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THREE METRIC RELATIONS IN A RIGHT KITE

MARTIN JOSEFSSON

Abstract. We derive two formulas for the area and one for the circumradius of a right kite. All formulas are expressed in terms of two of the three radii in the associated circles.

1. Introduction

There exist at least two different definitions of what a right kite is. We use the one that seems to be the most common, which can be stated as follows: a *right kite* is a quadrilateral with two pairs of non-overlapping adjacent congruent sides and with two opposite right angles (note that a square is a special case). An equivalent and shorter way of expressing this is that it is a cyclic kite. A right kite has the triple properties of having an incircle, a circumcircle and an excircle (see Figure 1).

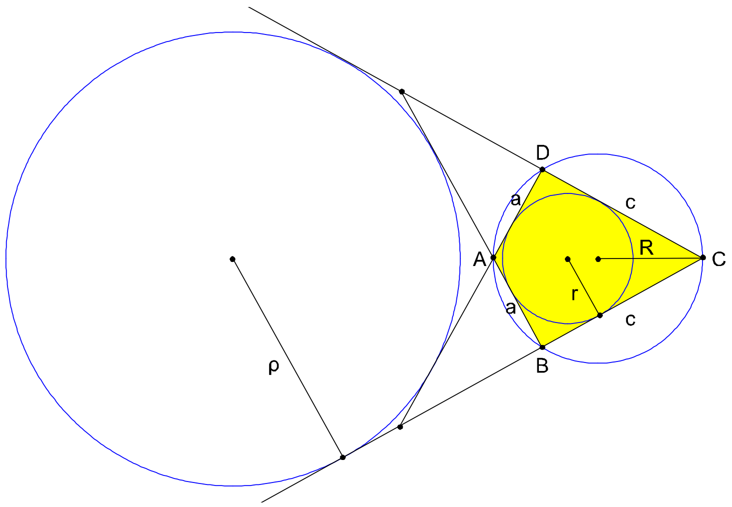

Figure 1

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Thus it could also be called a tricentric quadrilateral in comparison to a bicentric quadrilateral, which only has the first two circles.

2. Area

The area of a right kite can be expressed in terms of the inradius (the radius in the incircle) and the exradius (the radius in the excircle).

Theorem 2.1. *A right kite that has an incircle and an excircle with radii r and ρ respectively has the area*

$$K = \frac{4\rho^2 r^2}{\rho^2 - r^2}.$$

Proof. The exradius to an extangential quadrilateral is (see [1, Thm 8])

$$\rho = \frac{K}{|a-c|} \tag{1}$$

where a and c are two opposite sides. A well known formula for the area of a tangential quadrilateral is $K = r(a+c)$. It's easy to see that the area K of a right kite satisfy $K = ac$. Thus we get

$$\frac{K^2}{r^2} - \frac{K^2}{\rho^2} = (a^2+c^2+2ac) - (a^2+c^2-2ac) = 4ac = 4K.$$

Then

$$K\left(\frac{1}{r^2} - \frac{1}{\rho^2}\right) = 4$$

and the formula follows by solving for K.

The formula in the theorem also holds for squares even through they have an infinite exradius (according to (1)), since

$$\lim_{\rho\to\infty}\left(\frac{4\rho^2 r^2}{\rho^2 - r^2}\right) = 4r^2 \lim_{\rho\to\infty}\left(\frac{1}{1-\frac{r^2}{\rho^2}}\right) = 4r^2$$

which is the correct expression for the area of a square in terms of the inradius. □

In [2] we gave a short proof of a previously known formula for the area K of a right kite in terms of its circumradius R and inradius r. This formula is

$$K = r\left(r + \sqrt{4R^2 + r^2}\right). \tag{2}$$

Now we shall see that there is a very similar looking formula for the area in terms of the circumradius and the exradius.

Theorem 2.2. *A right kite that has a circumcircle and an excircle with radii R and ρ respectively has the area*

$$K = \rho\left(-\rho + \sqrt{4R^2 + \rho^2}\right).$$

Proof. The area and circumradius of a right kite satisfy $K = ac$ and $4R^2 = a^2 + c^2$, where a and c are two opposite sides. From (1) we get

$$K^2 = \rho^2(a^2+c^2-2ac) = \rho^2(4R^2 - 2K)$$

which yields the quadratic equation

$$K^2 + 2\rho^2 K - 4R^2\rho^2 = 0.$$

Its only positive solution is the formula in the theorem.

The formula does actually apply to squares as well. We can see this by calculating the limit

$$\lim_{\rho\to\infty} \rho\left(-\rho + \sqrt{4R^2+\rho^2}\right) = \lim_{\rho\to\infty}\left(\frac{4R^2}{1+\sqrt{\frac{4R^2}{\rho^2}+1}}\right) = \frac{4R^2}{2}$$

giving the correct formula $K = 2R^2$ for the area of a square in terms of the circumradius. □

3. Circumradius

We conclude by deriving a formula for the circumradius of a right kite in terms of the inradius and exradius, thus giving a connection between the three radii R, r and ρ.

Theorem 3.1. *In a right kite that has an incircle and an excircle with radii r and ρ respectively, the circumcircle has the radius*

$$R = \frac{r\rho}{\rho^2 - r^2}\sqrt{2(\rho^2+r^2)}\,.$$

Proof. Setting the area formulas in (2) and Theorem 2.2 equal results in the equality

$$-\rho^2 + \rho\sqrt{4R^2+\rho^2} = r^2 + r\sqrt{4R^2+r^2}$$

or equivalently

$$\rho\sqrt{x+\rho^2} - r\sqrt{x+r^2} = r^2 + \rho^2$$

where we put $x = 4R^2$. Squaring and rewriting it yields

$$\rho^2(x+\rho^2) + r^2(x+r^2) - (r^2+\rho^2)^2 = 2\rho r\sqrt{(x+\rho^2)(x+r^2)}$$

which is simplified to

$$x(\rho^2+r^2) - 2r^2\rho^2 = 2\rho r\sqrt{(x+\rho^2)(x+r^2)}.$$

Squaring again and simplifying results in

$$x^2\big((\rho^2+r^2)^2 - 4\rho^2 r^2\big) = 4r^2\rho^2 x\big((\rho^2+r^2) + r^2 + \rho^2\big)$$

where $x \neq 0$ can be canceled on both sides. By a further simplification and reinserting $x = 4R^2$, we get

$$4R^2(\rho^2 - r^2)^2 = 4r^2\rho^2 \cdot 2(\rho^2+r^2)$$

and the formula follows by taking the square root of both sides and solving for R. Since $\rho > r$, we don't need an absolute value in the denominator.

The formula applies to squares, although they have an infinite exradius, since

$$\lim_{\rho\to\infty}\left(\frac{r\rho}{\rho^2-r^2}\sqrt{2(\rho^2+r^2)}\right) = r\sqrt{2}\lim_{\rho\to\infty}\left(\frac{1}{1-\frac{r^2}{\rho^2}}\sqrt{1+\frac{r^2}{\rho^2}}\right) = r\sqrt{2}\cdot 1$$

which gives the well known identity $R = r\sqrt{2}$ in a square. □

References

[1] M. Josefsson, Similar metric characterizations of tangential and extangential quadrilaterals, *Forum Geom.*, **12 (2012)** 63–77.

[2] M. Josefsson, Maximal area of a bicentric quadrilateral, *Forum Geom.*, **12 (2012)** 237–241.

SECONDARY SCHOOL KCM
MARKARYD, SWEDEN
E-mail address: `martin.markaryd@hotmail.com`