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GRAPH-DIRECTED COUNTABLE IFS

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Abstract. In [10], the notion of iterated function system (IFS) is generalized to a sequence of contractions on a compact metric space so-called countable IFS (CIFS). On the other hand, graph-directed iterated function systems (GIFSs) can be thought as a generalization of IFSs. In this work, we define graph-directed "countable" iterated function systems (GCIFSs) as a generalization of GIFSs to the countable case in the sense of [10] and give new examples for each system.

1. Introduction

Most of fractals can be realized as an attractor of an iterated function system (IFS) which is a finite collection of contractions on a complete metric space. We first give a small brief for this notion.

Let (X, d) be a complete metric space and

$$\{w_i : X \to X \mid i = 1, 2, \dots, n\}$$

be a family of finite contractions with contractivity ratios $0 < r_i < 1$. The system $\{X; w_i, i = 1, 2, \dots, n\}$ is called iterated function system (IFS) (see [1]). Hutchinson, in his famous paper [5], proves by using the Banach Contraction Principle that there exists a unique nonempty compact set $A \subset X$ such that

$$A = \bigcup_{i=1}^{n} w_i(A)$$

which is the unique fixed point of the set valued contraction

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$$W : \mathcal{H}(X) \longrightarrow \mathcal{H}(X),\ W(B) = \bigcup_{i=1}^{n} w_i(B),$$

where $\mathcal{H}(X)$ is the set of nonempty compact subsets of X which is also complete metric space with the Hausdorff metric ([1]). The fixed point A of W is called the attractor of the IFS (or usually self-similar fractal).

On the other hand, there is a generalization of the notion of iterated function system which is called graph-directed iterated function system (GIFS). Let $\{X^\alpha \,|\, \alpha = 1, 2, \ldots, N\}$ be a finite collection of complete metric spaces and $w_i^{\alpha\beta} : X^\beta \to X^\alpha$ be contractions with contractivity ratio $0 < r_i^{\alpha\beta} < 1$ for each $i = 1, 2, \ldots, K^{\alpha\beta}$, where $K^{\alpha\beta}$ denotes the number of contractions from X^β to X^α. Assume that for each $\alpha = 1, 2, \ldots, N$, $K^{\alpha\beta} > 0$ for some β. It can be shown that there exists a unique family of nonempty compact sets $\{A^\alpha \,|\, \alpha = 1, 2, ..., N\}$ such that $A^\alpha \subset X^\alpha$ and

$$A^\alpha = \bigcup_{\beta=1}^{N} \bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)$$

for $\alpha = 1, 2, \ldots, N$ (see [4]). The system $\left\{X^\alpha; w_i^{\alpha\beta}\right\}$ is called a graph-directed iterated function system (GIFS) and the compact sets $A^\alpha, \alpha = 1, 2, \ldots, N$ are called the attractors of the system (or usually graph-directed fractals). Similar to the classical case, this collection of compact sets is the unique fixed point of the contraction

$$\begin{array}{rcl} W : \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N) & \longrightarrow & \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N) \\ B = (B^1, \ldots, B^N) & \mapsto & W(B) = (W_1(B), \ldots, W_N(B)), \end{array}$$

where the product space $\mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N)$ is a complete metric space with the maximum metric

$$h_{\max}(A, B) = \max\{h_1(A^1, B^1), \ldots, h_N(A^N, B^N)\},$$

where $A = (A^1, \ldots, A^N), B = (B^1, \ldots, B^N)$ and h_α is the corresponding Hausdorff metric on the complete metric space $\mathcal{H}(X^\alpha)$ and

$$W_\alpha(B) = W_\alpha(B^1, \ldots, B^N) = \bigcup_{\beta=1}^{N} \bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(B^\beta)$$

(see [4],[9] for more details).

To see this different kind of self-similarity see the example below.

