INTERNATIONAL
JOURNAL OF GEOMETRY
Vol. 4 (2015), No. 2, 47 - 68

CLASSIFYING SETS OF THREE CIRCUMFERENCES BASED ON POWER THEOREM

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Abstract. This paper presents the theoretical framework and implementation of a method based on the power theorem to identify sets of three circumferences α, β and γ in any topological arrangement named configurations. This method assigns a vector V to each set $\alpha\beta\gamma$, which consists of ten parameters that describe the lowest power in the subsets $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$. Also, describe the relatives powers of centers of them and the power of its radical center. Considering the potential occurrence of great topological variety of the sets $\alpha\beta\gamma$ in a largest cluster of circumferences, this method can be used in search and classification processes.

1. INTRODUCTION

When given three circumferences α, β and γ whose coordinates of centers are (x_1, y_1), (x_2, y_2) and (x_3, y_3) and whose radii are r_1, r_2 and r_3 respectively, the Apollonius problem is to find tangent circumferences to them. Naming the center and radius of the searched circumference for x, y and r, the most direct method to obtain the solutions of this problem is to solve the system of three quadratic equations in three unknowns described by equations 1,2 e 3.

$$(x - x_1)^2 + (y - y_1)^2 - (r \pm r_1)^2 = 0. \tag{1}$$

$$(x - x_2)^2 + (y - y_2)^2 - (r \pm r_2)^2 = 0. \tag{2}$$

$$(x - x_3)^2 + (y - y_3)^2 - (r \pm r_3)^2 = 0. \tag{3}$$

The conditions to be satisfied by the circumferences that are tangent to α, β and γ can be understood by noting that the distance between the centers of two circumferences tangent is equal to the sum or difference of their radii.

Keywords and phrases: Sets of three circumferences, Apollonius problem, Power theorem
(2010)Mathematics Subject Classification: 51M04,97R30,14N10
Received: 17.01.2015. In revised form: 28.08.2015. Accepted: 20.09.2015.

If tangents are internal the condition is subtraction, and if they are external the condition is sum.

As these two conditions can be checked for the three circumferences the problem may have $2^3 = 8$ solutions [2]. Others solution methods to this problem and its implications are mentioned in reference [11], and one of them was used by me in [16] and [17].

An important property of this set of solutions is that the center P of the smaller radius circumference, named σ, is the closest point to α, β and γ according to a specific distance orientation. Four possible orientations of distance are shown in Figure 1.

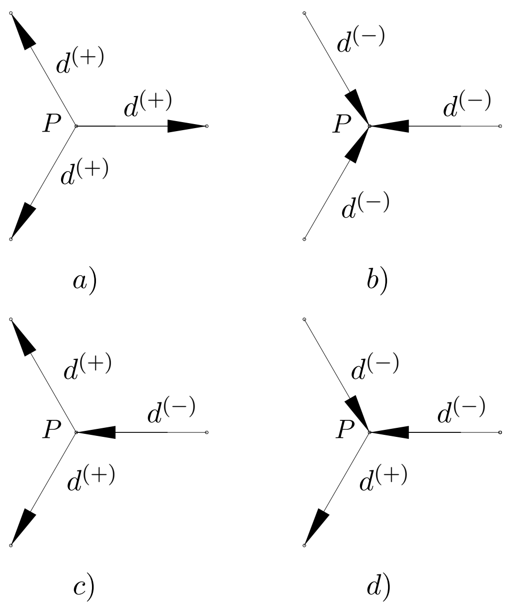

FIGURE 1. Shortest distance orientation: a) three positive orientations; b) three negative orientations; c) two positive orientations and one negative; d) two negative orientations and one positive.

The positive orientation indicates that σ and one of three given circumferences are external tangents. The negative orientation indicates that σ and one of three given circumferences are internal tangents, not considering for example if α is inside σ or σ is inside α.

Voronoi diagrams are geometric structures whose construction are closely related with the shortest distance and its orientation. These diagrams divide a cluster of circumferences in regions of influence of each one of them. The boundaries of these regions are arcs of hyperbolas that intersect at points which are, each one, three circumferences nearest, according to the considered orientation of shortest distance.

A recurring problem in the construction of these diagrams is that $\alpha\beta\gamma$ has a great topological and geometrical variety and, without a systematic method that considers many possible cases that can be occur, it is difficult filter specific type sets in a largest clusters. For example, in a sparse cluster, we may think that the most expected orientation to shortest distance is positive to the three given circumferences, but it is not necessarily true, as exemplified in Figure 2. In Figure 2*a*) the shortest distance has three positive orientations and in Figure 2*b*) it has two positive orientations and one negative.

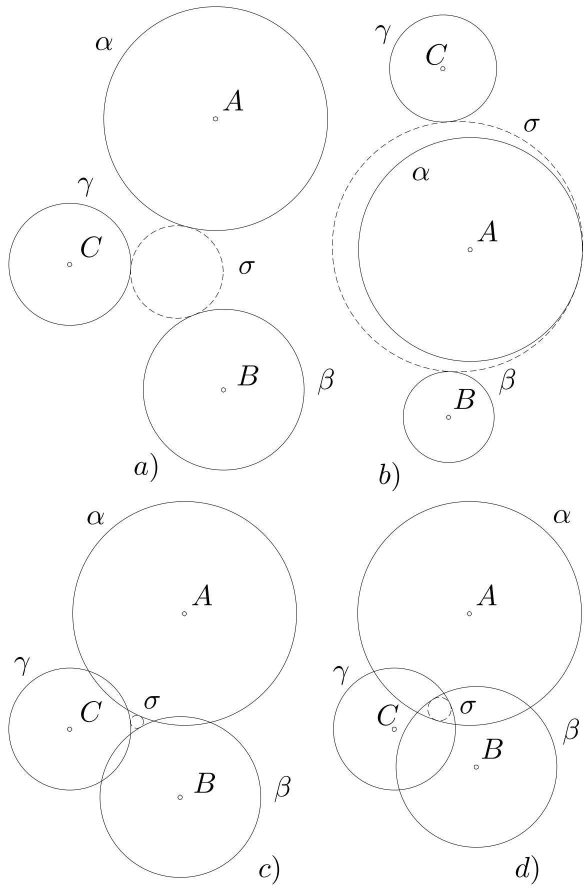

FIGURE 2. Orientations of shortest distances: a) three positive orientations; b) two positive orientations and one negative; c) three positive orientations and d) three negative orientations.

Considering a dense cluster we may think again that the most expected orientation to the shortest distance is negative to the three given circumferences but, it is not necessarily true as suggested in Figures 2*c*) and 2*d*). In Figure 2*c*) situation the shortest distance has three positive orientations and in Figure 2*d*) there is only one possibility to three negative orientations.

