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ON GENERALIZED EULER SPIRALS IN E^3

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Abstract. The Cornu spirals on plane are the curves whose curvatures are linear. Generalized planar cornu spirals and Euler spirals in E^3, the curves whose curvatures are linear are defined in [1,5]. In this study, these curves are presented as the ratio of two rational linear functions.

Also here, generalized Euler spirals in E^3 has been defined and given their some various characterizations. The approach we used in this paper is useful in understanding the role of Euler spirals in E^3 in differential geometry.

1. Introduction

Spirals are the curves that had been introduced in the 1700s. Privately, one of the most fascinating spiral in nature and science is Euler Spiral. This curve is defined by the main property that its curvature is equal to its arclength.

Euler Spirals were discovered independently by three researchers [5, 9]. In 1694, Bernoulli wrote the equations for the Euler spiral for the first time, but did not draw the spirals or compute them numerically. In 1744, Euler rediscovered the curve's equations, described their properties, and derived a series expansion to the curve's integrals. Later, in 1781, he also computed the spiral's end points. The curves were re-discovered in 1890 for the third time by Talbot, who used them to design railway tracks [5].

On the other hand, the Euler spiral, defined by the linear relationship between curvature and arclength, was first proposed as a problem of elasticity of James Bernoulli, then solved accurately by Leonhard Euler [9]. The Euler spiral, also well known as Clothoid or Cornu Spiral is a plane curve and defined as the curve in which the curvature increases linearly with arclength. Changing the constant of proportionality merely scales the entire curve [9,11]. This curve is known as an example of such an aesthetic curve and also we know that its curvature varies linearly with arclength [2,5,7, 9]. In [5], their proposed curve has both its curvature and torsion change linearly with length.

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In this paper, at the beginning, we give the basic concepts about the study then we deal with Euler spirals whose curvature and torsion are linear and generalized Euler spirals whose ratio between its curvature and torsion changes linearly in E^3.

Here, we give the definitions of the spirals in E^2 also referred to as a Cornu Spiral, then the spirals in E^3 with the name Euler Spirals. In addition, by giving the definition of logarithmic spirals in E^3, we seek that if the logarithmic spirals in E^3 are generalized Euler spirals or not. In [5], the curvature and torsion are taken linearly; by using some characterizations of [5] and also differently from [5], we have presented the ratio of the curvature and torsion change linearly. And finally, we have named these spirals as generalized euler spirals in E^3 by giving some different characterizations.

2. PRELIMINARIES

Now, the basic concepts have been recalled on classical differential geometry of space curves. References [1, 5, 6, 10] contain the concepts about the elements and properties of Euler spirals in E^2 and in E^3. Then we give the definition of generalized Euler Spirals in E^3.

In differential geometry of a regular curve in Euchlidean 3-space, it is well-known that, one of the important problems is characterization of a regular curve. The curvature κ and the torsion τ of a regular curve play an important role to determine the shape and size of the curve [5,10]

Let

$$\alpha : I \to E^3 \tag{1}$$

$$s \mapsto \alpha(s)$$

be unit speed curve and $\{T, N, B\}$ Frenet frame of α. T, N, B are the unit tangent, principal normal and binormal vectors respectively. Let κ and τ be the curvatures of the curve α.

A spatial curve $\alpha(s)$ is determined by its curvature $\kappa(s)$ and its torsion $\tau(s)$. Intuitively, a curve can be obtained from a straight line by bending (curvature) and twisting (torsion). The Frenet Serret Equations will be necessary in the derivation of Euler spirals in E^3. In the following

$$\overrightarrow{T}(s) = \frac{d\alpha}{ds}$$

is the unit tangent vector, $\overrightarrow{N}(s)$ is the unit normal vector, and $\overrightarrow{B}(s) = \overrightarrow{T}(s) \times \overrightarrow{N}(s)$ is the binormal vector. We assume an arc-length parametrization.

Given a curvature $\kappa(s) \rangle\ 0$ and a torsion $\tau(s)$, according to the fundamental theorem of the local theory of curves [10], there exists a unique (up to rigid motion) spatial curve, parametrized by the arc-length s, defined by its Frenet-Serret equations, as follows:

$$\frac{d\overrightarrow{T}(s)}{ds} = \kappa(s)\overrightarrow{N}(s), \quad \frac{d\overrightarrow{N}(s)}{ds} = -\kappa(s)\overrightarrow{T}(s) + \tau(s)\overrightarrow{B}(s), \quad \frac{d\overrightarrow{B}(s)}{ds} = -\tau(s)\overrightarrow{N}(s). \tag{2}$$

From [1], a unit speed curve which lies in a surface is said to be a Cornu spiral if its curvatures are non-constant linear function of the natural parameter s, that is if $\alpha(s) = as + b$, with $a \neq 0$. The classical Cornu spiral in E^2 was studied by J. Bernoulli and it appears in diffraction.