Example 1.1. *Let X^1 and X^2 be the complete metric spaces (with respect to the induced metric from Euclidean space) as shown in Figure 1 and let $w_i^{\alpha\beta} : X^\beta \to X^\alpha$ be the contractions (in fact, similitudes) where $K^{11} = 6, K^{12} = 4, K^{21} = 6$ and $K^{22} = 6$ as indicated in Figure 2. The attractors of this graph-directed system shown in Figure 3 (as the fourth stage of the iteration).*

It is important to generalize the notion of classical (finite) iterated function system to the infinite case since this notion has became more important in the last decades especially in the engineering and biological sciences. See

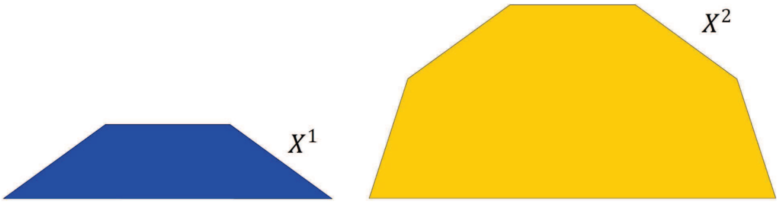

FIGURE 1. The complete metric spaces X^1 and X^2.

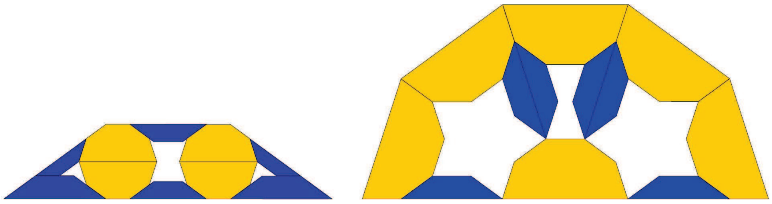

FIGURE 2. Pictorial representation of the contractions $w_i^{\alpha\beta}$.

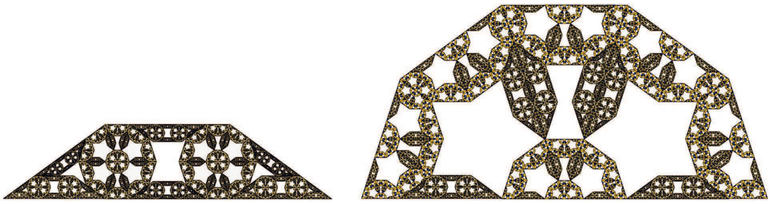

FIGURE 3. Fourth stage of the iteration of the GIFS.

[10],[3],[8],[6] and [7] for some works published last years. Secelean, in his 2001 paper [10], was defined "countable iterated function systems" as follows:

Let (X, d) be a "compact" metric space and let $\{w_i, i \in \mathbb{N}\}$ be a sequence of contraction mappings with contractivity ratios $0 < r_i < 1$ such that $\sup_{i \in \mathbb{N}}\{r_i\} < 1$ where $\mathbb{N}$ is the set of natural numbers starting from 1. The system $\{X; w_i, i \in \mathbb{N}\}$ is called countable iterated function system (CIFS). According to the Banach Contraction Principle, there exists a unique nonempty compact set $A \subset X$ (the attractor of the CIFS) that satisfies

$$A = \overline{\bigcup_{i=1}^{\infty} w_i(A)}$$

as the unique fixed point of the contraction

$$W : \mathcal{H}(X) \longrightarrow \mathcal{H}(X),\ W(B) = \overline{\bigcup_{i=1}^{\infty} w_i(B)}, \tag{1}$$

where $\mathcal{H}(X)$ is also compact (hence also complete) metric space with the Hausdorff metric since X is ([11, Theorem 1.16]) and the bar stands for the closure of the set (see [11] for more details).

Remark 1.1. *Secelean's construction works on a compact metric space. For the case X is complete, if $\{w_i \,|\, i \in \mathbb{N}\}$ is a bounded family of contractions (i.e., $\cup w_i(A)$ is bounded for every bounded subset A) such that the supremum of the contractivity ratios less than* 1*, then the mapping W (as in (1)) is again a contraction on $\mathcal{H}(X)$ and one can apply the Banach Contraction Principle to obtain a unique compact set as an attractor (see* [3] *for more details).*

Example 1.2. *Let $X \subset \mathbb{R}^2$ be the compact space as shown in Figure 4a. We indicate in Figure 4b the sequence of the similitudes which constitute a CIFS whose attractor (as the third stage of the iteration) is given in Figure 4c.*