The examples of Figure 2 shows that the shortest distance to α, β and γ depends on the topology and geometry of the set. Because of this, studies in which is necessary to calculate the closest point of three circumferences have been considered clusters composed by sets that are topologically and geometrically homogeneous. For example, chronologically, in references [9],[10], [11], [12] and [13] are considered disjoints and intercepting sets, but not those where one circumference is inside the other. These studies consider only sets that admit a tangent circumference whose center has three positive distance orientations as shown in Figures 1a) and 1c). Similarly, in reference [14] are considered sets where two circumferences are inside the other but, about these two, one cannot be inside the other.

The selection of particular type sets in a cluster first appeared in reference [15]. Here, are considered sets of circumferences where two of them are inside another, but these two cannot be one inside the other, because this case has no solution. The selection of sets is done so that they may be a valid configuration, that is, have a solution.

The greatest variety of sets of circumferences considered in a study appears in the references [5], [8] and [7]. In these studies are considered sets in which circumferences can be intersecting or not, have a different radii and, in [7], they can contain others. In [5] and [8] is considered the Inversion and radius adjustment operation to find the centers of the smaller radius tangent circumference. This method requires that at least one of the three circumferences have the smallest radius among the three given. In this method the topology of the desired solution is given as an initial condition but, this is not a garantee that this solution is really calculated. The proposed algorithm in [7], as in [15], uses a sweepline to identify in a cluster, specific sets of three circumferences. These sets may contain secant circumferences and one inside the other but, they must have a different radii.

Considering the growth potential of the studies related to sets of three circumferences, this paper presents a study to enumerate their topological variety and a automatic method to classify them. Section 2 gives an overview of methods for classifying sets of three circumferences in the literature and highlight aspects that are important for this study. In sections 3.1 and 3.2 we describe the use of power theorem to classify sets formed by two and three circumferences and we make them count. In section 4 we describe the proposed method as well, the vector V calculations for each one of the 27 kinds of sets identified. In addition, are given some examples of how to identify the position of the smallest radius tangent circumference at some type of sets. In section 5 we draw the final remarks and conclusions of this study.

2. CLASSIFICATION METHODS

The oldest method for classifying sets of three geometric elements including points, lines and circumferences found in literature dates from the eighteenth century and is due to L. Gaultier de Tours [6]. In this work are listed 33 cases to the problem to find circumferences tangent to three elements. The case where the three elements are circumferences is one among the 33 cases listed. The purpose of this classification is to associate the number of solutions to the tangency conditions. For example, in the case where the three conditions are "tangent to a circumference" the number of solutions is between 2 and 8.

In the method proposed in reference [18] there are 46 configurations for sets of 3 circumferences grouped into 13 categories. The first four categories are Division I, Division II, Division III and Division IV. The others are named A_1, A_2, A_3; B; C_1, C_2; D, E and F. Any category in this method is composed by an specific topological relationship among three circumferences. For example, Division I if composed by "three not secant circumferences". The purpose of this classification method is to enumerate from any configuration the number of solutions. For example, none of three configurations in B category have 5 solutions.

In the classification method formulated in reference [1] there are 27 configurations for sets of 3 circumferences. This classification method follow the formulation presented in reference [18]. The new element in this proposal is the use of geometric transformation Inversion to rearrange some configurations formed by three circumferences that, in this method, are now listed as formed by two lines and a circumference or three lines. Thus, the 46 configuration formed by three circumferences listed in the previous method are reduced in this method to 27. The other 19 cases are now listed as configurations formed by two lines and a circumference or 3 lines. Finally, the 27 configurations listed are grouped into 9 categories, according to the number of solutions ranging from "zero" to 8.

These two classification methods, references [18] and [1], follows the principles initiated in reference [6]. They improved the way the circumferences are grouped but, without altering its purpose, which is to enumerate the number of possible solutions for each configuration. These methods are important in studies and researches related with Plane Geometry. Therefore, to verify the number of solutions through a computer program it is first necessary identify this configuration unequivocally. In the references [18] and [1] there are only two configurations of three circumferences that have no solution but, as shown in Figure 3, intuitively we find three configurations. It is clear that a computer program to be able to analyze sets of three circumferences is necessary to know all of these variations.

From the computational point of view, in reference [8] the authors propose a method of classification using the fact that topologically a solution and one of the three given circumferences can be external or may be internal tangents. When these relationships are considered for the three circumferences it reaches up to 27 different configurations. After this, considering that the three circumferences have different radii, only 20 configurations are valid, since they admit solutions. These 20 configurations are organized into 10 groups called operators. The are three types of operators: O, X and Δ. The are operator O means that a circumference solution and a circumference configuration are external tangents. The operator X means that a circumference solution and a circumference configuration are internal tangents, but this circumference contain a circumference solution. The operator Δ means that a circumference configuration and a circumference solution are internal tangents, but this circumference contain a circumference configuration.

The algorithm proposed here operates with three circumferences with different radii and ordered in sequence. First, the algorithm performs the adjustment of radii so that a configuration is transformed in a problem where are given one point P and two circumferences. Then, taking the point P as an inversion center, the problem is transformed in the case where they build

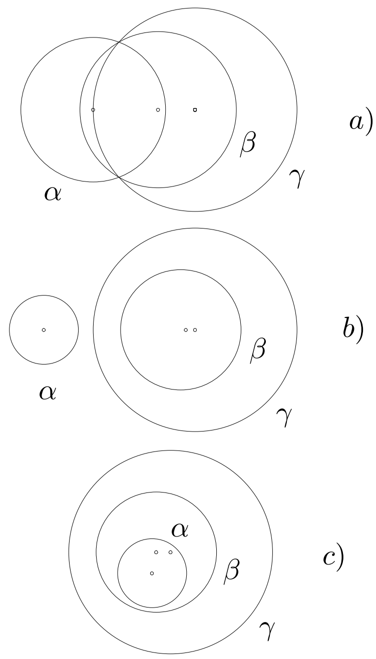

FIGURE 3. Configuration without solution a); configuration without solution b); configuration without solution c).

up the tangent lines to two circumferences and whose number of solutions is 2 or 4.

This algorithm requires two input informations: three circumferences G_0, G_1 and G_2 ordered according to their radii and the topological arrangement of the desired solution, for example, $\Delta\Delta\Delta$. Note that these symbols means that the desired solution is internal to the three generators but, in the paper it is not explained how the desired solution topology choices is made according to a given set of generators. For example, the generators G_0, G_1 and G_2 ordered and the topological arrangement $\Delta\Delta\Delta$ do not guarantee that the desired solution is one that has a smaller radius, as shown in Figure 4.

The solution of the smallest radius s_1 is inside the α, β and γ in Figure 4a) and is outside the α and the internal to β and γ in Figure 4b). Its shown that the topology of the desired solution is possible for multiple generator sets and it means that without a previous analysis of the geometry and topology of the three generators it can not guarantee that the obtained solution is the one that has a smallest radius. Also, in this method, the radii of the circumferences must to be distinct and, in practical situations this is not always possible.