Harary and Tall define the Euler spirals in E^3 the curve having both its curvature and torsion evolve linearly along the curve (Figure 1). Furthermore, they require that their curve conforms with the definition of a *Cornu spiral*.

Figure 1. Logarithmic spiral in E^3 [6]

Here, these curves whose curvatures and torsion evolve linearly are called Euler spirals in E^3. Thus for some constants $a, b, c, d \in \mathbb{R}$,

$$\kappa(s) = as + b, \quad \tau(s) = cs + d \tag{3}$$

On the other hand, in [13], another property of Euler spirals is underlined as : Euler spirals have a linear relation between the curvature and torsion, it is a special case of Bertrand curves.

An attempt to generalize Euler spirals to $3D$, maintaining the linearity of the curvature, is presented in [5, 8]. A given polygon is refined, such that the polygon satisfies both arc-length parametrization and linear distribution of the *discrete curvature binormal vector*. The logarithm ignores the torsion, despite being an important characteristic of $3D$ curves [5]. Harary and Tal prove that their curve satisfies properties that characterize fair and appealing curves and reduces to the $2D$ Euler spiral in the planar case. Accordingly the Euler spirals in E^2 and in E^3 that satisfy by a set of differential equations is the curve α for which the following conditions hold:

$$\frac{d\overrightarrow{T}(s)}{ds} = \left(\frac{1}{(as+b)}\right)\overrightarrow{N}(s),$$
$$\frac{d\overrightarrow{N}(s)}{ds} = -\left(\frac{1}{(as+b)}\right)\overrightarrow{T}(s) + \left(\frac{1}{(cs+d)}\right)\overrightarrow{B}(s),$$
$$\frac{d\overrightarrow{B}(s)}{ds} = -\left(\frac{1}{(cs+d)}\right)\overrightarrow{N}(s).$$

Next, we define logarithmic spiral having a linear radius of curvature and a linear radius of torsion from [6]. They seek a spiral that has both a linear radius of curvature and a linear radius of torsion in the arc-length parametrization s :

$$\kappa(s) = \frac{1}{as+b}$$
$$\tau(s) = \frac{1}{cs+d}$$

where a, b, c and d are constants.

Figure 2. Euler spirals in E^3

In addition to these, we want to give our definition that Euler spirals in E^3 whose ratio between its curvature and torsion evolve linearly is called generalized Euler spirals in E^3. Thus for some constants $a, b, c, d \in \mathbb{R}$,

$$\frac{\kappa}{\tau} = \frac{as+b}{cs+d}.$$

3. EULER SPIRALS IN E^3

In this section, we study some characterizations of Euler spirals in E^3 by giving some theorems with using the definitions in section 2.

Proposition 3.1. *If the curvature τ is zero then $\kappa = as + b$ and then the curve is planar cornu spiral.*

Proof. *If $\tau = 0$ and the curvature is linear, then the ratio*

$$\frac{\tau}{\kappa} = 0.$$

Therefore, we see that the curve is planar cornu spiral.

Proposition 3.2. *If the curvatures are*

$$\begin{aligned}\tau &= as + b\\ \kappa &= c\end{aligned}$$

then the euler spirals are rectifying curves.

Proof. *If we take the ratio*

$$\frac{\tau}{\kappa} = \frac{as + b}{c}$$

where λ_1 and λ_2 , *with* $\lambda_1 \neq 0$ *are constants, then*

$$\frac{\tau}{\kappa} = \lambda_1 s + \lambda_2.$$

It shows us that the Euler spirals are rectifying curves. It can be easily seen from [3] that rectifying curves have very simple characterization in terms of the ratio $\frac{\tau}{\kappa}$.

Proposition 3.3. *Euler spirals in* E^3*are Bertrand curves.*

Proof. *From the definition of Euler spiral in* E^3 *and the equation* (3) in *Section.*2, *the equations can be taken as*

$$\begin{aligned}\tau(s) &= c_1 s + c_2,\\ \kappa(s) &= d_1 s + d_2, \text{ with } c_1 \neq 0 \text{ and } d_1 \neq 0.\end{aligned}$$

Here,

$$s = \frac{1}{c_1}(\tau - c_2)$$

and then,

$$\begin{aligned}\kappa &= \frac{d_1}{c_1}(\tau - c_2) + d_2,\\ c_1\kappa &= d_1(\tau - c_2) + c_1 d_2,\\ c_1\kappa + d_1\tau &= c_3,\\ \frac{c_1}{c_3}\kappa + \frac{d_1}{c_3}\tau &= 1.\end{aligned}$$

Thus, we obtain

$$\lambda\kappa + \mu\tau = 1.$$

It shows us that there is a Bertrand curve pair that corresponds to Euler spirals in E^3.

