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SYNTHETIC FOUNDATIONS OF CEVIAN GEOMETRY, II: THE CENTER OF THE CEVIAN CONIC

IGOR MINEVICH and PATRICK MORTON[1]

Abstract. We study the conic $\mathcal{C}_P$ on the five points $ABCPQ$, where ABC is a given ordinary triangle and Q is the isotomcomplement of P, defined as the complement of the isotomic conjugate P' of P with respect to triangle ABC. The properties of $\mathcal{C}_P$ are shown to be related to the affine mapping $\lambda = T_{P'} \circ T_P^{-1}$, where T_P and $T_{P'}$ are the unique affine maps taking ABC to the cevian triangles of P and P', respectively. We characterize the center Z of $\mathcal{C}_P$ as the unique fixed point of λ in the extended plane, when $\mathcal{C}_P$ is a parabola or an ellipse; and as the unique *ordinary* fixed point of λ, when $\mathcal{C}_P$ is a hyperbola. When P is the Gergonne point of ABC, this gives a new characterization of the Feuerbach point Z. All of our arguments are purely synthetic.

1. Introduction.

In this paper we continue the investigation begun in Part I [9] , and study the conic $\mathcal{C}_P = ABCPQ$ on the five points A, B, C, P, Q, where $Q = K \circ \iota(P)$ is the isotomcomplement of P, defined to be the complement of the isotomic conjugate of P with respect to ABC. Here, as in Part I, P is any point not on the extended sides of ABC or its anticomplementary triangle. This conic is defined whenever P does not lie on a median of triangle ABC. We show first that the symmetrically defined points $P' = \iota(P)$ and $Q' = K \circ \iota(P') = K(P)$ also lie on this conic, as well as 6 other points that can be given explicitly (see Theorem 2.1 and Figure 1).

Recall from Part I that the map T_P is the unique affine map which takes triangle ABC to the cevian triangle DEF of P with respect to ABC. The map

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$T_{P'}$ is defined in the same way for the point P'. In Theorem 2.4 we show that if P and P' are ordinary points and do not lie on a median of ABC, then

$$\eta T_P = T_{P'}\eta, \tag{1}$$

where η is the harmonic homology (affine reflection, see [4]) whose center is the infinite point V_∞ on the line PP' and whose line of fixed points is the line GV, where G is the centroid of ABC and $V = PQ \cdot P'Q'$. Thus, T_P and $T_{P'}$ are conjugate maps in the affine group. We prove this formula synthetically by proving an interesting relationship between the centroids G_1 and G_2 of the cevian triangles DEF and $D_3E_3F_3$ of the points P and P', respectively. Lemma 2.5 shows that G is the midpoint of the segment G_1G_2 and $\eta(G_1) = G_2$.

After we introduce the affine map

$$\lambda = T_{P'} \circ T_P^{-1}$$

in Section 3, we show that the 6 points on $\mathcal{C}_P$ mentioned above are the images of the vertices A, B, C under λ and λ^{-1} (Theorem 3.4). The mapping λ leaves the conic $\mathcal{C}_P$ invariant as a set (Theorem 3.2), and is the main affine map considered in this paper. The relation (1) allows us to write the map λ as

$$\lambda = \eta \circ (T_P \circ \eta \circ T_P^{-1}) = \eta_1 \circ \eta_2,$$

where both maps η_1 and η_2 are harmonic homologies. Using this representation we prove in Theorem 4.1 that the center $Z = Z_P$ of the conic $\mathcal{C}_P$ is the intersection

$$Z = GV \cdot T_P(GV),$$

and that when $\mathcal{C}_P$ is a parabola or an ellipse, Z is the unique fixed point of λ in the extended plane. When $\mathcal{C}_P$ is a hyperbola, Z is the unique ordinary fixed point of λ. In the latter case, λ also fixes the two points at infinity on the asymptotes of $\mathcal{C}_P$.

At the end of the paper we interpret the mapping λ as an isometry on the model of hyperbolic geometry whose points are the interior points of the conic $\mathcal{C}_P$ and whose lines are Euclidean chords.

In Part III [10] of this series of papers we will prove that the point Z is a generalized Feuerbach point. In particular, this will show that Z is the Feuerbach point of triangle ABC when P is the Gergonne point and P' is the Nagel point of ABC (see [1] and [8]). Our Theorem 4.1 therefore gives a representation of the Feuerbach point as the intersection of two lines. Corollary 4.2 shows that these two lines are GV and $T_P(GV) = G_1J$, where J is the midpoint of PQ. A third line through Z is the line G_2J', where $G_2 = T_{P'}(G)$ and J' is the midpoint of $P'Q'$. (See Figure 5 in Section 4.) In this case $\mathcal{C}_P$ is a hyperbola, and the Feuerbach point Z is the unique ordinary fixed point of $\lambda = T_{P'} \circ T_P^{-1}$.

We mention one more fact that we prove along the way. The mapping

$$\mathcal{S}' = T_{P'}T_PT_{P'}^{-1}T_P^{-1}$$

is always a translation, which implies (Corollary 2.8 and Figure 4) that the triangles $A_3B_3C_3 = T_P(D_3E_3F_3)$ and $A_3'B_3'C_3' = T_{P'}(DEF)$ are always congruent triangles.

We refer to Part I [9] for the notation that we use throughout this series of papers; to [1], [11], [12], or [13] for definitions in triangle geometry; and to [2] and [3] for classical results from projective geometry.

2. The conic $ABCPQ$ and the affine mapping T_P.

We start by proving

Theorem 2.1. *If P does not lie on the sides of triangle ABC or its anticomplementary triangle, and not on a median of ABC, then there is a conic $\mathcal{C}_P$ on the points A, B, C, P, Q, P', Q'. This conic also passes through the points*

$$A_0P \cdot D_0Q', \quad B_0P \cdot E_0Q', \quad C_0P \cdot F_0Q', \tag{2}$$

and

$$A_0'P' \cdot D_0Q, \quad B_0'P' \cdot E_0Q, \quad C_0'P' \cdot F_0Q,$$

where D_0, E_0, F_0 are the midpoints of the sides BC, CA, AB, and $A_0B_0C_0 = T_P(D_0E_0G_0)$ and $A_0'B_0'C_0' = T_{P'}(D_0E_0F_0)$ are the medial triangles of DEF and $D_3E_3F_3$, respectively.

Proof. The condition that P is not on a median of ABC ensures that the points P and Q are not collinear with one of the vertices. If P, Q, and A are collinear, for example, then the points $D = AP \cdot BC$ and $D_2 = AQ \cdot BC$ coincide, so $A_1 = T_P(D)$ and $A_2 = T_P(D_2)$ coincide, meaning that Q' is collinear with A and G (by I, Theorem 3.5). But P is collinear with $K(P) = Q'$ and G, so P would lie on AG. By the same reasoning, P, Q', and A are not collinear and neither are P, P' and A.