3. PROPOSED METHOD

The mentioned methods previously are based on geometrical and some topological aspects of the sets but they do not consider that it can have many configurations. The method we are looking must be able to identify any configuration of three circumference unequivocally, must be fast and

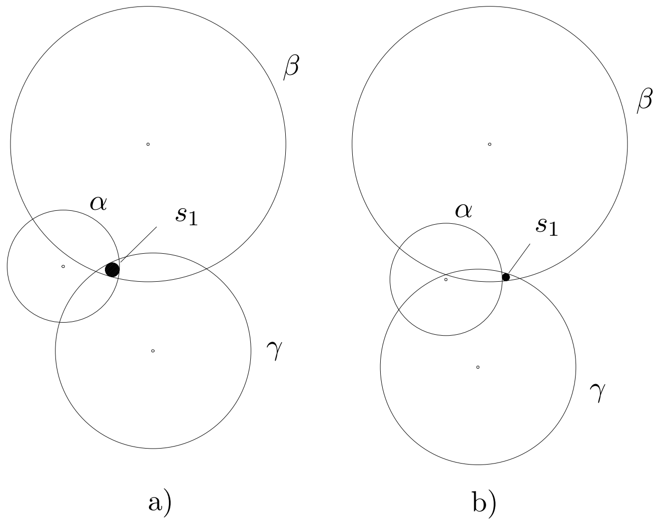

FIGURE 4. *a*) Solution S_1 internal to α, β and γ; *b*) Solution S_1 external to α and internal to β and γ.

simple so that the information that specifies the configurations must be minimal and quickly obtained.

In pursuit of this goal, a method proposed here has beginnings, with the analysis of subsets and sets of circumferences in a way that any of them can be unambiguously characterized by a fewest number of parameters. Also, we indicate how to calculate them in a non explicit way.

3.1. **Two circumferences sets.** If we take two tangent circumferences, secant, not secant or concentric and added others conveniently we get two types of sets called coaxial and radial. According to these relative positions, the sets formed are shown in Figure 5. Sets a, b and c are coaxial and d is radial [3], [4].

The generative principle and definitions of coaxial and radial sets are 3.1 and 3.2 respectively.

Definition 3.1. *Let* Δ *a line and* α *a circumference. It is called coaxial the set composed by circumferences that admit with* α, Δ *as its radical axis.*

Definition 3.2. *Let* α *a circumference with radius* r *and center* O*. It is called radial the set composed by concentric circumferences with* α*.*

Coaxial sets have two subsets named branches, left and right, which definition is 3.3.

Definition 3.3. *Given a coaxial set, it is called branches the subsets of circumferences which centers are on the right or the left to the radical line according to the considered direction.*

The circumferences α and β are in different branches as shown in Figure 5*a* and Figure 5*b*. The circumferences α and β are in the same branch in Figure 5*c*. In a intercepting set there is a neutral circumference whose center

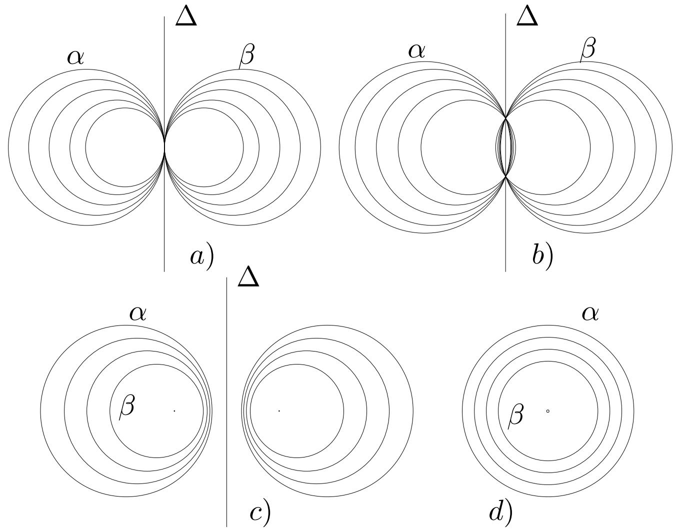

FIGURE 5. Relative positions of circumferences and the formed type sets.

belong to it radical line Δ. A coaxial set can be defined by one or two of their branches.

3.1.1. *The lowest power property.* The coaxial sets shown in Figure 6 can be identified by a property named Lowest Power Point expressed by a point named L_p. The existence of this point and its properties can be demonstrated by the theorem 3.1.

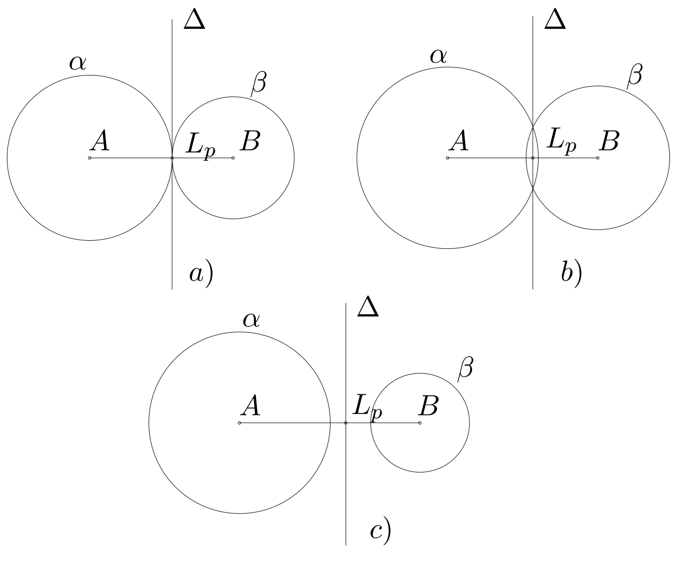

FIGURE 6. Lowest power point's of coaxial sets.

Theorem 3.1. *Given a coaxial set $\alpha\beta$ which centers are the points A and B respectively, there is a point, named L_p, that has the lowest power among* *all points of the radical axis* Δ*. This point results from the intersection of the straights* Δ *and* AB*.*

Proof. Let P the point where the straight AB intersects the radical axis Δ. The power of P relative to the set is negative if it's internal, positive if it's external to the circumferences and null if it's belong to circumferences. Let $Q \neq P$ another point in Δ. The line segment $\overline{QP}$ is the edge of a triangle which other side is the line segment $\overline{AP}$ or $\overline{BP}$. By the Pythagorean theorem, the hypotenuses $\overline{QA}$ and $\overline{QB}$ are larger than the edges $\overline{QP}$ or $\overline{AP}$ respectively. Thus, by the power definition $Pw_{(Q,\alpha\beta)} > Pw_{(P,\alpha\beta)}, \forall Q \in \Delta$.

Corrolary 3.1. *If* L_p *has a null, positive or negative power to* $\alpha\beta$ *then the coaxial set is tangent, not secant or secant respectively.*

3.1.2. *Power of once circumference center to another.* The lowest power property characterize coaxial sets but does not indicate whether the circumferences which define the set are in the same branch or not. To accurately identify the relative position between two circumferences this method considers the relative position of their centers.

