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Three Mutually Tangent Congruent Circles Tangent to the Sidelines of a Triangle

Francisco Javier García Capitán and Paul Yiu

Abstract. We present an animation of three mutually tangent congruent circles each tangent to one side of a given triangle.

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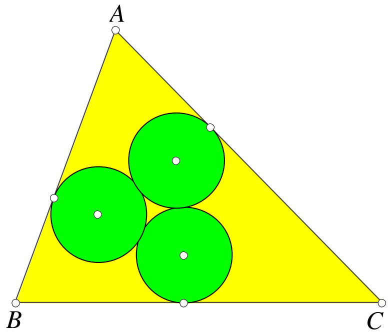

FIGURE 1

Consider a triangle ABC with incircle $I(r)$ tangent to the sidelines BC, CA, AB at D, E, F respectively. Let XYZ be an inscribed equilateral triangle of sides 2ℓ. Clearly the circles of radii ℓ and centers X, Y, Z are mutually tangent. For each circle, consider the tangent parallel to the corresponding sideline, farther from the opposite vertex. These tangents bound a triangle $A'B'C'$ homothetic to ABC (see Figure 1). We make use of the folklore theorem below. For basic information on barycentric coordinates with reference to a triangle, see [2].

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Theorem 1. *The images of the sidelines BC, CA, AB of triangle ABC under the homotheties $\mathsf{h}(A,\ 1+u)$, $\mathsf{h}(B,\ 1+v)$, $\mathsf{h}(C,\ 1+w)$ bound a triangle homothetic to ABC at $(u:v:w)$ (in homogeneous coordinates) and homothety ratio $1+u+v+w$.*

Proof. These homothetic images are the lines

$$\begin{aligned} (1+u)x + \quad uy + \quad uz &= 0,\\ vx + (1+v)y + \quad vz &= 0,\\ wx + \quad wy + (1+w)z &= 0. \end{aligned}$$

They bound a triangle with vertices

$$(1+v+w:-v:-w),\quad (-u:1+u+w:-w),\quad (-u:-v:1+u+v).$$

Since

$$\begin{aligned} (1+v+w,-v,-w) =& \ (1+u+v+w)(1,0,0)+(-1)(u,v,w)\\ =& \ (1+u+v+w)(1,0,0)-(u+v+w)\cdot\frac{(u,v,w)}{u+v+w}, \end{aligned}$$

the homothetic ratio is $1+u+v+w$.

Proposition 2. *Triangle $A'B'C'$ is the image of triangle ABC under the homothety with center I and ratio $1+\frac{\ell}{r}$.*

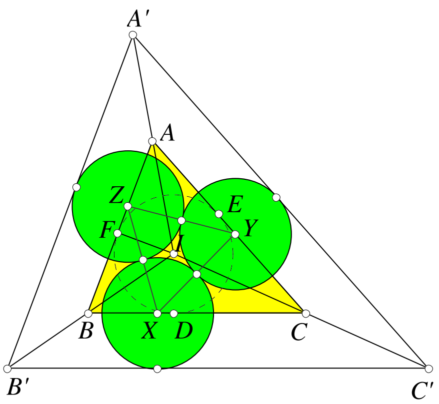

FIGURE 2

Proof. The tangent to the circle $X(\ell)$ parallel to BC, and external to the triangle, is the image of the line BC under the homothety, center A, and ratio

$$\frac{\frac{S}{a}+\ell}{\frac{S}{a}}=\frac{S+a\ell}{S}=1+\frac{a\ell}{S},$$

where S is twice the area of triangle ABC. For the other two tangents, the homothetic ratios are $1+\frac{b\ell}{S}$ and $1+\frac{c\ell}{S}$. By Theorem 1, the three parallels bound a triangle homothetic to ABC with homothetic center $\left(\frac{a\ell}{S}:\frac{b\ell}{S}:\frac{c\ell}{S}\right)=(a:b:c)$ in homogenous barycentric coordinates. This is the incenter I of triangle ABC. The ratio of homothety is

$$1+\frac{a\ell}{S}+\frac{b\ell}{S}+\frac{c\ell}{S}=1+\frac{(a+b+c)\ell}{S}=1+\frac{\ell}{r},$$

since $S = r(a+b+c)$.