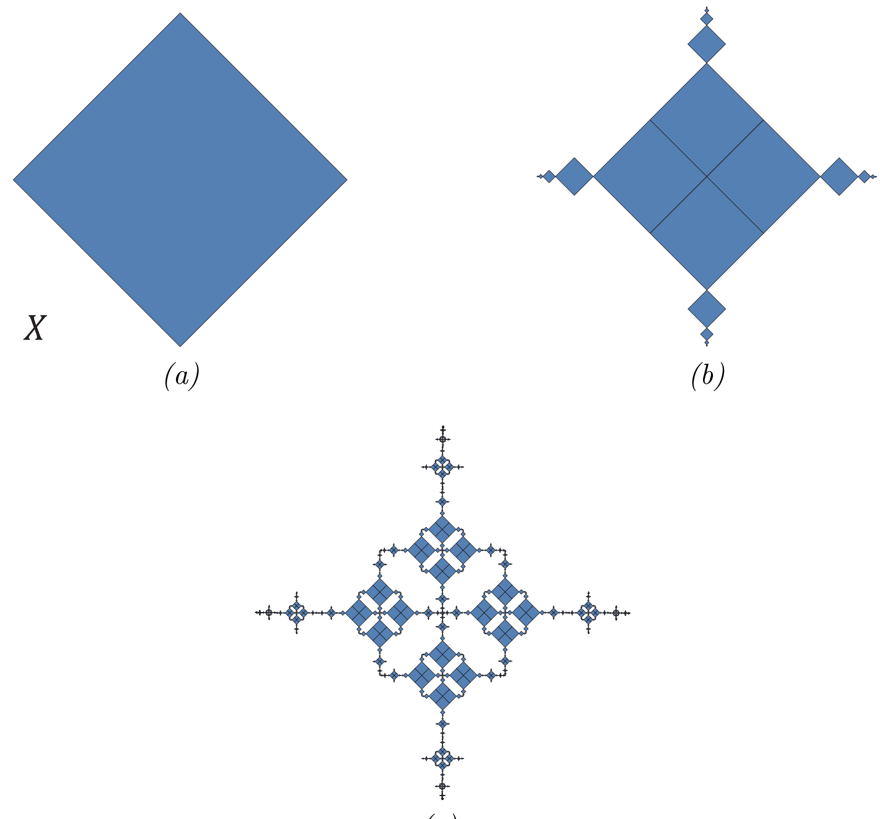

Figure 4. (a) The compact metric space X, (b) pictorial representation of the contractions and (c) the third stage of the iteration of the CIFS.

In the next section, we generalize the graph-directed system by allowing countable number of contractions between any pair of metric spaces (in the sense of [10]). (There exists a generalization of infinite IFSs to the graph-directed case (see [2]) but in the sense of [6] which is a quite different generalization of IFSs to the infinite case).

2. Graph-Directed Countable IFSs

Lemma 2.1. *Let $\{A_i\}_{i\in I}$ and $\{B_i\}_{i\in I}$ be two sets of family of nonempty compact subsets of a metric space X. Then*

$$h\left(\overline{\bigcup_{i\in I} A_i}, \overline{\bigcup_{i\in I} B_i}\right) \leq \sup_i h(A_i, B_i)$$

where h is the Hausdorff metric on $\mathcal{H}(X)$ ([11, Theorem 1.13]*).*

Lemma 2.2. *Let $\{X^\alpha \mid \alpha = 1, 2, \ldots, N\}$ be a family of compact metric spaces and the mappings $w_i^{\alpha\beta} : X^\beta \to X^\alpha$, $(\alpha, \beta = 1, 2, \ldots, N, i = 1, 2, \ldots, K^{\alpha\beta})$ be contractions with contractivity ratios $0 < r_i^{\alpha\beta} < 1$ such that $\sup\{r_i^{\alpha\beta}\} < 1$ where $K^{\alpha\beta} \in \mathbb{N} \cup \{\infty\}$ denotes the number of contractions from X^β to X^α. Assume that for each $\alpha = 1, 2, \ldots, N$, $K^{\alpha\beta} > 0$ for some β. Let*

$$\begin{aligned} W : \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N) &\longrightarrow \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N) \\ A = (A^1 \ldots, A^N) &\mapsto W(A) = (W_1(A), \ldots, W_N(A)) \end{aligned}$$

where

$$W_\alpha(A) = \overline{\bigcup_{\beta=1}^{N} \bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)}$$

for $\alpha = 1, 2, \ldots, N$. Then, W is a contraction with contractivity ratio r satisfying

$$r \leq \max_{1\leq\alpha,\beta\leq N} \left\{\sup_i\{r_i^{\alpha\beta}\}\right\}.$$