When considering the relative positions of the two tangent circumferences as well the dimensions of their radii we find four possible situations illustrated in Figure 7 and described below.

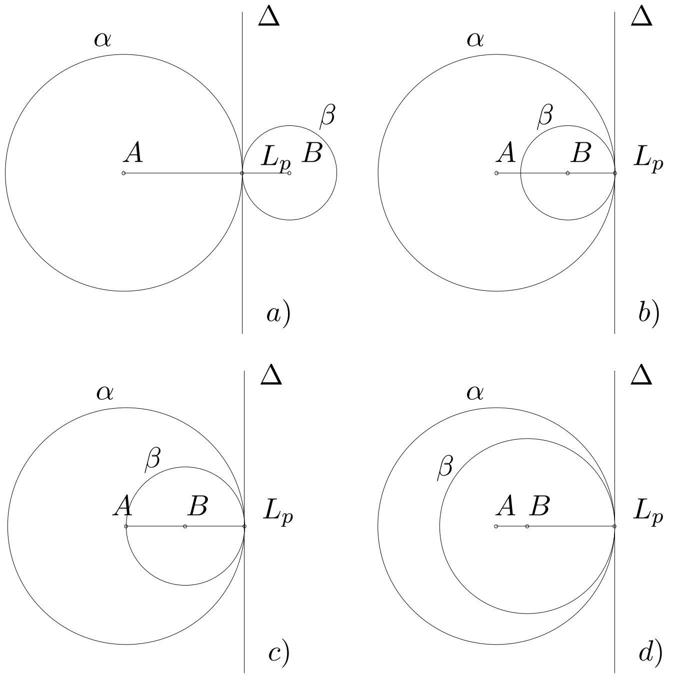

FIGURE 7. Tangent set: relative positions of two circumferences centers.

(1) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} > 0$. This indicates that α and β are in different branches and that A and B are external to β and α respectively. In this situation, they may have equal or different radii as shown in Figure 7a.

(2) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} < 0$. This indicates that α and β are in the same branch and that B is internal to α and A is external to β. In this situation, the radius of α is better than β as shown in Figure 7*b*.

(3) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} = 0$. This indicates that α and β are in the same branch and that A belongs to β and B is internal to α. In this situation, the α radius is equal to the diameter of β as shown in Figure 7*c*.

(4) $Pw_{(A;\beta)} < 0 \wedge Pw_{(B;\alpha)} < 0$. This indicates that α and β are in the same branch and that B is internal to α and A is internal to β. In this situation, the α radius is shorter than the diameter of β as shown in Figure 7*d*.

When considering the relative positions of the two secant circumferences as well the dimensions of their radii we find six possible situations illustrated in Figure 8 and described below.

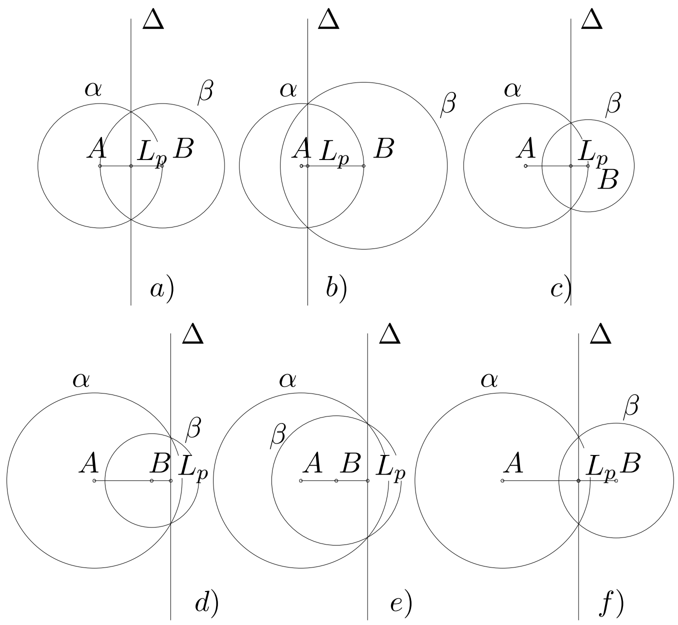

FIGURE 8. Secant set: relative positions of two circumferences centers.

(1) $Pw_{(A;\beta)} = 0 \wedge Pw_{(B;\alpha)} = 0$. This indicates that α and β are in different branches and that $B \in \alpha$ while $A \in \beta$. In this situation, the α and β radii are equal as shown in figure 8*a*.

(2) $Pw_{(A;\beta)} < 0 \wedge Pw_{(B;\alpha)} = 0$. This indicate that α and β are in different branches and that $B \in \alpha$ while A is internal to β. In this situation, the radius of α is shorter than β radius as shown in figure 8*b*.

(3) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} = 0$. This indicates that α and β are in different branches and that $B \in \alpha$ while A is external to β. In this situation, the radius of α is bigger than β radius as shown in figure 8*c*.

(4) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} < 0$. This indicates that α and β are in different branches and that B is internal to α while A is external to β. In this situation, the radius of α is bigger than β radius as shown in figure 8d.
(5) $Pw_{(A;\beta)} < 0 \wedge Pw_{(B;\alpha)} < 0$. This indicates that α and β are in the same or in different branches and that B is internal to α while A is internal to β. In the case where α and β are in the same branch, the radius of α is bigger than β radius. In the case where α and β are in different branches, the radii of the α and β are equal as shown in figure 8e.
(6) $Pw_{(A;\beta)} > 0 \wedge Pow_{(B;\alpha)} > 0$. This indicates that α and β are in different branches and that A and B are external to β and α respectively. In this situation, the circumferences α and β can have equal or different radii as shown in figure 8f.

When considering the relative positions of the two not secant circumferences as well the dimensions of their radii we find four possible situations illustrated in figure 9 and described below.

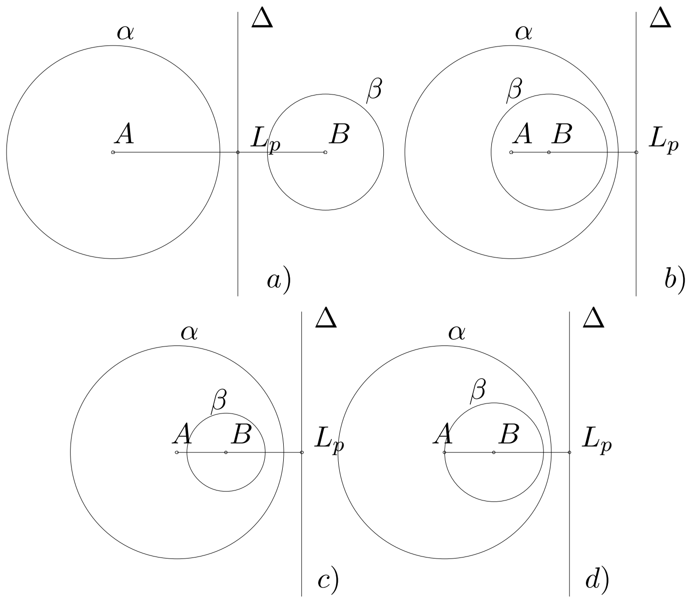

FIGURE 9. Not secant set: relative positions of two circumferences centers.