Now the mapping T_P is a projective mapping which takes the pencil of lines x on Q' to the pencil $y = T_P(x)$ on P, since $T_P(Q') = P$ (I, Theorem 3.7). For the lines $x = AQ'$ and $y = DP = AP$ we have $x \cdot y = A$; while $x = BQ'$ and $y = EP = BP$ give $x \cdot y = B$; and $x = CQ'$ and $y = FP = CP$ give $x \cdot y = C$. Thus the pencil x is not perspective to the pencil y, so Steiner's theorem [3], p. 80 implies that the locus of points $x \cdot y$ is the conic $ABCPQ'$. If $x = QQ'$ then $y = QP$, so $x \cdot y = Q$ is also on this conic. Hence the conic $ABCPQ' = ABCQQ'$. Arguing the same with the mapping $T_{P'}$ shows that there is a conic $ABCP'Q = ABCQQ'$. It follows that the conic $\mathcal{C}_P = ABCPQ'$ lies on P' and Q, so that $\mathcal{C}_P = ABCPQ$. This proves the first assertion. Letting $x = D_0Q'$ gives $y = A_0P$, so $A_0P \cdot D_0Q'$ is on $\mathcal{C}_P$, as are all the other listed intersections. □

Corollary 2.2. *(a) If Y is any point on the conic $\mathcal{C}_P$ other than P, Q' (respectively Q, P'), then $T_P(Q'Y) = PY$ (resp. $T_{P'}(QY) = P'Y$).*
(b) The conic $\mathcal{C}_P$ is the locus of points Y for which P, Y, and $T_P(Y)$ are collinear.
(c) In particular, $P, P', T_P(P')$, and $T_{P'}(P)$ are collinear (whether P lies on a median or not).

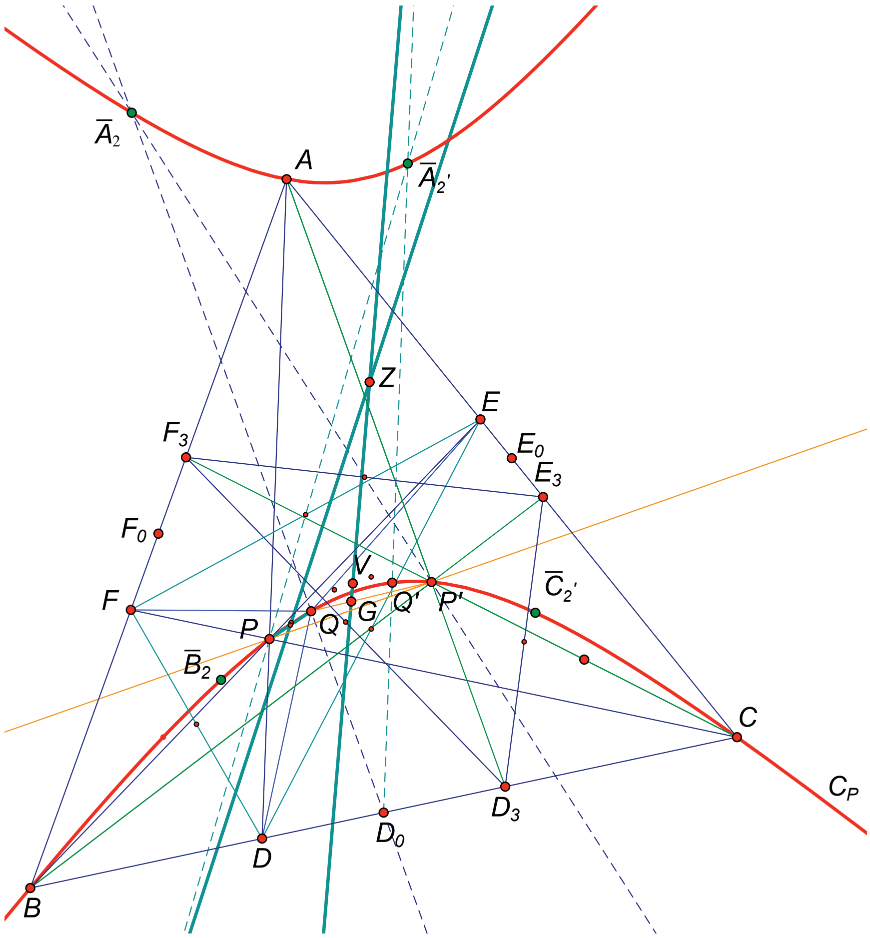

Figure 1. Cevian conic, with points $\bar{A}_2 = A_0'P' \cdot D_0Q, \bar{A}_2' = A_0P \cdot D_0Q'$, etc.

Proof. Part (a) follows from the definition of the conic $\mathcal{C}_P$ as the locus of intersections $x \cdot y$. For part (b), if Y lies on $\mathcal{C}_P$, then by part (a), P, Y, and $T_P(Y)$ are collinear. This also clearly holds for $Y = P$ and $Y = Q'$. Conversely, suppose P, Y, and $T_P(Y)$ are collinear. If $Y \notin \{P, Q'\}$, then T_P maps the line $x = Q'Y$ to the line $y = PT_P(Y) = PY$, so $x \cdot y = Y$ lies on $\mathcal{C}_P$. Part (c) follows from part (b) with $Y = P'$ and from the analogous statement obtained by switching P and P', as long as P does not lie on a median of ABC. Now assume that $P \neq G$ lies on the median AG. Then P, P', Q, and Q' are all on AG. From the Collinearity Theorem (I, Theorem 3.5) the points D_i for $0 \le i \le 4$ are all the same point, so $A_0 = A_2 = A_3$, the midpoint $EF \cdot AG$ of EF; and similarly, the points A_i' equal $A_0' = A_3'$, the midpoint $E_3F_3 \cdot AG$ of E_3F_3. From P' on AD_3 it follows that $T_P(P')$ is on $DA_3 = D_0A_2 = AG$. Similarly, $T_{P'}(P)$ lies on $AG = PP'$. □

Remarks. 1. We know from I, Theorems 3.13 and 2.4 that the lines D_0Q', E_0Q', F_0Q' in (1) are the cevians for the point Q' with respect to the anticevian triangle $A'B'C'$ of Q, since $A'Q'$, $B'Q'$, and $C'Q'$ pass through the midpoints

of the sides of ABC. See Theorem 3.4 below.
2. Part (b) of the corollary allows for an easy computation of an equation for the conic $\mathcal{C}_P$.

We will assume the hypothesis of Theorem 2.1 anytime we make use of the conic $\mathcal{C}_P = ABCPQ$. Alternatively, we could define $\mathcal{C}_P$ to be a degenerate conic, the union of a median and a side of ABC, when P does lie on a median. In that case, P, Q, P', Q' are all on the median.

Assume now that the point P is ordinary and does not lie on a median of triangle ABC or on $\iota(l_\infty)$. Then the points P' and Q are also ordinary. We shall use the conic $\mathcal{C}_P$ to prove an interesting relationship between the maps T_P and $T_{P'}$. First note that the lines PP' and QQ' are parallel, since $K(PGP') = Q'GQ$. The midpoint of QQ' is clearly the complement of the midpoint of PP', so the line joining them passes through the centroid G. Since the quadrangle $PP'QQ'$ is inscribed in $\mathcal{C}_P$, its diagonal triangle is a self-polar triangle for $\mathcal{C}_P$ ([3], p. 75). The vertices of this self-polar triangle are $PQ' \cdot P'Q = G, PQ \cdot P'Q' = V$, and $PP' \cdot QQ' = V_\infty$, a point on the line at infinity. Hence, the polar of G is the line VV_∞, which is the line through V parallel to PP'. The polar of V_∞ is GV. Since the segment QQ' is parallel to PP' and half its length, Q is the midpoint of segment PV, so V is the reflection of P in Q, and the reflection of P' in Q'. Considering the quadrangle $VQGQ'$, it is not hard to see that GV passes through the midpoints of segments PP' and QQ', since these midpoints are harmonic conjugates of V_∞ with respect to the point pairs (QQ') and (PP').

