12). Then we get

$$1 = \frac{J'A}{AI_2} \cdot \frac{I_2P}{PI_1} \cdot \frac{I_1C}{CJ'} = \frac{d(J', AD)}{r_{ABD}} \cdot \frac{r_{ABD}}{r_{BCD}} \cdot \frac{r_{BCD}}{d(J', BC)} = \frac{d(J', AD)}{d(J', BC)}.$$

Whence $d(J', AD) = d(J', BC)$. A third option is to draw one normal to AD and one to CD. The final result from Menelaus' theorem this time is the equality $d(J', AD) = d(J', CD)$. Combining the three equalities regarding those distances, we have

$$d(J', AB) = d(J', CD) = d(J', AD) = d(J', BC).$$

This means that the point J' is equidistant from the extensions of the sides in the convex quadrilateral $ABCD$. Hence J' is the center in a circle tangent to the side extensions, which proves that $ABCD$ is an extangential quadrilateral. □

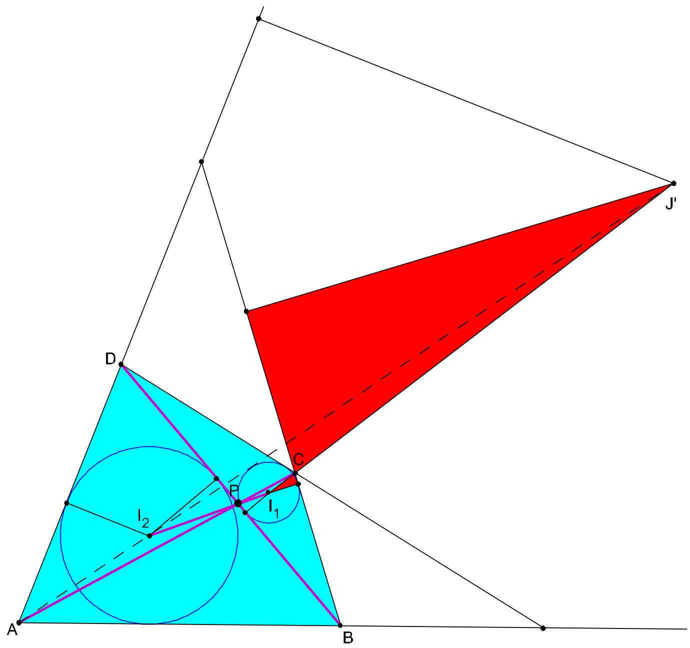

Figure 12. Here AC, BD and I_1I_2 are concurrent at P

By symmetry there is a similar necessary and sufficient condition for an excircle outside one of the other two vertices.

We note that the same configuration with two subtriangle incircles and an extangential quadrilateral was the subject of the final problem at the International Mathematical Olympiad in 2008 (problem G7 on the short list). The problem, which was proposed by Vladimir Shmarov from Russia, can be reformulated in the following way (with notations as in Figure 11):

Suppose that $ABCD$ is an extangential quadrilateral with an excircle outside of C. Let the incircles in triangles ABD and CBD be tangent to the diagonal BD at P_1 and P_2, and have incenters I_1 and I_2 respectively. Prove that the lines AP_2, CP_1 and I_1I_2 concur in a point on the circumference of the excircle.

The official solution to this beautiful problem appears in [1, pp.40–41], where you can also find the original formulation of the problem. A similar solution was given in [2, pp.175–177].

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