(1) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} > 0$. This indicates that α and β are in different branches and that B and A are external to α and β respectively. In this situation, α and β can have equal or different radii as shown in figure 9a.
(2) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} < 0$. This indicates that α and β are in different branches and that B is internal to α while A is external to β. In this situation, the radius of α is bigger than β radius as shown in figure 9b.

(3) $Pw_{(A;\beta)} > 0 \wedge Pw_{(B;\alpha)} = 0$. This indicates that α and β are in different branches and that and that$B \in \alpha$ while A is external to β. In this situation, the radius of α is necessarily bigger than β radius as shown in figure 9c.

(4) $Pw_{(A;\beta)} < 0 \wedge Pw_{(B;\alpha)} < 0$. This indicates that α and β are in the same branch and that B is internal to α while A is internal to β. In this situation, the radius of α is bigger than β radius as shown in figure 9d.

When considering the relative positions of the two circumferences centers in a radial set is possible the only case illustrated in figure 10. In this set, for any two circumferences α and β, $Pw_{(A;\beta)} < 0 \wedge Pow_{(B;\alpha)} < 0$. This indicate that B and A points are internal to α and β respectively but, this does not indicate that A and B are coincident.

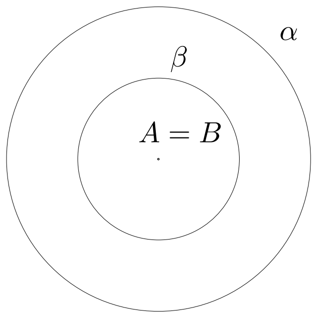

Figure 10. Radial set: relative positions of two circumferences centers.

3.2. **Three circumferences sets.** In this method, configuration is a set of three circumferences and, cases are its variations depending on the relative positions of their centers. The number of configurations is obtained by combining the coaxial sets and radial set.

The configurations are grouped into two main groups. The group sets for which the radical center is defined and the group sets for which the radical center is not defined. The group sets for wich the radical center is not defined is composed by two subgroups: configurations that have aligned centers and that have concentricities. The configurations that defines three parallel radicals axis have aligned centers of three circumferences. If the configurations have only one radical axis the circumferences belong to the same coaxial set. Similarly, the configurations that defines two parallel radical axis have one concentricity while those that do not define any radical axis have three concentricity.

The definition of radical center, R_c, used in this study is 3.4. The existence conditions for which the radical center is defined is shown in proposition 3.1.

Definition 3.4. *Given three or more circumferences, if there is a point which has the same power to them, it is called radical center.*

Proposition 3.1. *If three circumferences they do have not centers aligned or have concentricity their radical center is defined.*

Proof. Let Δ_1, Δ_2 and Δ_3 the radical axis of the sets $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$ respectively. Let O the point where Δ_1 and Δ_2 intersect. Let P the point where Δ_1 and Δ_3 intersect. Let Q the points where Δ_2 and Δ_3 intersect. According to definition 3.4, the point O has the same power for α, β and γ. The point P has the same power for α, β and γ. The point Q has the same power for α, β and γ. Then, $O = P = Q$.

3.2.1. *Defined radical center.* The configurations for which the radical center is defined are listed in Table 1. They are the result of combinations of not secant, secant and tangent pairs. The radical center of configurations 1 to 8 has a exclusive power. In configuration 9 there are cases where the power of the radical center is positive or null. In configuration 10 there are cases where the power of the radical center is positive, negative or null. These properties are stated in the propositions 3.2 to 3.6.

TABLE 1. Configurations with defined radical center.

Configuration	Not secant	Tangent	Secant	Concentric	Cases
1	3	0	0	0	20
2	2	1	0	0	40
3	2	0	1	0	60
4	1	2	0	0	40
5	1	0	2	0	84
6	1	1	1	0	96
7	0	3	0	0	20
8	0	2	1	0	60
9	0	1	2	0	84
10	0	0	3	0	56

Proposition 3.2. *Let a set of three circumferences formed by a secant pair, a tangent pair and not secant pair. If there is a radical center it has positive power.*

Proof. The radical axis in a secant pair has positive power in all points except in the common chord where it has a negative power and null power at intersection points. The radical axis of the tangent pair has positive power in all its points except at the tangency point where it is null. The radical axis of the not secant pair has positive power in all of its points. So,if there a radical center its have a positive power.

Corrolary 3.2. *Given three coaxial sets, if one of them is not secant, the power of the radical center, if it is defined, is positive.*

This corollary applies to configurations 1,2,3,4,5 of table 1.

Proposition 3.3. *Given three coaxial tangents sets the power of the radical center, if it is defined, is positive.*

Proof. The radical axis of the tangent sets has a null power at the tangency point and positive in the remaining points. This type pair have only one common point. Then,if it is defined, the power of the radical center is positive.

Proposition 3.4. *Given two tangent sets and one secant, the power of the radical center is positive.*

Proof. The radical axis of the tangent sets has a null power at the point of tangency and positive in the remaining points. The radical axis of the secant set has negative power in the common chord and has a null power at the intersection points and positive in the remaining points. As the three sets have no common area or point, if there is a radical center defined, its power is positive.

Proposition 3.5. *Given two secant sets and one tangent the power of the radical center, if it is defined, is positive or null to them.*

Proof. The radical axes of secant sets has a negative power in the common chord, has a null power at the intersection points and positive in the remaining points. The radical axis of tangent set has a null power at the intersection point and positive power in remaining points. In the first case, the sets have no common point. In this case, the radical center has a positive power. In another case, the three sets have a common point, the radical center. In this case, the radical center has a null power.

Proposition 3.6. *Given three secant sets, the power of the radical center, if it is defined, is positive, negative or null.*

Proof. The radical axis of secant sets has a negative power in the common chord, has a null power at the intersection points and positive in the remaining points. In the first and second possibilities, the sets share a common area. In the first situation, the chords intersect within in the common area. In this case, the power of the radical center is negative for the sets. In the second situation, the radical axis intersects in a border of the common area, the radical center. In this case, the power of the radical center is null. In the third case, the sets do not share a common area and the radical axes intersects at a common point. In this case, the power of the radical center is positive.

The propositions 3.2 to 3.6 explain the power of radical center in all cases where it is defined. The positive power of the radical center occurs in ten configurations and in eight exclusively. There are two configurations in which occur null power and in only one case where power of the radical center is negative.

Configurations 1 to 8 have an exclusive power. Any of these configurations have a single type of power for the radical center if it is defined. The configuration 9 has two possibilities for the power of its radical center: null or positive and configuration 10 have three possibilities: positive, negative or null. In the case where $Pw_{(rc)} < 0$, the three sets are secant and there is only one possibility to a topological arrangement.