Theorem 3.1. *Let* M *be a surface in* E^3 *and* $\alpha : I \to M$ *be unit speed curve but not a general helix. If the Darboux curve*

$$W(s) = \tau T + \kappa B$$

is geodesic curve on the surface M, *then the curve* α *is Euler spirals in* E^3.

We have

$$W(s) = \tau T + \kappa B, \tag{4}$$

$$W'(s) = \tau' T + \kappa' B, \tag{5}$$

$$W''(s) = \tau'' T + \kappa'' B + (\kappa\tau' - \tau\kappa')N, \tag{6}$$

$$W''(s) = \tau'' T + \kappa'' B + \left[\left(\frac{\tau}{\kappa}\right)' \kappa^2\right] N. \tag{7}$$

Here, $n(s)$ is the unit normal vector field of the surface M. Darboux curve is geodesic on surface M, therefore

$$W''(s) = \lambda(s) n(s).$$

And also $n = N$, then we have

$$W''(s) = \lambda(s) N(s).$$

From (7), if we take

$$\tau'' = 0, \ \kappa'' = 0$$

and also

$$\kappa = as + b,$$
$$\tau = cs + d,$$

then the curve α is generalized Euler spiral in E^3.

4. GENERALIZED EULER SPIRALS IN E^3

In this section, we investigate generalized Euler spirals in E^3 by using the definitions in *section*.2.

Theorem 4.1. *In E^3, all logarithmic spirals are generalized euler spirals.*

Proof. As it is known that in all logarithmic spirals, the curvatures are linear as:

$$\kappa(s) = \frac{1}{as + b},$$
$$\tau(s) = \frac{1}{cs + d}.$$

In that case, it is clear that the ratio between the curvatures can be given as:

$$\frac{\kappa}{\tau} = \frac{cs + d}{as + b}.$$

Thus, it can be easily seen all logarithmic spirals are generalized euler spirals.

Theorem 4.2. *Euler spirals are generalized Euler spirals in E^3.*

Proof. *It is clear from the property of curvature, torsion and the ratio that are linear as:*

$$\kappa(s) = as + b,$$
$$\tau(s) = cs + d$$

and then

$$\frac{\kappa}{\tau} = \frac{as + b}{cs + d}.$$

That shows us Euler spirals are generalized Euler spirals.

Proposition 4.1. *All generalized Euler spirals in E^3 that have the property*

$$\frac{\tau}{\kappa} = d_1 s + d_2$$

are rectifying curves.

Proof. *If the curvatures $\kappa(s)$ and $\tau(s)$ are taken as*

$$\kappa(s) = c,$$
$$\tau(s) = d_1 s + d_2 \text{ with } d_1 \neq 0,$$

then

$$\frac{\tau}{\kappa} = \frac{d_1 s + d_2}{c} = \lambda_1 s + \lambda_2,$$

where λ_1 and λ_2 are constants. This gives us that if the curve α is generalized Euler spiral then it is also in rectifying plane.

Result 1. *General helices are generalized euler spirals.*

Proof. *It can be seen from the property of curvatures that are linear and the ratio is also constant as it is shown:*

$$\frac{\tau}{\kappa} = \lambda.$$

Theorem 4.3. *Let*

$$\alpha : I \to E^3 \tag{8}$$
$$s \mapsto \alpha(s)$$

be unit speed curve and let κ and τ be the curvatures of the Frenet vectors of the curve α. For $a, b, c, d, \lambda \in \mathbb{R}$, let take the curve β as

$$\beta(s) = \alpha(s) + (as + b)T + (cs + d)B + \lambda N. \tag{9}$$

In this case, the curve α is generalized Euler spiral which has the property

$$\frac{\kappa}{\tau} = \frac{cs + d}{as + b}$$

if and only if the curves β and (T) are the involute-evolute pair. Here, the curve (T) is the tangent indicatrix of α.

Proof. *The tangent of the curve β is*

$$\beta'(s) = ((1 - \lambda)\kappa + a)T + (c + \lambda\tau)B + (\kappa(as + b) - \tau(cs + d))N. \tag{10}$$

The tangent of the curve (T) is

$$\frac{dT}{ds_T} = N.$$

Here, s_T is the arc parameter of the curve (T).

$$\langle \beta', N \rangle = \kappa(as + b) - \tau(cs + d). \tag{11}$$

If the curves β and (T) are the involute-evolute pair then

$$\langle \beta', N \rangle = 0.$$

From (11), it can be easily obtained

$$\frac{\kappa}{\tau} = \frac{cs + d}{as + b}.$$

This means that the curve α is generalized Euler spiral.