Let $Z = Z_P$ be the center of the conic $\mathcal{C}_P = ABCPQ$, so Z is the pole of the line at infinity. Since V_∞ lies on the polar of Z, Z must lie on GV. Since PP' and QQ' are parallel chords on the conic $\mathcal{C}_P$, the line through their midpoints passes through Z ([2], p. 111). Hence we have:

Proposition 2.3. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$.*

(a) The points $G, V = PQ \cdot P'Q'$, and $V_\infty = PP' \cdot QQ'$ form a self-polar triangle with respect to the conic $\mathcal{C}_P = ABCPQ$.
(b) The center Z of the conic $\mathcal{C}_P$ lies on the line $GV = G(PQ \cdot P'Q')$, which is the polar of the point V_∞.
(c) The line $GV = GZ$ passes through the midpoints of the parallel chords PP' and QQ'.
(d) The harmonic homology μ_G with center G and axis VV_∞ maps the conic $\mathcal{C}_P$ to itself. In other words, if a line GY intersects the conic in points X_1 and X_2 and $GY \cdot VV_\infty = X_3$, then the cross-ratio $(X_1X_2, GX_3) = -1$.
(e) The line VV_∞ is the same as the line joining the anti-complements of P and P'. Thus, the polar of G is the line $K^{-1}(PP')$.
(f) The point V is the midpoint of the segment joining $K^{-1}(P)$ and $K^{-1}(P')$.

Proof. We have already proven parts (a)-(c). Part (d) is immediate from the fact that a conic is mapped into itself by any homology whose center is the pole of its axis ([3], p. 76, ex. 4). For part (e), note that $\mu_G(P) = Q'$, since PQ' is a chord of the conic containing G. Hence $(PQ', GX_3) = -1$, where $GP \cdot VV_\infty = X_3$; this implies $PX_3 = -2X_3Q'$, which means that Q'

is the midpoint of PX_3. But $Q' = K(P)$, so $K(X_3) = P$. Thus X_3 is the anti-complement of P. Similarly, the anti-complement of P' lies on VV_∞, and this proves part (e). Part (f) follows from the fact that the midpoint M of PP' lies on the line GV, so that

$$K^{-1}(M) = K^{-1}(PP') \cdot K^{-1}(GV) = K^{-1}(PP') \cdot GV = V,$$

since the line GV is an invariant line of the complement map. □

Let $\eta = \eta_P$ be the harmonic homology whose center is V_∞ and whose axis is its polar GV. The map η is an affine reflection ([4], p. 203), since it fixes the line GV and maps a point Y to the point Y' with the property that YY' is parallel to VV_∞ (or PP') and $YY' \cdot GV$ is the midpoint of YY'. The map η is an involution on the extended plane. It takes the conic $\mathcal{C}_P$ to itself and interchanges the point pairs (PP') and (QQ'), since the line GV passes through the midpoints of the chords PP' and QQ' and both lines PP' and QQ' lie on V_∞. Hence this homology induces an involution of points on $\mathcal{C}_P$.

Remark. It is not hard to show that the map η commutes with the complement map: $K\eta = \eta K$.

We shall now prove

Theorem 2.4. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$. Then the maps T_P and $T_{P'}$ satisfy the equation $\eta T_P = T_{P'}\eta$, and so are conjugate to each other in the affine group.*

To prove this theorem we need a lemma, which is of interest in its own right.

Lemma 2.5. *Let $G_1 = T_P(G)$ and $G_2 = T_{P'}(G)$ be the centroids of the cevian triangles DEF and $D_3E_3F_3$ of P and P'. Then the centroid G of ABC is the midpoint of the segment G_1G_2, which is parallel to PP'. In other words, $\eta(G_1) = G_2$ (when P and P' are ordinary).*

Proof. We first show that the line G_1G_2 lies on G and is parallel to PP'. To begin with, assume P and P' are ordinary. Using I, Corollary 3.3, we know that the points $Q, G_1, T_P(P')$ are collinear, with G_1 one-third of the way from Q to $T_P(P')$. Since G is collinear with Q and P' and one-third of the way along QP', the triangles G_1QG and $T_P(P')QP'$ are similar (SAS). Hence the line G_1G is parallel to the line $T_P(P')P' = PP'$ (Corollary 2.2). Switching the roles of P and P' gives that G_2G is also parallel to PP', which implies that $G_1G = G_2G$, proving the claim.

If $P' = Q$ is infinite, then P and Q' are ordinary, so we still get $G_2G \parallel PP'$. Applying the map T_P and using I, Theorem 3.14 gives that $T_P(G_2G) \parallel T_P(PP')$, i.e., $GG_1 \parallel PP'$, because $T_P(G_2) = T_PT_{P'}(G) = K^{-1}(G) = G$ and $T_P(PP') = T_P(PQ) = T_P(P)Q \parallel PQ = PP'$. Hence we get $G_1G = G_2G$, as before. A similar argument works if $P = Q'$ is infinite.

We now show that G is the midpoint of G_1G_2. (Cf. Figure 3.) Consider the sequence of triangles DEF, D_0EF, D_3EF with centroids G_1, G_{01}, G_{11} lying on the respective lines A_0D, A_0D_0, A_0D_3 (since A_0 is the midpoint of EF). By the properties of the centroid we know that the segment G_1G_{11} is the image of the segment DD_3 under a dilatation with center A_0 and ratio 1/3. Since

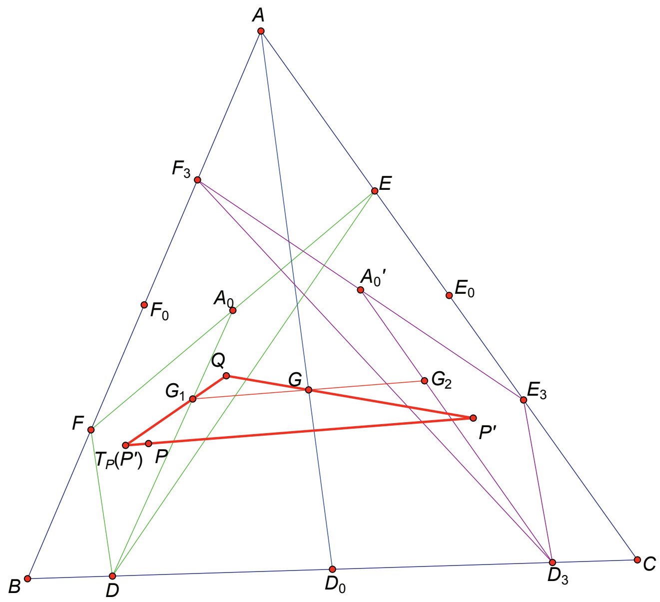

Figure 2. $G_1G_2 \| PP'$

D_0 is the midpoint of DD_3 it follows that the vector $\overrightarrow{G_1G_{01}}$ is $\frac{1}{2}$ the vector $\overrightarrow{G_1G_{11}}$.

Next consider passing from triangle D_3EF with centroid G_{11} to the triangle D_3E_3F with centroid G_{12}. Considering the dilatation with ratio $1/3$ from the midpoint of D_3F shows that the vector $\overrightarrow{G_{11}G_{12}}$ is $\frac{1}{3}$ the vector $\overrightarrow{EE_3}$. In the same way, if we move from triangle D_0EF with centroid G_{01} to triangle D_0E_0F with centroid G_{02}, the vector $\overrightarrow{G_{01}G_{02}}$ is $\frac{1}{3}$ the vector $\overrightarrow{EE_0}$, and therefore $\frac{1}{2}$ the vector $\overrightarrow{G_{11}G_{12}}$.

Finally, in passing from triangle D_3E_3F with centroid G_{12} to triangle $D_3E_3F_3$ with centroid G_2, the vector $\overrightarrow{G_{12}G_2}$ is $\frac{1}{3}$ the vector $\overrightarrow{FF_3}$. Similarly, in passing from D_0E_0F with centroid G_{02} to triangle $D_0E_0F_0$ with centroid G, the vector $\overrightarrow{G_{02}G}$ is $\frac{1}{3}$ of $\overrightarrow{FF_0}$. Hence, the vector $\overrightarrow{G_{02}G}$ is $\frac{1}{2}$ the vector $\overrightarrow{G_{12}G_2}$.