Therefore, a homothety with center I and ratio $\frac{r}{r+\ell}$ will map $A'B'C'$ into ABC, and the three circles with centers X, Y, Z into three congruent circles tangent to the sidelines of ABC. The three image circles have radii $\rho = \ell \cdot \frac{r}{r+\ell}$. Equivalently,

$$\frac{1}{\rho} = \frac{1}{r} + \frac{1}{\ell}. \tag{1}$$

Here is a simple construction of ρ (see [3, §2.2]). On the perpendicular to BC at X, let P be a point on the same side of BC as the incenter I, such that $PX = \ell$. Join PD and IX to intersect at X' (see Figure 3). If X'' is the orthogonal projection of X' on BC, then (1) is satisfied, and $DX'' : X''X = r : \ell$. From this,

$$\frac{IX'}{IX} = \frac{DX''}{DX} = \frac{r}{r+\ell}.$$

This means that X' is the image of X under the homothety mapping $A'B'C'$ to ABC. The homothety sends the circle $X(\ell)$ into the circle $X'(\rho)$ which is tangent to the sideline BC.

The same homothety maps Y and Z into Y' and Z' respectively, which can be constructed analogously as X. Here is a simpler construction. The parallels to XY and XZ through X' intersect IY and IZ at Y' and Z' respectively (see Figure 3).

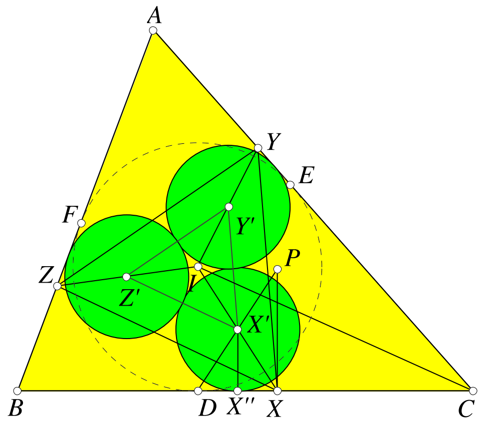

Figure 3

We conclude this note with a remark on the construction of inscribed equilateral triangles. Let J be an isodynamic point of ABC, with pedal triangle $X_0Y_0Z_0$. It is well known that $X_0Y_0Z_0$ is equilateral, with the same or opposite orientation of ABC according as $J = X(15)$ or $X(16)$ in [1]. Furthermore, every inscribed equilateral triangle XYZ is obtained by rotating about J the lines JX_0, JY_0, JZ_0 by the same oriented angle (see Figure 4).

This remark, coupled with the construction of the circles (X'), (Y'), (Z'), furnishes the animation of two families of three mutually tangent congruent circles each tangent to one sideline of a given triangle ABC.

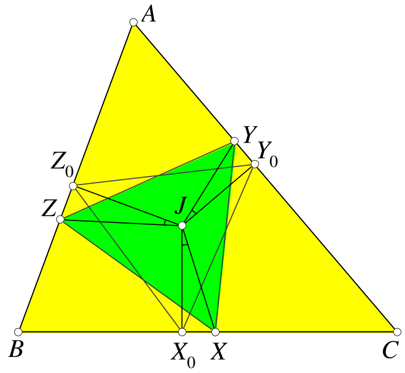

FIGURE 4

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DEPARTAMENTO DE MATEMÁTICAS
I.E.S. ÁLVAREZ CUBERO
PDTE. ALCALÁ ZAMORA s/n, 14800 PRIEGO DE CRDOBA, SPAIN
E-mail address: garciacapitan@gmail.com

DEPARTAMENT OF MATHEMATICAL SCIENCES
FLORIDA ATLANTIC UNIVERSITY
777 GLADES ROAD, BOCA RATON, FLORIDA 33431-0991, USA
E-mail address: yiu@fau.edu