Proof. Using the facts that the finite union of closed sets and the closure of any set are closed, the set

$$\bigcup_{\beta=1}^{N} \overline{\bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)} \subset X^\alpha \tag{2}$$

is also closed. Moreover, the subset given in (2) is compact and nonempty since X^α is compact and $K^{\alpha\beta} > 0$ for some β by the assumption, that is, $W_\alpha(A)$ belongs to $\mathcal{H}(X)$ for all $\alpha = 1, 2, \ldots, N$. Thus,

$$W(A) \in \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N)$$

so W is well-defined.

In order to show that W is a contraction on $\mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N)$, it is necessary to find a real number $0 < r < 1$ such that

$$h_{\max}(W(A), W(B)) \leq r\, h_{\max}(A, B)$$

for all $A, B \in \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N)$. Using Lemma 2.1, we obtain

$$\begin{aligned}
h_{\max}(W(A), W(B)) &= \max_{1\le\alpha\le N} \{h_\alpha(W_\alpha(A), W_\alpha(B))\} \\
&= \max_{1\le\alpha\le N} \left\{ h_\alpha \left(\overline{\bigcup_{\beta=1}^{N} \bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)}, \overline{\bigcup_{\beta=1}^{N} \bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(B^\beta)} \right) \right\} \\
&\le \max_{1\le\alpha\le N} \left\{ \max_{1\le\beta\le N} \left\{ h_\alpha \left(\overline{\bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)}, \overline{\bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(B^\beta)} \right) \right\} \right\} \\
&= \max_{1\le\alpha,\beta\le N} \left\{ h_\alpha \left(\overline{\bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)}, \overline{\bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(B^\beta)} \right) \right\} \\
&\le \max_{1\le\alpha,\beta\le N} \left\{ \sup_i h_\alpha \left\{ w_i^{\alpha\beta}(A^\beta), w_i^{\alpha\beta}(B^\beta) \right\} \right\}
\end{aligned}$$

since $w_i^{\alpha\beta} : X^\beta \to X^\alpha$ is a contraction with contractivity ratio $r_i^{\alpha\beta}$. Thus we get

$$\begin{aligned}
h_{\max}(W(A), W(B)) &\le \max_{1\le\alpha,\beta\le N} \left\{ \sup_i \{r_i^{\alpha\beta}\} h_\beta(A^\beta, B^\beta) \right\} \\
&= \max_{1\le\alpha,\beta\le N} \left\{ \sup_i \{r_i^{\alpha\beta}\} \right\} \max_{1\le\beta\le N} \left\{ h_\beta(A^\beta, B^\beta) \right\} \\
&= \max_{1\le\alpha,\beta\le N} \left\{ \sup_i \{r_i^{\alpha\beta}\} \right\} h_{\max}(A, B)
\end{aligned}$$

which gives

$$h_{\max}(W(A), W(B)) \le r\, h_{\max}(A, B)$$

for some

$$r \le \max_{1\le\alpha,\beta\le N} \left\{ \sup_i \{r_i^{\alpha\beta}\} \right\}.$$

Definition 2.1. *The system $\{X^\alpha; w_i^{\alpha\beta}\}$ described in Lemma 2.2 is called a graph-directed countable iterated function system, abbreviated GCIFS.*

We then obtain Theorem 2.1 as the main result of this paper.

Theorem 2.1. *Let $\left\{X^\alpha; w_i^{\alpha\beta}\right\}$ be a GCIFS. Then, there exists a unique family of nonempty compact sets $\{A^\alpha \in \mathcal{H}(X^\alpha) \,|\, \alpha = 1, 2, \ldots, N\}$ such that*

$$A^\alpha = \overline{\bigcup_{\beta=1}^{N} \bigcup_{i=1}^{K^{\alpha\beta}} w_i^{\alpha\beta}(A^\beta)}$$

for $\alpha = 1, 2, \ldots, N$. Moreover for every $B \in \mathcal{H}(X^1) \times \cdots \times \mathcal{H}(X^N)$, the sequence $(W^k(B))_{k\in\mathbb{N}}$ converges to $(A^1, \ldots, A^N)$ with respect to $h_{\max}$ where W is the corresponding map of the GCIFS as defined in Lemma 2.2.

