THE NEW INEQUALITY IN A CYCLIC POLYGON

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Abstract. In this paper we give a proof of new inequality in arbitrary cyclic polygon.

1. Introduction

Let \( n \) points \( A_1, A_2, \ldots, A_n \) lie on a circle, find structure geometry of \( A_1, A_2, \ldots, A_n \) such that \( \sum_{i<j} A_iA_j \) is maximum. The answer of this question in the statement of the theorem as followings:

![Diagram of cyclic polygon](image)

Figure 1

Keywords and phrases: geometry inequality, cyclic polygon,

(2010)Mathematics Subject Classification: 51M15

Received: 5.05.2016. In revised form: 18.09.2016. Accepted: 26.01.2017.
Theorem 1.1. Let $n$-regular polygon $X_1X_2\cdots X_n$ with the circumcribed circle $(O)$. Let $n$ points $A_1, A_2, \cdots, A_n$ lie on the circle $(O)$ then:

$$
\sum_{i<j} A_i A_j \leq \sum_{i<j} X_i X_j
$$

Equality holds if only if $A_1, A_2, \ldots, A_n$ are the vertices of a $n$ regular polygon with the circumcribed circle $(O)$.

We give the proof of theorem 1.1 in item 2 below:

2. PROOF OF THE ISOPERIMETRIC INEQUALITY

Proof. The case $n = 3$ is trivial and well-known [1].

Case 1: $n$ odd, $n \geq 5$ denote $n = 2m + 1, m \geq 2$.

We choose WLOG the circle $x^2 + y^2 = 1$ and the points $A_i(\cos 2t_i, \sin 2t_i)$ for $i = 1, 2, \cdots, 2m + 1$, such that $0 = t_1 < t_2 < \cdots < t_{2m+1} < \pi$. From here we deduce that $A_i A_j = 2 \sin (t_j - t_i)$ for $i = 1, 2, \cdots, 2m + 1$. So we need to maximize the sum $\sum_{1 \leq i < j \leq 2m+1} \sin (t_j - t_i)$. Since the function $f: (0, \pi) \to R, f(x) = \sin x$ is strictly concav, so by the Jensen’s inequality we have:

$$
\sum_{k=2}^{2m+1} \sin (t_k - t_{k-1}) \leq 2m \sin \left( \frac{1}{2m} \sum_{k=2}^{2m+1} (t_k - t_{k-1}) \right) = 2m \sin \left( \frac{1}{2m} \cdot t_{2m+1} \right)
$$

$$
\sum_{k=2m+1}^{2m+1} \sin (t_k - t_{k-1}) = \sin (t_{2m+1})
$$

$$
\sum_{k=3}^{2m+1} \sin (t_k - t_{k-2}) \leq (2m - 1) \sin \left( \frac{1}{2m - 1} \sum_{k=3}^{2m+1} (t_k - t_{k-2}) \right) = (2m - 1) \sin \left( \frac{1}{2m - 1} (t_{2m+1} + t_{2m} - t_2) \right)
$$

$$
\sum_{k=2m}^{2m+1} \sin (t_k - t_{k-2m+1}) \leq 2 \sin \left( \frac{1}{2} (t_{2m+1} + t_{2m} - t_2) \right)
$$

In general, if $2 \leq i \leq m + 1$, then we have:

$$
\sum_{k=i}^{2m+1} \sin (t_k - t_{k-i+1}) \leq (2m + 2 - i) \sin \left( \frac{1}{2m + 2 - i} \sum_{k=i}^{2m+1} (t_k - t_{k-i+1}) \right) = (2m + 2 - i) \sin \left( \frac{1}{2m + 2 - i} (t_{2m+1} + t_{2m} + \cdots + t_{2m+3-i} - (t_{i-1} + \cdots + t_1) \right)
$$

$$
\sum_{k=2m+3-i}^{2m+1} \sin (t_k - t_{k-2m-2+i}) \leq (i-1) \sin \left( \frac{1}{i-1} \sum_{k=i}^{2m+1} (t_k - t_{k-2m-2+i+1}) \right)
$$
\[= (i-1) \sin \left( \frac{1}{i-1} \left( t_{2m+1} + t_{2m} + \cdots + t_{2m+3-i} - (t_{i-1} + \cdots + t_1) \right) \right) \]