In the case where $Pw_{(rc)} = 0$, there are two possibilities for topological arrangement. In the first case, the three sets are secant. In the second case, two sets are tangent and one is secant. In these two possibilities they have a common point. In the case where $Pw_{(rc)} > 0$, the three sets have no common area or point. The Table 2 shows the percentage of configurations according to the power of the radical center.

Table 2. Number of configurations according to radical center power.

Radical center power	Number of config.	Percentage
$Pw_{(rc)} < 0$	1/3	3,3 %
$Pw_{(rc)} = 0$	(1/2)+(1/3)	8,3 %
$Pw_{(rc)} > 0$	8+(1/2)+(1/3)	88,3 %

The data in Table 2 means that the sets of three circumferences that have positive power are much more abundant than other types. The numbers 1/2 and 1/3 means half part and the third part of cases in a configuration.

3.2.2. *The relative power between coaxial sets.* In a set of three circumferences one coaxial set can have its relative position to another described by the lowest power point. For example, let a configuration 2 and $Lp_{(\alpha\beta)}$ the lowest power point of coaxial set $\alpha\beta$. The position of this set relative to γ circumference is described by the $Lp_{(\alpha\beta)}\gamma$ parameter as shown in Figure 11. In Figure 11*a*, $Lp_{(\alpha\beta)}\gamma$ has a positive power and in Figure 11*b* it has a negative power. Considering the circumferences α, β and γ one can have parameters $Lp_{(\alpha\beta)}\gamma$, $Lp_{(\alpha\gamma)}\beta$ and $Lp_{(\beta\gamma)}\alpha$. These parameters are considered secondary.

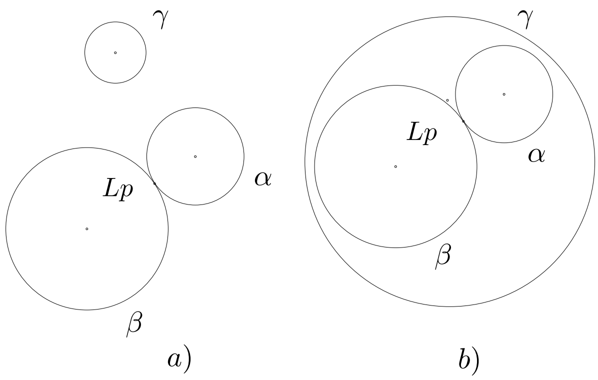

Figure 11. Relative power between a coaxial set and the another circumference: *a*) positive power and *b*) negative power.

3.2.3. *Counting cases.* The count of cases in each configuration listed in the tables 1 and 3 were made by combining, in a appropriate way, the topological types of sets listed in Figures 7, 8, 9.

The counting process for the configurations formed by three coaxial sets of the same type, which have 4 topological types is obtained by combining $C = (p + n - 1, n)$ where $p = 4$ and $n = 3$. Then, configurations 1,7,11 and 17 have $C = 20$ cases.

The counting process for the configurations formed by two equal coaxial sets having 4 topological types and one coaxial set with 4 topological types is obtained by combining $C = (p + n - 1, n) \times 4$ where $p = 4$ and $n = 2$. Then, configurations 2,4,12 and 14 have $C = 40$ cases.

The counting process for the configurations formed by three coaxial sets sets of the same type, which have 6 topological types is obtained by combining $C = (p + n - 1, n)$ where $p = 6$ and $n = 3$. Then, configurations 10 and 20 have $C = 56$ cases.

The counting process for the configurations formed by two equal coaxial sets sets having 4 topological types and one coaxial set with 6 topological types is obtained by combining $C = (p + n - 1, n) \times 6$ where $p = 4$ and $n = 2$. Then, configurations 3,8,13 and 18 have $C = 60$ cases.

The counting process for the configurations formed by two equal coaxial sets having 6 topological types and one coaxial set with 4 topological types is obtained by combining $C = (p + n - 1, n) \times 4$ where $p = 6$ and $n = 2$. Then, configurations 5,9,15 and 19 have $C = 84$ cases.

The counting process for the configurations formed by three different coaxial sets is obtained multiplying the number of topological cases in each one. Then, configurations 6 and 16 have $C = 4 \times 4 \times 6 = 96$ cases.

3.2.4. *Undefined radical center.* The configurations for which the radical center is not defined have the circumferences with aligned centers or have at least one concentricity. In table 3 are listed the configurations with aligned centers. The number of cases of these configurations is the same as those listed in Table 1. In table 4 are listed the configurations with concentricities. The number of cases of the configurations with concentricity was obtained by exhaustive enumeration.

Table 3. Configurations with aligned centers.

Configuration	Not secant	Tangent	Secant	Concentric	Cases
11	3	0	0	0	20
12	2	1	0	0	40
13	2	0	1	0	60
14	1	2	0	0	40
15	1	0	2	0	84
16	1	1	1	0	96
17	0	3	0	0	20
18	0	2	1	0	60
19	0	1	2	0	84
20	0	0	3	0	56

4. IMPLEMENTATION OF METHOD

In this method, each set of three circumferences is identified by a vector V composed of ten parameters according to the expression 4.

$$V = [P_{(A,\beta)}, P_{(B,\alpha)}, P_{(\alpha\beta)}, P_{(A,\gamma)}, P_{(C,\alpha)}, P_{(\alpha\gamma)}, P_{(B,\gamma)}, P_{(C,\beta)}, P_{(\beta\gamma)}, P_{(rc)}] \quad (4)$$

TABLE 4. Configurations with concentricities.

Configuration	Not secant	Tangent	Secant	Concentric	Cases
21	0	0	0	3	1
22	0	0	2	1	12
23	0	2	0	1	2
24	2	0	0	1	6
25	0	1	1	1	14
26	1	0	1	1	6
27	1	1	0	1	8

The parameters $P_{(A,\beta)}, P_{(B,\alpha)}, P_{(A,\gamma)}, P_{(C,\alpha)}, P_{(B,\gamma)}$ and $P_{(C,\beta)}$ can take positive, negative and null values. In vector V, to indicate these conditions are assigned values 1,2 and 3 respectively. The parameters $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ and $Lp_{(\beta\gamma)}$ can take positive, negative, null and undefined values. In vector V, to indicate these conditions are assigned values 1,2,3 and 4 respectively. The parameter $Pw_{(rc)}$ can take positive, negative, null and undefined values. Also, to indicate these conditions in vector V are assigned values 1,2,3 and 4 respectively.

4.1. **Calculations of vector V parameters.** The calculations of parameter values $P_{(A,\beta)}, P_{(B,\alpha)}, P_{(A,\gamma)}, P_{(C,\alpha)}, P_{(B,\gamma)}$ and $P_{(C,\beta)}$ are made by substituting the values of the coordinates of the center of a circumference in the equation of another. For example, checking the condition of A relative to β it is determinated by substituting its coordinates in the equation 2.