It follows that in passing from triangle DEF to $D_0E_0F_0$, the centroid experiences a displacement represented by the vector

$$\overrightarrow{G_1G_{01}} + \overrightarrow{G_{01}G_{02}} + \overrightarrow{G_{02}G} = \frac{1}{2}(\overrightarrow{G_1G_{11}} + \overrightarrow{G_{11}G_{12}} + \overrightarrow{G_{12}G_2}),$$

which is $\frac{1}{2}$ the vector displacement from G_1 to G_2. Hence, $G_1G = \frac{1}{2}G_1G_2$, which proves that G is the midpoint of G_1G_2. □

Proof of Theorem 2.4. We check that $\eta T_P(Y) = T_{P'}\eta(Y)$ for three non-collinear ordinary points Y. This holds for $Y = G$ by the lemma. It also holds for the points Q and Q', since I, Theorems 3.2 and 3.7 imply that

$$\eta T_P(Q) = \eta(Q) = Q' = T_{P'}(Q') = T_{P'}\eta(Q)$$

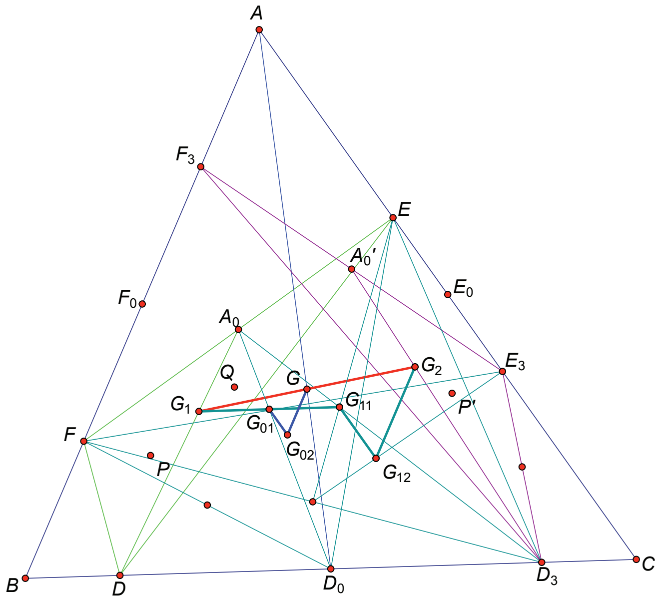

FIGURE 3. G is the midpoint of G_1G_2

and

$$\eta T_P(Q') = \eta(P) = P' = T_{P'}(Q) = T_{P'}\eta(Q').$$

The points Q, Q', and G are clearly not collinear, since G is the intersection of the diagonals in the trapezoid $PP'Q'Q$ (and no three of these points lie on a line because they all lie on the conic C_P). This implies the theorem, since ηT_P and $T_{P'}\eta$ are affine maps. □

Remark. The map η is the unique involution ψ in the affine group satisfying $\psi T_P = T_{P'}\psi$.

For the corollary, recall that the point $X = AA_3 \cdot BB_3$ is the fixed point of the map $\mathcal{S}_1 = T_P T_{P'}$, and $X' = AA_3' \cdot BB_3'$ is the fixed point of $\mathcal{S}_2 = T_{P'}T_P$.

Corollary 2.6. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$. If X is an ordinary point, then $\eta(X) = X'$ and XX' is parallel to PP'. Thus the line joining the P-ceva conjugate of Q and the P'-ceva conjugate of Q' is parallel to PP'.*

Proof. We have $T_{P'}T_P(\eta(X)) = T_{P'}\eta(T_{P'}(X)) = \eta T_P T_{P'}(X) = \eta(X)$, which shows that $\eta(X)$ is an ordinary fixed point of $\mathcal{S}_2 = T_{P'}T_P$. Hence, $\eta(X) = X'$. This implies the assertion, by I, Theorems 3.8 and 3.10. □

Theorem 2.7. *The commutator $\mathcal{S}' = T_{P'}T_P T_{P'}^{-1}T_P^{-1}$ is always a translation, in the direction PP' by the distance $T_P(P')P' \cong PT_{P'}(P)$, if P and P' are ordinary; and by the distance $3|G_1G|$, if P or P' is infinite.*

Proof. For notational convenience write T_1 for T_P and T_2 for $T_{P'}$. We first note that

$$\mathcal{S}'(T_1(P')) = T_2T_1T_2^{-1}(P') = T_2T_1(Q) = T_2(Q) = P'$$

and similarly

$$\mathcal{S}'(P) = T_2 T_1 T_2^{-1}(Q') = T_2 T_1(Q') = T_2(P).$$

From this computation and Corollary 2.2, the mapping $\mathcal{S}'$ fixes the line PP'.

By I, Theorem 3.8 and Corollary 3.11(c) we may assume that both $\mathcal{S}_1 = T_1T_2$ and $\mathcal{S}_2 = T_2T_1$ are homotheties, since otherwise the assertion is trivial. If P and P' are ordinary points, then $\mathcal{S}_2(Q) = P'$ implies that X', P', and Q are collinear, so part (b) of the same corollary implies that

$$\frac{X'P'}{X'Q} = \frac{T_1(X'P')}{T_1(X'Q)} = \frac{XT_1(P')}{XQ} = \frac{XP}{XQ'};$$

the last equality being a consequence of $\mathcal{S}_1(Q) = T_1(P')$ and $\mathcal{S}_1(Q') = P$. This equation shows that the similarity ratios of $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal and the mapping $\mathcal{S}' = \mathcal{S}_2\mathcal{S}_1^{-1}$ is an isometry which fixes l_∞ pointwise. It follows that $\mathcal{S}'$ is either a half-turn or a translation.

First assume that P does not lie on a median of ABC. Then $\eta(T_1(P')) = T_2(\eta(P')) = T_2(P)$, so that $T_1(P')$ and $T_2(P)$ are both inside or both outside the segment PP', and $T_1(P')P' \cong PT_2(P)$ since η preserves lengths along PP'. If the midpoints of segments $T_1(P')P'$ and $PT_2(P)$ are M_1 and M_2, then $\eta(M_1) = M_2$. If $M_1 = M_2$ then M_1 is the midpoint of PP', impossible since $T_1(P') \neq P = T_1(Q')$. Thus, $M_1 \neq M_2$, and $\mathcal{S}'$ fixes the line PP' but is not a half-turn. Hence, $\mathcal{S}'$ is a translation. This proves the assertion in the case that P is not on a median.

Now assume that $P \neq G$ lies on the median AG. Then P, P', Q, and Q' are distinct points on AG. (This follows easily from the fact that P, Q are on the opposite side of G on line AG from P', Q' and $T_P(Q) = Q$ and $T_P(Q') = P$. See I, Theorems 3.2 and 3.7.) From I, Theorem 3.5 the points A_i for $0 \le i \le 4$ are all the same point $A_0 = A_3$, the midpoint $EF \cdot AG$ of EF; and similarly, the points A_i' equal $A_0' = A_3'$, the midpoint $E_3F_3 \cdot AG$ of E_3F_3. Also, $T_1(P')$ and $T_2(P)$ are on $PP' = AG$, so T_1, T_2 and $\mathcal{S}'$ all fix the line $AG = AD = PP'$.