Proof. It follows by Theorem 2.2 and the Banach Conraction Priciple.

Again, we call these invariant compact sets as the attractors of the GCIFS. Finally we give an example below in which we use finite number of contractions to obtain the sets relatively close to the attractors.

Example 2.1. *Let* $X^1, X^2 \subset \mathbb{R}^2$ *be two compact metric spaces (with the metric induced by Euclidean metric) as given in Figure 5 and* $w_i^{\alpha\beta} : X^\beta \to X^\alpha$ *be the contractions (in fact, similitudes) where* $K^{11} = \infty, K^{12} = 4, K^{21} = \infty$ *and* $K^{22} = \infty$ *as indicated in Figure 6. The attractors of this graph-directed system shown (as the third stage of the iteration) in Figure 7.*

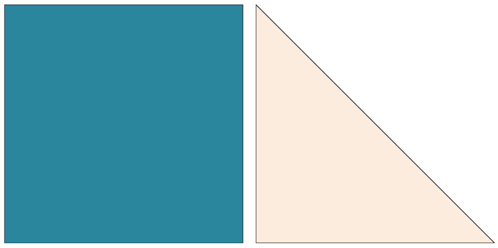

FIGURE 5. The complete metric spaces X^1 (left) and X^2 (right).

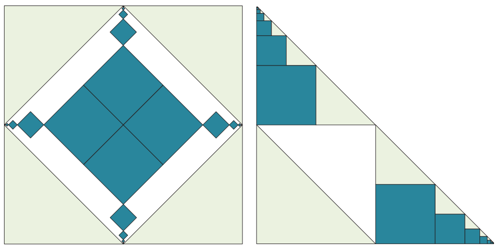

FIGURE 6. Pictorial representation of the contractions $w_i^{\alpha\beta}$.

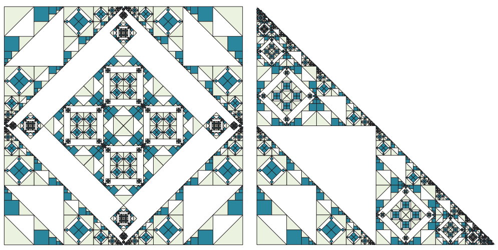

FIGURE 7. Third stage of the iteration of the GCIFS.

References

[1] Barnsley, M.F., *Fractals everywhere*, Academic Press, 1988.

[2] Bernardi M.P., Bondioli, C. and Quaglia, M., *On the graph directed constructions of fractals*, Electronic Notes in Discrete Mathematics, **26(2006)** 9–14.

[3] Dimitru D., *Attractors of Infinite Iterated Function Systems Containing Contraction Type Functions*, Analele Stiintifice ale Universitatii Al I Cuza din Iasi – Matematica, **59(2)(1993)** 281–297.

[4] Edgar, G., *Measure, Topology and Fractal Geometry* Springer, New York, 2008.

[5] J. Hutchinson, *Fractals and Self Similarity*,Indiana Univ. J. Math., **30(1981)** 713–747.

[6] Follo, G., *Some remarks on fractals generated by a sequences of finite systems of contractions*, Georgian Math., **8(4)(2001)** 733–752.

[7] Lewellen, G.B., *Self-similarity*, Rocky Mountain J. Math., **23(3)(1993)** 1023–1040.

[8] Mauldin, R.D., *Infinite iterated function systems: theory and applications*, Progr. Probab., **37(1995)** 91–110.

[9] Mauldin, R.D. and Williams, S.C., *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc., **309(2)(1988)** 811–829.

[10] Secelean, N.A., *Countable Iterated Function Systems*, Far East Journal of Dynamical Systems, Pushpa Publishing House, **3(2)(2001)** 149–167.

[11] Secelean, N.A., *Countable Iterated Function System*, Lambert Academic Publishing, 2013.

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