The parameter values $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ and $Lp_{(\beta\gamma)}$ substituting the Lp coordinates in one equation of circumference set. For example, the power of $Lp_{(\alpha\beta)}$ parameter is determinated by substituting its coordinates in the equation 1 or 2. The coordinates of Lp are determined by the intersections of the radical axis of the sets $\alpha\beta$, $\alpha\gamma$ and $\beta\gamma$ with the the straights AB, AC and BC respectively.

The parameter value $Pw_{(rc)}$ is determined by the intersections of any of two radical axis of the sets. For example, $\alpha\beta$ and $\alpha\gamma$, $\alpha\beta$ and $\beta\gamma$ or $\alpha\gamma$ and $\beta\gamma$. The equations of the radicals axes are obtained by subtracting equations of two circumferences. For example from $\alpha\beta$ $(2-1)$, to $\alpha\gamma$ $(3-1)$ and to $\beta\gamma$ $(3-2)$ [4].

4.2. **Algorithm flowchart.** This algorithm is quite simple. First, are read the coordinates of the centers and radii of three circumferences and begins the calculations of the parameters vector V. Then, the parameters for the first two read circumferences $P_{(A,\beta)}$, $P_{(B,\alpha)}$ and $Lp_{(\alpha\beta)}$ are calculated. The calculation of the parameters $P_{(A,\beta)}$, $P_{(B,\alpha)}$ does not depend on none prior verification but $Lp_{(\alpha\beta)}$ is defined only if A and B are not coincidents. This part of algorithm flowchart is shown in Figure 12.

In the second part are calculated parameters $P_{(A,\gamma)}$, $P_{(C,\alpha)}$ and $Lp_{(\alpha\gamma)}$. These calculations does not depend on none prior verification but $Lp_{(\alpha\gamma)}$ is defined only if A and C are not coincidents. If $Lp_{(\alpha\gamma)}$ and $Lp_{(\alpha\beta)}$ are not defined the program is terminated because α, β and γ are concentric. Of course, $Pw_{(rc)}$ is not defined.

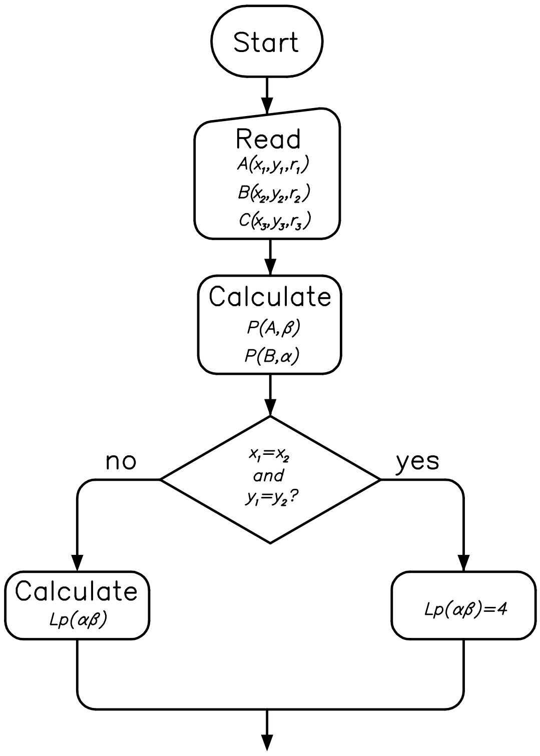

FIGURE 12. Algorithm flowchart:first part.

At least, the parameters $P_{(B,\gamma)}$, $P_{(C,\beta)}$ and $P_{(\beta\gamma)}$ are calculated. These calculations does not depend on none prior verification but $Lp_{(\beta\gamma)}$ is defined only if B and C are not coincidents. If none of two $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ and $Lp_{(\beta\gamma)}$ are defined and these are not parallel the parameter $Pw_{(rc)}$ is defined. This third part of the flowchart algorithm is shown in Figure 13.

This algorithm was implemented in C language and it was conducted a series of tests to characterize the processing time of sets formed by three circumferences. The obtained times reflects the complexity of the calculations carried out for each one of the configurations according to the algorithm flowcharts.

4.3. **Applications of the proposed method.** In this section is exemplified the uses of vector V from the calculation the topology of the smallest circumference tangent to the three given in some configurations.

A full analysis of the **configuration 7** shows that the smallest tangent circumference is external to the α,β and γ or is external of two of them and internal to the other. In this configuration the radical center, $Pw_{(rc)}$, has a positive power and lowest powers $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ or $Lp_{(\beta\gamma)}$ has a null power. Hence, the decision criterion depends on the values of $P_{(A,\beta)}$, $P_{(B,\alpha)}$, $P_{(A,\gamma)}$, $P_{(C,\alpha)}$, $P_{(B,\gamma)}$ and $P_{(C,\beta)}$.

Then, if the sum of these variables values is "6", the smallest circumference tangent is determined by the condition $(+r_1)$, $(+r_2)$ and $(+r_3)$ in the equations system 1, 2 and 3. In another situation, at least one of the these

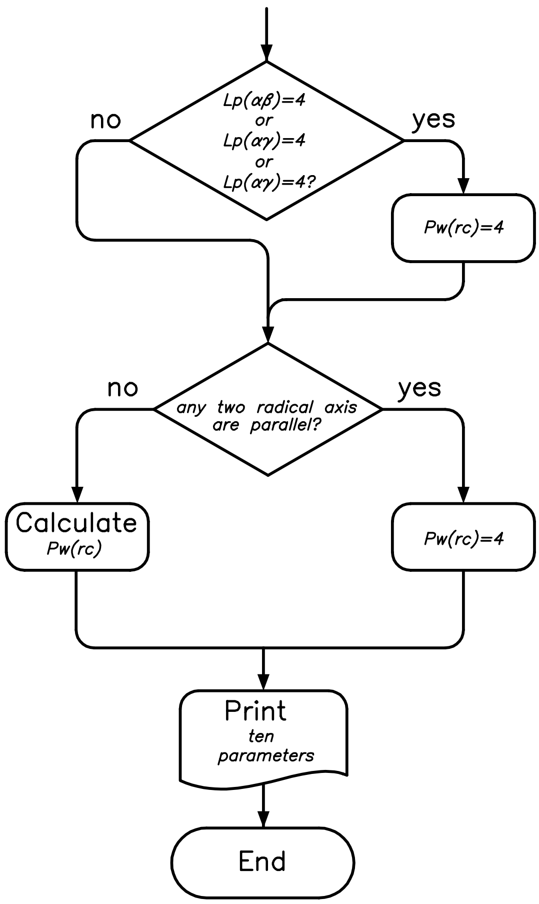

FIGURE 13. Algorithm flowchart:second part.

variables, for example $P_{(C,\alpha)}$, has the value "2" indicating that this circumference radius should decrease. Another way to check this is to consider the secondary parameters $Lp_{(\alpha\beta)}\gamma$, $Lp_{(\alpha\gamma)}\beta$ and $Lp_{(\beta\gamma)}\alpha$ as shown in section 3.2.2. If one of these, for example $Lp_{(\alpha\beta)}\gamma$, has value "2", this indicates that this circumference radius, γ, should decrease. Thus, the algorithm description of this procedure can be as follows:

`if` (**Sum** $P_{(i,j)} = 6$);
    r_1=positive, r_2=positive, r_3=positive,
  `else if` ($P_{(A,\beta)} = 2$ or $P_{(A,\gamma)} = 2$);
    r_1=negative, r_2=positive, r_3=positive,
  `else if` ($P_{(B,\alpha)} = 2$ or $P_{(B,\gamma)} = 2$);
    r_1=positive, r_2=negative, r_3=positive,
  `else`;
    r_1=positive, r_2=positive, r_3=negative.