On the other hand, if the curve α is generalized Euler spiral which has the property

$$\frac{\kappa}{\tau} = \frac{cs+d}{as+b},$$

then from (11)

$$\langle \beta', N \rangle = 0.$$

This means that β and (T) are the involute-evolute pair.

Result 2. *From the hypothesis of the theorem above, the curve α is generalized Euler spiral which has the property*

$$\frac{\kappa}{\tau} = \frac{cs+d}{as+b}$$

if and only if β and (B) are the involute-evolute pair. Here, (B) is the binormal of the curve α.

Proof. *The tangent of the curve (B) is*

$$\frac{dB}{ds_B} = -N.$$

The smiliar proof above in Theorem 4.3. can be given for (B) again.

Theorem 4.4. *Let the ruled surface Φ be*

$$\Phi : I \times \mathbb{R} \longrightarrow E^3 \tag{12}$$

$$(s,v) \to \Phi(s,v) = \alpha(s) + v[(as+b)T + (cs+d)B].$$

The ruled surface Φ is developable if and only if the curve α is generalized Euler spiral which has the property

$$\frac{\kappa}{\tau} = \frac{cs+d}{as+b}.$$

Here, α is the base curve and T, B are the tangent and binormal of the curve α, respectively.

Proof. *For the directrix of the surface*

$$X(s) = (as+b)T + (cs+d)B,$$

and also for

$$X'(s) = aT + [(as+b)\kappa - (cs+d)\tau]N + cB,$$

we can easily give

$$\det(T, X, X') = \begin{vmatrix} 1 & 0 & 0 \\ as+b & 0 & cs+d \\ a & (as+b)\kappa - \tau(cs+d) & c \end{vmatrix}.$$

In this case, the ruled surface is developable if and only if $\det(T, X, X') = 0$ *then*

$$(cs+d)(as+b)\kappa - (cs+d)\tau = 0$$

then for $cs+d \neq 0$

$$(as+b)\kappa - (cs+d)\tau = 0.$$

Thus, the curve α is generalized Euler spiral which has the property

$$\frac{\tau}{\kappa} = \frac{as+b}{cs+d}.$$

Theorem 4.5. *Let $\alpha : I \to M$ be unit speed curve but not a general helix. If the curve*

$$U(s) = \frac{1}{\kappa}T + \frac{1}{\tau}B$$

is ta geodesic curve then the curve α is a logarithmic spiral in E^3.

Proof. *We have*

$$U' = \left(\frac{1}{\kappa}\right)' T + \left(\frac{1}{\tau}\right)' B$$
$$U'' = \left(\frac{1}{\kappa}\right)'' T + \left(\frac{1}{\tau}\right)'' B + \left[\left(\frac{1}{\kappa}\right)' \kappa - \left(\frac{1}{\tau}\right)' \tau\right] N.$$

Here,

$$\left[\left(\frac{1}{\kappa}\right)' \kappa - \left(\frac{1}{\tau}\right)' \tau\right] \neq 0$$

and then

$$\begin{aligned}\left(\frac{1}{\kappa}\right)' \kappa - \left(\frac{1}{\tau}\right)' \tau &= -\frac{\kappa'}{\kappa} + \frac{\tau'}{\tau} \\ &= \frac{-\tau\kappa' + \kappa\tau'}{\kappa\tau} \\ &= \left(\frac{\kappa}{\tau}\right)' \left(\frac{\tau}{\kappa}\right).\end{aligned}$$

If we take

$$\left(\frac{1}{\kappa}\right)'' = 0 \text{ and } \left(\frac{1}{\tau}\right)'' = 0$$

then the curve α is a logarithmic spiral. Here, it can be easily seen that

$$\frac{1}{\kappa} = as + b$$
$$\frac{1}{\tau} = cs + d$$

Thus, it is clear that the curve α is a logarithmic spiral.

5. CONCLUSIONS

The starting point of this study is to extend the notion of Euler spirals to E^3 by using some important properties. First, I introduce Euler spirals in E^2 and in E^3 then define the generalized Euler spirals in E^3. Using these concepts, some necessary conditions for these curves to be Euler spirals and generalized Euler spirals in E^3 are presented. Also, some different characterizations of these curves are expressed by giving theorems with their results. At this time, it is obtained that Euler spirals in E^3 are Bertrand curves. Additionally, I show that all Euler spirals are generalized Euler spirals in E^3 and also general helices are generalized Euler spirals in E^3. Moreover many different approaches about generalized Euler spirals in Euchlidean 3-space are presented in this paper.

We hope that this study will gain different interpretation to the other studies in this field.

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