If P lies in the interior of triangle ABC, then it is easy to see that $E = T_1(B)$ and $E_3 = T_2(B)$ are on the same side of the line AD as point C, and hence that T_1 and T_2 interchange the sides of this line. Therefore, $B_3 = T_1(E_3)$ and $B_3' = T_2(E)$ are on the same side of line $AD = AG$, which implies that segments B_3B_3' and A_3A_3' do not intersect. Since

$$\mathcal{S}'(A_3B_3C_3) = T_2T_1T_2^{-1}(D_3E_3F_3) = T_2T_1(ABC) = A_3'B_3'C_3',$$

we see that $\mathcal{S}'$ is not a half-turn, and is therefore a translation. On the other hand, if P is exterior to ABC, points E and E_3 are on opposite sides of the line AD, so one of T_1 and T_2 interchanges the sides of AD and the other leaves both sides of AD invariant. Hence $B_3 = T_1(E_3)$ and $B_3' = T_2(E)$ are on the same side of the line AD and we get the same conclusion as before.

Finally, assume that $P' = Q$ is infinite. Using I, Theorem 3.14, we have as in the proof of Lemma 2.5 that $T_1(G) = G_1, T_1(G_2) = T_1T_2(G) = K^{-1}(G) = G$ and $T_1(Q) = Q$. Since G is the midpoint of the segment G_1G_2, the fundamental theorem of projective geometry implies that T_1 acts as translation along the line GG_1. In this situation the map $\mathcal{S}_1 = K^{-1}$ fixes G, so $X = G$, while

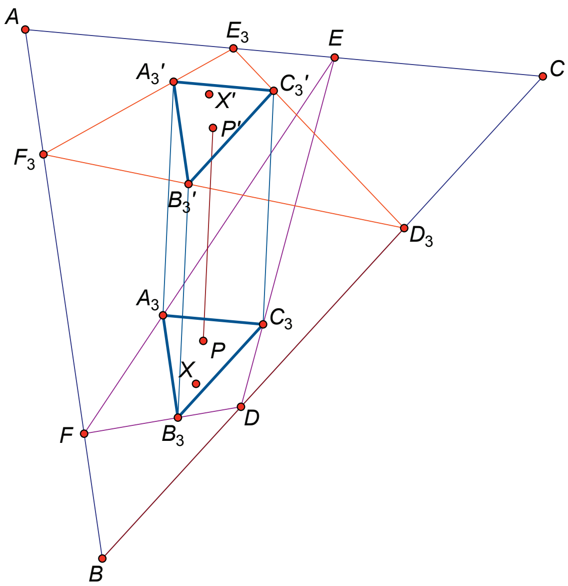

Figure 4. $A_3B_3C_3 \cong A_3'B_3'C_3'$

$\mathcal{S}_2 = T_2T_1$ fixes $G_2 = X'$. (See I, Theorem 3.10 and Corollary 3.11.) Furthermore, $\mathcal{S}_2(G) = T_2T_1(G) = T_1^{-1}K^{-1}(G_1) = T_1^{-1}(G_3)$, where $G_3 = K^{-1}(G_1) = T_1^{-1}(G_2)$. Hence, $\mathcal{S}_2(G) = T_1^{-2}(G_2)$, and the similarity ratio of the homothety $\mathcal{S}_2$ is -2, which is equal to the similarity ratio of $\mathcal{S}_1 = K^{-1}$. Once again, $\mathcal{S}' = \mathcal{S}_2\mathcal{S}_1^{-1} = \mathcal{S}_2K$ is an isometry. It is now straightforward to verify that the map $\mathcal{S}'$, which is the commutator of T_1^{-1} and K^{-1}, is equal to the translation T_1^{-3} on the line GG_1. For example, $\mathcal{S}'(G) = \mathcal{S}_2K(G) = T_1^{-2}(G_2) = T_1^{-3}(G)$, while $\mathcal{S}'(G_3) = \mathcal{S}_2(G_1) = T_1^{-1}K^{-1}T_1(G_1) = T_1^{-4}(G_2) = T_1^{-3}(G_3)$. Hence, the points on the line GG_1 experience a translation and not a half-turn. If instead, $P = Q'$ is infinite, we apply the same argument to the inverse of $\mathcal{S}'$. This completes the proof. □

Corollary 2.8. *(a) The triangles $A_3B_3C_3$ and $A_3'B_3'C_3'$ are always congruent to each other. (See Figure 4.)*

(b) If P and P' are ordinary, the length $|T_P(P')P|$ of the segment $T_P(P')P$ is equal to $|QQ'||XP|/|XQ'|$.

Proof. Part (a) follows from $\mathcal{S}'(A_3B_3C_3) = A_3'B_3'C_3'$. Part (b) follows from $\mathcal{S}_1(Q) = T_1T_2(Q) = T_1(P')$ and $\mathcal{S}_1(Q') = P$, since $\mathcal{S}_1$ is a homothety with fixed point X. □

3. The affine map λ.

We now set $\lambda = T_{P'}T_P^{-1}$, the unique affine map taking the cevian triangle DEF for P to the cevian triangle $D_3E_3F_3$ for P'. We will show that λ maps the conic $\mathcal{C}_P$ to itself!

We use the fact that the diagonal triangle of any quadrangle inscribed in $\mathcal{C}_P$ is self-polar ([2], p. 73). Since A, B, C are on $\mathcal{C}_P$, if Y is any other point on $\mathcal{C}_P$, the cevian triangle whose vertices are the traces of Y on the sides of ABC is a self-polar triangle. We apply this to the point $Y = P$, obtaining that DEF is a self-polar triangle.

Associated to the vertex D of the self-polar triangle DEF is the harmonic homology η_D whose center is D and whose axis is EF. We know P and A are collinear with D and A_4 (on EF) by the Collinearity Theorem. Furthermore, η_D maps the conic to itself. Since P and A are on the conic $\mathcal{C}_P$, the definition of η_D implies that $\eta_D(A) = P$ and hence that $(AP, DA_4) = -1$. This proves

Proposition 3.1. *The point sets $AA_4PD_1, BB_4PE_1, CC_4PF_1$ are harmonic sets.*

This is the key fact we use to prove the following.

Theorem 3.2. *If P does not lie on a median of triangle ABC, then the map $\lambda = T_{P'}T_P^{-1}$ takes the conic $\mathcal{C}_P$ to itself: $\lambda(\mathcal{C}_P) = \mathcal{C}_P$.*

Proof. Consider the anticevian triangle of Q, which, by I, Corollary 3.11, is $A'B'C' = T_{P'}^{-1}(ABC)$. The cevian triangle for Q with respect to this triangle is ABC, so Q plays the role of P in the above discussion and ABC plays the role of DEF. We have that $AQ \cdot BC = D_2$, so Proposition 3.1 implies that $A'D_2QA$ is a harmonic set. Now we map this set by T_P. We have that $T_P(A'D_2QA) = T_P(A')A_2QD$, so $T_P(A')$ is the harmonic conjugate of Q with respect to D and A_2. But A_2 is on EF, so the above discussion implies that $\eta_D(Q) = T_P(A')$. Since Q is on $\mathcal{C}_P$, so is $T_P(A') = T_P T_{P'}^{-1}(A) = \lambda^{-1}(A)$. Similar arguments apply to the other vertices, so λ^{-1} maps A, B, C to points on $\mathcal{C}_P$. Now it is easy to see that $\lambda^{-1}(P') = Q$ and $\lambda^{-1}(Q') = P$, using I, Theorems 3.2 and 3.7. Hence λ^{-1} maps 5 points on $\mathcal{C}_P$ to 5 other points on $\mathcal{C}_P$, so we must have $\lambda^{-1}(\mathcal{C}_P) = \mathcal{C}_P$. This implies the assertion. □

If P is ordinary and not on $\iota(l_\infty)$, then the map $\lambda = T_{P'}T_P^{-1} = \eta T_P \eta T_P^{-1}$ is the product of η and $\eta\lambda = T_P\eta T_P^{-1}$, both of which are involutions (harmonic homologies) on the extended plane, and both of which fix the conic $\mathcal{C}_P$. The involution $T_P\eta T_P^{-1}$ interchanges P and Q. Its axis of fixed points is the line $T_P(GV)$ and its center $T_P(V_\infty) = T_P(PP' \cdot QQ')$ lies on $T_P(QQ') = PQ$, so that the midpoint of segment PQ lies on $T_P(GV)$.