A full analysis of **configuration 8** shows that the smallest tangent circumference is external to the α,β and γ or is external of two of them and internal to the other. In this configuration the radical center, $Pw_{(rc)}$, has a positive power and two lowest powers $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ or $Lp_{(\beta\gamma)}$ are null and one of them is negative. Again, the decision criterion depends on the values of $P_{(A,\beta)}$, $P_{(B,\alpha)}$, $P_{(A,\gamma)}$, $P_{(C,\alpha)}$, $P_{(B,\gamma)}$ and $P_{(C,\beta)}$. Thus, the algorithm description of this procedure can be as follows:

```
if (Sum P(i,j) = 6);
    r1=positive, r2=positive, r3=positive,
  else if (P(B,α) = 2 or P(C,α) = 2);
    r1=negative, r2=positive, r3=positive,
  else if (P(A,β) = 2 or P(C,β) = 2);
    r1=positive, r2=negative, r3=positive,
  else;
    r1=positive, r2=positive, r3=negative.
```

Another way to do this verification is using the secondary parameters $Lp_{(\alpha\beta)}\gamma$, $Lp_{(\alpha\gamma)}\beta$ and $Lp_{(\beta\gamma)}\alpha$. Thus, the algorithm description of this procedure can be as follows:

```
if (Lp(αβ)γ = 2);
    r1=positive, r2=positive, r3=negative,
  else if (Lp(αγ)β = 2);
    r1=positive, r2=negative, r3=positive,
  else if (Lp(βγ)α = 2);
    r1=negative, r2=positive, r3=positive,
  else;
    r1=positive, r2=positive, r3=positive.
```

A full analysis of the **configuration 9** shows that the power of radical center, $Pw_{(rc)}$, is positive or null. In both cases, two lowest powers $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ or $Lp_{(\beta\gamma)}$ are negative and the other is null. Again, the decision criterion depends on the values of $P_{(A,\beta)}$, $P_{(B,\alpha)}$, $P_{(A,\gamma)}$, $P_{(C,\alpha)}$, $P_{(B,\gamma)}$ and $P_{(C,\beta)}$.

The smallest tangent circumference is outside of two circumferences in a tangent set if they belong to different branches or is internal to one of them if they belong to the same branch. Now, suppose that the tangent set is $\alpha\beta$. It is easy to see that the decision criterion depends only on the parameters $P_{(A,\beta)}$, $P_{(B,\alpha)}$ and $Lp_{(\alpha\beta)}\gamma$. Thus, the smallest tangent circumference can be find as follows:

```
if (Sum P(ij) =!2) and (r1 > r2) and (Lp(αβ)γ =!1);
    r1=positive, r2=negative, r3=negative,
  else if (Sum P(ij) =!2) and (r1 < r2) and (Lp(αβ)γ =!1);
    r1=negative, r2=positive, r3=negative,
  else if (Sum P(ij) =!2) and (r1 > r2)and (Lp(αβ)γ = 1);
    r1=positive, r2=negative, r3=positive,
  else if (Sum P(ij) =!2) and (r1 < r2)and (Lp(αβ)γ = 1);
    r1=negative, r2=positive, r3=positive,
  else if (Sum P(ij) =!2) and (Lp(αβ)γ =!1);
    r1=positive, r2=positive, r3=negative,
  else;
    r1=positive, r2=positive, r3=positive.
```

In the **configuration 10** the power of radical center $Pw_{(rc)}$ is positive, negative or null and the lowest powers points $Lp_{(\alpha\beta)}$, $Lp_{(\alpha\gamma)}$ and $Lp_{(\beta\gamma)}$ have a negative power to its circumferences.

In the first situation the power of radical center is positive and a full analysis of this shows that are only one possibility for the smallest tangent circumference. This case of this configuration has eight solutions but, the smallest is always external to the three given.

In the second situation the power of radical center is negative and a full analysis of this shows that are three possibilities for the smallest tangent circumference. It can be internal to the three given; it can be internal of the two of them and external to the one and it can be external of the two of them and internal to the one. This cases of this configuration has eight solutions and the smallest can be in seven different positions.

In the third situation the power of radical center is null and a full analysis of this situation shows that are two possibilities for the smallest tangent circumference. It can be internal to the two circumferences and external to one of them or it can be external to the two circumferences and internal to one of them. This cases of this configuration has four solutions and the smallest can be in three different positions.

The algorithmic procedures for the latter two cases of the configuration 9 may be constructed in a analogous way to the examples given for the configurations 7 and 8. The configurations from 1 to 6 have at least one not-secant set and a full analysis of them shows that is necessary more than topology and geometry analysis to deduce the smallest tangent circumference topology. The exceptions is the cases where a not-secant set has circumferences in the same branch. In these situations, the smallest tangent circumference are internal to one of them and internal or external to another's circumferences and in this condition is not difficult to determine.

5. FINAL REMARKS

A method shown here can correctly identify any set of three circumferences by the vector V. This vector is constructed by calculating the relative positions of the centers of the circumferences, the powers of L_p points and at last, the power of radical center if the radicals axis are defined or are not parallels. As one can see, its theoretical basement is the power concept but in the calculating of vector V are used only operations such as checks the relative positions between points and circumferences, parallelism between lines and calculations of intersections between them.

In the method, any set of three circumferences correspond to one of the 1331 identified cases and the only situation in which is needed additional information to a vector V to distinguish cases occurs in a subgroup that has the aligned centers. Is this subgroup, we need to know if the three tangent, not secant or secant circumferences belongs to a same coaxial set or not. To know this, is just necessary verify if the any two L_p points are coincident or not.

The vector V generated in the method was used to identify the topology of the smallest tangent circumference in some configurations. Therefore, it cannot be applied in all of them and this indicates that more studies on the geometry and topology of this type set are necessary. In a general way, the main advantage of the method is it conceptual simplicity. Therefore, its brings new light to a classification of sets of three circumferences and induces new questions about selection sets in largest cluster of circumferences.

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