Also, let’s observe that:

\[
\sum_{i=2}^{m+1} \sum_{k=i}^{2m+1} \sin(t_k - t_{k-i+1}) + \sum_{i=2}^{m+1} \sum_{k=2m+3-i}^{2m+1} \sin(t_k - t_{k-2m-2+i}) = \\
= \sum_{1 \leq j < k \leq 2m+1} \sin(t_j - t_i)
\]

For every \( i \in \{2, 3, \cdots, m+1 \} \) define the function \( f_i : (0, (i-1)\pi) \to R \), as

\[ f_i(x) = (2m + 2 - i) \sin \left( \frac{x}{2m+2-i} \right) + (i-1) \sin \left( \frac{x}{i-1} \right) \]

we have:

\[
f_i'(x) = \cos \left( \frac{x}{2m + 2 - i} \right) + \cos \left( \frac{x}{i-1} \right) = 2 \cos \left( \frac{(2m+1)x}{2(2m+2-i)(i-1)} \right) \cos \left( \frac{(2m+3-2i)x}{2(2m+2-i)(i-1)} \right)\]

But,

\[
\frac{(2m+3-2i)x}{2(2m+2-i)(i-1)} < \frac{(2m+3-2i)(i-1)\pi}{2(2m+2-i)(i-1)} = \frac{(2m+3-2i)\pi}{2(2m+2-i)} < \frac{\pi}{2},
\]

for \( i \in \{2, 3, \cdots, m+1 \} \) and \( x \in (0, (i-1)\pi) \) and

\[
\frac{(2m+1)x}{2(2m+2-i)(i-1)} < \frac{(2m+1)\pi}{2(2m+2-i)} < \pi,
\]

for \( i \in \{2, 3, \cdots, m+1 \} \) and \( x \in (0, (i-1)\pi) \).

In conclusion, \( \cos \left( \frac{(2m+1)x}{2(2m+2-i)(i-1)} \right) > 0 \) with \( \forall x \in (0, (i-1)\pi) \) and the function \( \cos \left( \frac{(2m+1)x}{2(2m+2-i)(i-1)} \right) \) is strictly decreasing on \((0, (i-1)\pi)\). So that \( x_0 \)

is critical point for \( f_i \) if only if

\[
\frac{(2m+1)x}{2(2m+2-i)(i-1)} = \frac{\pi}{2} \iff x_0 = \frac{(2m+2-i)(i-1)\pi}{(2m+1)}.
\]

From the above considerations, \( 0 < \frac{(2m+2-i)(i-1)\pi}{(2m+1)} < (i-1)\pi \). Thus,

\[
\text{max} \ f_i = f_i \left( \frac{(2m+2-i)(i-1)\pi}{2m+1} \right) = (2m+1) \sin \left( \frac{(i-1)\pi}{2m+1} \right).
\]

In conclusion,

\[
\sum_{k=i}^{2m+1} \sin(t_k - t_{k-i+1}) + \sum_{k=2m+3-i}^{2m+1} \sin(t_k - t_{k-2m-2+i}) \leq (2m+1) \sin \left( \frac{(i-1)\pi}{2m+1} \right).
\]

Combining with the lemma, we deduce that equality holds if and only if

\[ t_i - t_1 = t_{i+1} - t_2 = \cdots = t_{2m+1} - t_{2m+2-i} = \frac{(i-1)\pi}{2m+1} \]

and

\[ t_{2m+3-i} - t_1 = t_{2m+4-i} - t_2 = \cdots = t_{2m+1} - t_{i-1} = \frac{(2m+2-i)\pi}{2m+1}. \]

Since \( t_1 = 0 \), then we deduce that \( t_k = \frac{(k-1)\pi}{2m+1} \), for \( k = 1, 2, \cdots, 2m+1 \).
Hence the maximum value is achieved iff the polygon is regular. And note that if the polygon is regular, then we deduce from above that:

$$\sum_{1 \leq i < j \leq 2m+1} \sin(t_j - t_i) = (2m + 1) \sum_{i=2}^{m+1} \sin\left(\frac{(i - 1)\pi}{2m + 1}\right).$$

In closed form:

$$\sum_{1 \leq i < j \leq 2m+1} \sin(t_j - t_i) = (2m + 1) \frac{\sin\left(\frac{m\pi}{2(2m+1)}\right) \sin\left(\frac{(m+1)\pi}{2(2m+1)}\right)}{\sin\left(\frac{\pi}{2(2m+1)}\right)}.$$

**Case 2:** $n$ is even and $n \geq 4$. Analogously to case 1.

**References**


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