The map $T_P\eta T_P^{-1}$ is the map corresponding to η for the conic $T_P(\mathcal{C}_P) = T_P(ABCQ'Q) = DEFPQ$, which also lies on the points $T_P(P)$ and $T_P(P')$. Since Q is the complement of $T_P(P')$ with respect to the triangle DEF (I, Corollary 3.3), the conic $DEFPQ$ equals $\mathcal{C}_R$ for triangle DEF and the point $R = T_P(P)$. This is because the cevian triangle for $T_P(P)$ in DEF is $A_1B_1C_1$, the cevian triangle for $T_P(P')$ in DEF is $A_3B_3C_3$, and the points $T_P(P)$ and $T_P(P')$ are isotomic conjugates with respect to DEF.

Remark. It is easy to show that $\eta\lambda\eta = \lambda^{-1}$. Moreover, the map λ is never a projective homology. (Hint: show $T_P(V_\infty)$ is never on the line GV and use [3], p. 56, ex. 4.)

The last theorem has several interesting consequences.

Theorem 3.3. *If P does not lie on a median of triangle ABC, then the 6 vertices of the anticevian triangles for Q and Q' (with respect to ABC) lie on the conic $T_P^{-1}(C_P) = T_{P'}^{-1}(C_P)$, along with the points Q and Q'. This conic is $C_Q = A'B'C'QQ'$ for the anticevian triangle of Q, which is the same as $C_{Q'}$ for the anticevian triangle of Q'. Moreover, the vertices of the anticevian triangle of R with respect to ABC, for any point R on $T_P^{-1}(C_P)$, lie on $T_P^{-1}(C_P)$.*

Proof. In the proof of the previous theorem we showed that $T_P(A') = \lambda^{-1}(A)$ lies on the conic C_P, so A' lies on the conic $T_P^{-1}(C_P)$, as do B' and C', the other vertices of the anticevian triangle for Q. Similarly, the vertices of the anticevian triangle for Q' lie on the conic $T_{P'}^{-1}(C_P)$. But these two conics are the same conic, since $T_{P'}T_P^{-1}(C_P) = C_P$ implies $T_P^{-1}(C_P) = T_{P'}^{-1}(C_P)$. That Q and Q' also lie on this conic follows from $T_P^{-1}(Q) = Q$ and $T_P^{-1}(P) = Q'$. Theorem 3.12 of Part I implies the second assertion. Since the triangle DEF is self-polar for C_P, it follows that ABC is a self-polar triangle for $T_P^{-1}(C_P)$. The last assertion of the theorem follows from the general projective fact, that for any point R on a conic C, the vertices of the anticevian triangle of R, with respect to any self-polar triangle for the conic, also lie on C. (See [3], p. 91, ex. 3.) □

Note that the map η fixes the conic $T_P^{-1}(C_P)$, since $\eta T_P^{-1}(C_P) = T_{P'}^{-1}\eta(C_P) = T_{P'}^{-1}(C_P)$.

Remark. If P is the orthocenter of ABC, then Q' is the circumcenter, Q is the symmedian point ([7], Thm. 7), and P' is the point $X(69)$, the symmedian point of the anticomplementary triangle (see [8]). In this case, the anticevian triangle of Q is the tangential triangle [11], so Theorem 3.3 implies that $T_P^{-1}(C_P)$ is the Stammler hyperbola ([13], p. 21).

Theorem 3.4. *Assume that the ordinary point P does not lie on a median of triangle ABC or on $\iota(l_\infty)$.*

(a) The last-named points of Theorem 2.1 are

$$A_0P \cdot D_0Q' = \lambda^{-1}(A),\ B_0P \cdot E_0Q' = \lambda^{-1}(B),\ C_0P \cdot F_0Q' = \lambda^{-1}(C);$$

$$A_0'P' \cdot D_0Q = \lambda(A),\ B_0'P' \cdot E_0Q = \lambda(B),\ C_0'P' \cdot F_0Q = \lambda(C).$$

(b) Moreover, the lines $A_0P, D_0Q', DQ, A_3'P'$ are concurrent at the point $\lambda^{-1}(A)$, and $A_0'P', D_0Q, D_3Q', A_3P$ are concurrent at the point $\lambda(A)$, with similar statements holding for the other points in (a).

Proof. From the proof of Theorem 3.2 we have that $\lambda^{-1}(A)$ is on the intersection of the line DQ with the conic C_P and is distinct from Q. The proof of Theorem 2.1 together with $T_{P'}(DQ) = T_{P'}(D_3'Q) = A_3'P'$ shows that $DQ \cdot A_3'P'$ is also on C_P. Now $DQ \cdot A_3'P'$ is not the point Q; otherwise Q would lie on $A_3'P'$, which would imply the point X' is on $A_3'P'$, since X' is collinear with Q and P' (I, Theorem 3.8). However, the point A is on $X'A_3'$ by the definition of X' (I, Theorem 3.5), so Q and P' would be collinear with A. This contradicts the assumption that P is not on AG. Hence, $DQ \cdot A_3'P' = \lambda^{-1}(A)$, since DQ intersects the conic in exactly two points.

On the other hand, I, Corollary 3.11 and Theorems 3.12 and 2.4 give that the point $A' = T_{P'}^{-1}(A)$ is on both lines AQ and D_0Q', so we have that

$$\lambda^{-1}(A) = T_P T_{P'}^{-1}(A) = T_P(AQ \cdot D_0Q') = DQ \cdot A_0P.$$

Note that $DQ \cdot A_0P \neq P$ since Q does not lie on $PD = AP$ (see the proof of Theorem 2.1). Finally, $A_0P \cdot D_0Q'$ is on $\mathcal{C}_P$, by Theorem 2.1, and this intersection is not P, since P and Q' are collinear with G and G is not on D_0Q'. Hence, $A_0P \cdot D_0Q' = \lambda^{-1}(A)$. Therefore, the lines A_0P, D_0Q', DQ, and $A_3'P'$ meet at the point $\lambda^{-1}(A)$. This proves the first assertion in (b). The second assertion follows immediately upon reversing the roles of P and P'. This gives two of the equalities in part (a), and the others follow by the same reasoning applied to the other vertices. □

Corollary 3.5. *Under the assumptions of the theorem, the two quadrangles $PQQ'P'$ and $A_0DD_0A_3'$ are perspective, as are quadrangles $PQQ'P'$ and $A_3D_0D_3A_0'$. (See Part I, Figure 5, where $Y = \lambda(A)$.)*

Theorem 3.6. *The translation $\mathcal{S}' = T_{P'}T_PT_{P'}^{-1}T_P^{-1}$ of Theorem 2.7 maps the conic $DEFPQ$ to the conic $D_3E_3F_3P'Q'$. In other words, these two conics are congruent.*

Proof. From Theorem 3.3 we have, again with $T_1 = T_P$ and $T_2 = T_{P'}$, that

$$\mathcal{S}'(DEFPQ) = T_2T_1T_2^{-1}(ABCQ'Q) = T_2T_1(T_2^{-1}(\mathcal{C}_P))$$

$$= T_2T_1(T_1^{-1}(\mathcal{C}_P)) = T_2(\mathcal{C}_P) = D_3E_3F_3P'Q'.$$

□

4. The center Z of $\mathcal{C}_P$.

In this section we study the center $Z = Z_P$ of the conic $\mathcal{C}_P$, which we will recognize as a generalized Feuerbach point in Part III.

Theorem 4.1. *Assume that the ordinary point P does not lie on a median of ABC or on $\iota(l_\infty)$. Then the center Z of the conic $\mathcal{C}_P$ is given by $Z = GV \cdot T_P(GV)$. If $\mathcal{C}_P$ is a parabola or an ellipse, Z is the unique fixed point in the extended plane of the affine mapping $\lambda = T_{P'}T_P^{-1}$. If $\mathcal{C}_P$ is a hyperbola, the infinite points on the asymptotes are also fixed, and these are the only other invariant points of λ.*

Proof. Since the map λ leaves invariant the conic and the line at infinity, it fixes the pole of this line, which is Z.

To prove uniqueness write the map $\lambda = \eta_1\eta_2$ as the product of the harmonic homologies $\eta_1 = \eta$ and $\eta_2 = \eta\lambda = T_P\eta T_P^{-1}$ (see the discussion following Theorem 3.2). The center of η_1 is V_∞, lying on the line PP', and the center of η_2 is $T_P(V_\infty)$, lying on the line $T_P(QQ') = QP$. If R is any ordinary fixed point of λ, then $\eta_1(R) = \eta_2(R) = R'$. If R is distinct from R', this implies that RR' is parallel to both lines PP' and QP, which is impossible. If $R = R'$, then R is fixed by both η_1 and η_2, so R must be the intersection $GV \cdot T_P(GV)$ of the axes of the two maps. This proves that $Z = GV \cdot T_P(GV)$ if Z is ordinary.

Suppose now that the point Z (= $GV \cdot l_\infty$) is an infinite point (so $\mathcal{C}_P$ is a parabola) and R is another infinite fixed point. Then $\eta_1(R) = \eta_2(R) = R'$ is also an infinite fixed point of λ, since $\lambda\eta(R) = \eta\lambda^{-1}(R) = \eta(R)$. If $R \neq Z, R'$, then λ fixes three points on l_∞ and is therefore the identity on l_∞. But $\lambda(PQ) = Q'P'$, and $PQ \cdot P'Q'$ is the ordinary point V (Proposition 2.3f), implying that PQ cannot be parallel to $Q'P'$ and the line at infinity cannot be a range of fixed points. Thus, since Z on GV is fixed by η_1 we have $R = R'$, implying that R is fixed by η_1 and η_2. We know that $V_\infty \neq T_P(V_\infty)$ (because $T_P(QQ') = QP$). Since R is fixed by η_1 but different from Z, then $R = V_\infty$, which must lie on the axis $T_P(GV)$ in order to be fixed by η_2. Now $T_P(GV)$ lies on the point $G_1 = T_P(G)$ and on the midpoint J of segment PQ (see the comments following Theorem 3.2). The point G_1 lies on the line GV_∞ through G parallel to PP' (Lemma 2.5), but J is not on GV_∞, since this line divides the segment PQ in the ratio 2:1. Hence, $T_P(GV) = G_1J$ cannot possibly lie on the point V_∞. Thus $R = Z$ and Z is the only infinite invariant point of λ.

We have already shown that the only possible ordinary fixed point of λ is the point $Y = GV \cdot T_P(GV)$. We now show this point is infinite when Z is infinite. Assume Y is an ordinary point. Then Y is on the line GV with Z, so $GV = YZ$ is an invariant line; hence its pole V_∞ is also invariant. Now use the fact that $\lambda(P) = Q'$ and $\lambda(Q) = P'$. Since V_∞ is invariant the line PP' must be mapped to the line QQ'. Since the conic is also invariant, P' must map to the second point of intersection of QQ' with the conic, which is Q. Hence, λ interchanges P' and Q, implying that the line $P'Q$ is invariant; hence the infinite point on this line is invariant. This point is clearly not V_∞; it is also not Z because $P'Q \cdot GV = G$, so that if Z were on $P'Q$ then $P'Q = GV$, which is not the case since P' and Q are not fixed by the map η. This shows that λ has three invariant points on the line at infinity, which contradicts the fact that λ is not the identity on l_∞. Therefore, λ has no ordinary fixed point in this case, and we conclude that $Z = GV \cdot T_P(GV)$ if Z is infinite. Hence, $Z = GV \cdot T_P(GV)$ in all cases.

Suppose next that Z is ordinary and $\mathcal{C}_P$ is a hyperbola. Let R_1 and R_2 be the infinite points on the two asymptotes. Since the point V_∞ is the pole of the axis GV of the map η_1, the point $I_1 = GV \cdot l_\infty$ is conjugate to V_∞ on l_∞ and $GV = ZI_1$ and ZV_∞ are conjugate diameters. By a theorem in Coxeter ([2], p. 111, 8.82), these two conjugate diameters are harmonic conjugates with respect to the asymptotes. (Note that GV is never an asymptote: since GVV_∞ is the diagonal triangle of the quadrangle $PQQ'P'$, the pole V_∞ of GV never lies on GV.) Hence we have the harmonic relation $H(R_1R_2, I_1V_\infty)$. By the definition of η_1 this implies that $\eta_1(R_1) = R_2$. In the same way, $T_P(V_\infty)$ is the center and $T_P(GV)$ is the axis of the harmonic homology η_2. Since η_2 fixes the conic, it fixes the pole of the line $T_P(GV)$ with respect to $\mathcal{C}_P$; this pole is the center $T_P(V_\infty)$ since $T_P(V_\infty)$ is the only fixed point of η_2 off the line $T_P(GV)$. Hence we have the harmonic relation $H(R_1R_2, I_2T_P(V_\infty))$, where $I_2 = T_P(GV) \cdot l_\infty$, and this implies that $\eta_2(R_1) = R_2$. Therefore, $\lambda(R_1) = \eta_1\eta_2(R_1) = \eta_1(R_2) = R_1$ and $\lambda(R_2) = R_2$. There cannot be any other fixed points since λ induces a non-trivial map on the line l_∞.

The only case left to consider is the case when $\mathcal{C}_P$ is an ellipse. In this case Z is an interior point of the conic and any line through Z intersects the conic in two points. Suppose that the infinite point on the line ZR_1 is fixed

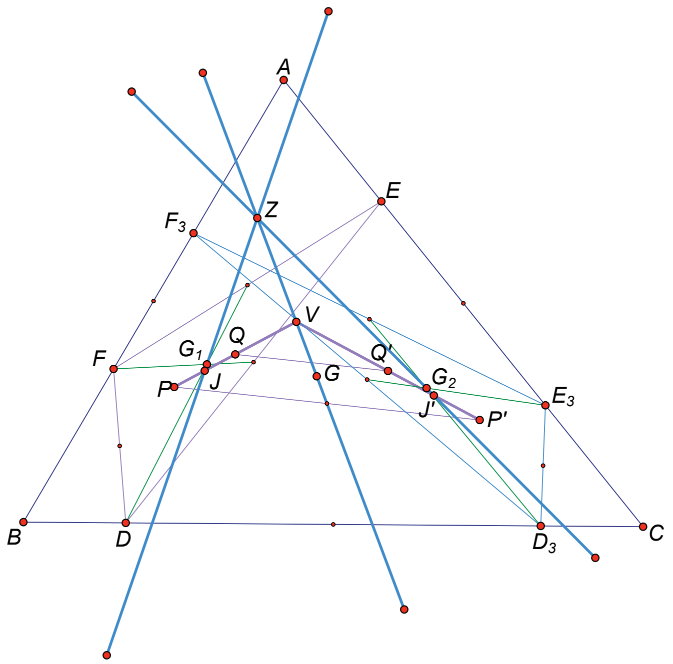

Figure 5. The center Z of the conic C_P

by λ, where R_1 lies on the conic. As before we have $\eta_1(ZR_1) = \eta_2(ZR_1)$, so $R_2 = \eta_1(R_1)$ lies on the conic, along with $R_3 = \eta_2(R_1)$, and Z, R_2, and R_3 are collinear. The points R_2 and R_3 are distinct because Z is the only ordinary fixed point of λ. Now line R_1R_2 is on the point V_∞ and R_1R_3 is on the point $T_P(V_\infty)$, so $R_1R_2R_3$ is a triangle. Furthermore, $T_P(GV)$ is on the midpoint of segment R_1R_3. Since Z is the center, it is the midpoint of the segment R_2R_3, and therefore the line $T_P(GV)$, which lies on Z and the midpoint of R_1R_3, is parallel to R_1R_2. This implies that $T_P(GV)$ lies on V_∞, which we showed above is never the case. Therefore, Z is the only fixed point of λ in this case. This completes the proof of the theorem. □

Corollary 4.2. *(a) The point $Z = GV \cdot G_1J$, where G_1 is the centroid of the cevian triangle of P and J is the midpoint of segment PQ.*
(b) The lines $GV, T_P(GV) = G_1J$, and $T_{P'}(GV) = G_2J' = \eta(G_1J)$ are concurrent at Z. (See Figure 5.)

Theorem 4.3. *Assume that the ordinary point P does not lie on a median of ABC but does lie on the Steiner circumellipse $\iota(l_\infty)$. Then the point Z is one-third of the distance from the point G to the point $G_1 = T_P(G)$ on the line GG_1 and is the only ordinary fixed point of the mapping λ. In this case C_P is a hyperbola and the line $l = GG_1$ is an asymptote.*

Proof. This will follow from the discussion in the last paragraph of the proof of Theorem 2.7, according to which T_P acts as translation on $l = GG_1$ and $\lambda = T_{P'}T_P^{-1} = T_P^{-1}K^{-1}T_P^{-1}$. First of all, it follows easily from this representation of λ that λ fixes the point Z_1 on l for which $GZ_1 : GG_1 = 1 : 3$, and that this Z_1 is the only fixed point of λ on this line.

Furthermore, the mapping T_P interchanges the sides of the line GG_1. This is because the point D_2 is on the line $AQ = AP' = AD_2$, which is parallel to the fixed line l; and $A_2 = T_P(D_2)$ is on the opposite side of l, because A_2 is a vertex of the anticomplementary triangle (I, Corollary 3.14) and the point G lies on segment AA_2. Since K interchanges the sides of the line l, it follows that the mapping λ has the same property. Thus, λ has no ordinary fixed points off of l, and Z_1 is its only ordinary fixed point.

On the other hand, λ fixes the center Z of the conic $\mathcal{C}_P$. If $Z \neq Z_1$, then Z is infinite and $\mathcal{C}_P$ is a parabola. Now the infinite point Q lies on l and $\mathcal{C}_P$, so $Z = Q$, and the line l must intersect the parabola in a second point, which would have to be a fixed point of λ, since l and $\mathcal{C}_P$ are both invariant under λ. This second point on $l \cap \mathcal{C}_P$ is therefore the point Z_1. Now all the points (other than Z_1) of the conic $\mathcal{C}_P$ lie on one side of the tangent line l_1 of Z_1. Note that $l \neq l_1$; otherwise Q on l_1 would imply l_∞ is on Z_1 by the polarity. If U and V are points on $\mathcal{C}_P$ on either side of Z_1, then they lie on opposite sides of l and the intersection $UV \cdot l$ is mapped to another point on l by λ, which must lie on the same side of l_1. This would say that λ, which fixes the tangent l_1, leaves invariant both sides of this line. However,

$$\lambda(G) = T_P^{-1}K^{-1}T_P^{-1}(G) = T_P^{-1}K^{-1}(G_2) = T_P^{-1}T_P^2(G) = G_1,$$

yet G and G_1 are on opposite sides of l_1. This contradiction shows that $Z = Z_1$, proving the first assertion. This implies that $\mathcal{C}_P$ is a hyperbola and the line $l = GG_1$ is an asymptote, since l is on Z and cannot intersect $\mathcal{C}_P$ at a point other than Q for the same reason as before - such an intersection would have to be a second ordinary fixed point of λ. This completes the proof. □

To view Theorem 4.1 from a different perspective, consider the model of hyperbolic geometry whose points are the interior points of the conic $\mathcal{C}_P$ and whose lines are the intersections of Euclidean chords on the conic with the interior of $\mathcal{C}_P$. Consider the case when $\mathcal{C}_P$ is an ellipse. The involutions $\eta_1 = \eta$ and $\eta_2 = T_P\eta T_P^{-1}$ fix the conic $\mathcal{C}_P$ and map the interior of $\mathcal{C}_P$ to itself. Furthermore, the points R and $\eta(R) = R'$ lie on a line "perpendicular" to GV in this model, since RR' lies on the pole of GV. (See Greenberg [6], p. 309.) Note that GV contains points that are interior to $\mathcal{C}_P$, because Z is an interior point in this case. Since η preserves cross-ratios it is a hyperbolic isometry. Thus, η represents reflection across the diameter GZ in this model ([6], pp. 341-343). The mapping $\eta_2 = \eta\lambda = T_P\eta T_P^{-1}$ is also a hyperbolic isometry fixing at least two points and thus a reflection across the diameter $T_P(GV) = G_1J = G_1Z$ ([6], p. 412). Thus, the map $\lambda = \eta_1\eta_2$ represents a hyperbolic rotation about the point Z.

When $\mathcal{C}_P$ is a hyperbola, then Z is an exterior point. If the lines GV and $T_P(GV)$ are secant lines, they both pass through the pole of the line at infinity (also a secant line) and are therefore perpendicular to l_∞ in the hyperbolic model. In this case λ represents a translation along l_∞. If the axis GV does not intersect the conic, then the homology η_1 interchanges the sides of the line l_∞ (in the model). Since the asymptotes separate the lines GV and ZV_∞ (see the proof of Theorem 4.1), the point V_∞ is a fixed point of η_1 in the hyperbolic plane, and all lines through V_∞ are invariant. Hence, η_1 is a half-turn about V_∞. If the axis $T_P(GV)$ is a secant line, then λ is an indirect isometry with no fixed points and an invariant line, and is therefore a glide ([6], p. 430). If neither axis is a secant line, λ is the product of two half-turns and is therefore a translation. When $\mathcal{C}_P$ is a parabola, Z is a point on the conic and therefore an ideal point in the hyperbolic model. In this case every line through Z intersects the conic, so GV and $T_P(GV)$ are secants and λ is a parallel displacement ([6], p. 424).

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DEPARTMENT OF MATHEMATICS, MALONEY HALL
BOSTON COLLEGE, 140 COMMONWEALTH AVENUE
CHESTNUT HILL, MASSACHUSETTS, 02467-3806
E-mail address: `igor.minevich@bc.edu`

DEPARTMENT OF MATHEMATICAL SCIENCES
INDIANA UNIVERSITY - PURDUE UNIVERSITY AT INDIANAPOLIS
402 N. BLACKFORD STREET, INDIANAPOLIS, INDIANA, 46202
E-mail address: `pmorton@iupui.edu`