INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 4 (2015), No. 2, 26 - 46

CALABI TRIANGLES FOR REGULAR POLYGONS

MICHAEL MCASEY and LIBIN MOU

Abstract. Just as with the in-circle for a given triangle, there are maximal inscribed regular polygons in a triangle. Rather than tangent to a side of the triangle, a side of the triangle might contain either a side of the polygon or just a vertex. This paper asks when the maximal polygon can be inscribed in different ways. Jerrard and Wetzel found a non-equilateral triangle that has a unique inscribed equilateral triangle. Prior to this Calabi found a non-equilateral triangle that has a unique inscribed square. In both these cases these triangles, called here Calabi triangles, are unique. In this paper we show that there are many such Calabi triangles and show that their number grows as the number of sides of the regular polygon is grows. A necessary and sufficient condition is given for an isosceles triangle to have the Calabi property.

1. Introduction

Let ΔABC be a triangle with sides of lengths a, b, c opposite the angles A, B, C. Consider the problem of finding the largest regular polygon, with a fixed number of sides, inscribed in ΔABC. That is, the polygon lies inside the triangle and touches all three sides of the triangle. Sullivan [8] has shown that this polygon lying in ΔABC can always be realized so that one of its sides lies along one of the sides of the triangle.

One approach to solving the problem of finding the largest inscribed polygon would be to solve the problem three times – finding the largest polygon with side lying, in turn, on each edge of the surrounding triangle and then taking the largest of these three polygons. Sometimes these three 'maximal' polygons will all be different and sometimes not. We are interested in the 'sometimes not' case.

Keywords and phrases: Triangle, Calabi Triangle, Maximal Inscribed Polygon, Maximality
(2010)Mathematics Subject Classification: 51M04, 51M16, 51M25
Received: 1.01.2015. In revised form: 13.06.2015. Accepted: 10.09.2015.

For fixed n, consider all the regular n-gons contained in ΔABC with one side lying on BC, that is, on the side with length a. Denote by r_a the maximum radius of such regular n-gons. It is possible, and even likely, that the maximum radius of the regular n-gon with side lying on AB is different from the radius with side lying on BC which in turn is different from the radius with side lying on AC. Clearly, in the isosceles case when $a = b$, then $r_a = r_b$. And if ΔABC is equilateral, then automatically $r_a = r_b = r_c$. In the equilateral case, because of the symmetry, the three maximal inscribed polygons are all the same. Does this ever happen in the non-equilateral case?

If $n = 3$, Jerrard and Wetzel [3] proved that there is a unique non-equilateral triangle ΔABC so that the maximal inscribed equilateral triangle can be inscribed in different ways. Since the maximal equilateral triangle can be inscribed so that one of its sides lies along one side of the surrounding triangle, this equilateral triangle can be inscribed in three different ways. Because the surrounding triangle found by Jerrard and Wetzel is in fact isosceles, two of these ways are symmetric, of course. Prior to this result, in the case $n = 4$, Calabi posed as a problem and proved that there is a unique non-equilateral triangle that has a maximal inscribed square that can be inscribed in different ways. In both the $n = 3$ and $n = 4$ cases, the surrounding triangles are isosceles and unique (up to scaling). See [9] for details.

We will call a non-equilateral triangle ΔABC a **Calabi triangle** for regular n-gons if $r_a = r_b = r_c$. Thus a Calabi triangle has the property that the maximal inscribed n-gon can be inscribed in possibly different ways in the given triangle. (See Figure 1.) We have explicitly excluded equilateral triangles here because by their symmetry, we always have $r_a = r_b = r_c$ and so is an uninteresting case.

In view of the results of Calabi and Jerrard and Wetzel there are some natural questions about triangles and their maximal inscribed regular n-gons.

(1) Does a Calabi triangle exist for all $n \geq 5$? If yes, how many are there?
(2) Is every Calabi triangle isosceles?

We will prove in Sections 2-5 the following results with regular n-sided polygons.

(a) For each $n \geq 5$, there are multiple isosceles Calabi triangles and the number of isosceles Calabi triangles approaches ∞ as $n \to \infty$.

(b) There is a unique scalene Calabi triangle for $n = 5$; the number of scalene Calabi triangles also approaches ∞ as $n \to \infty$.

(c) If n is odd, there is no isosceles Calabi triangle whose "repeated" angle is smaller than π/n.

(d) If n is even, there is always an isosceles Calabi triangle whose repeated angle is smaller than π/n.

In Section 6, we will pose several unsolved problems for interested readers.

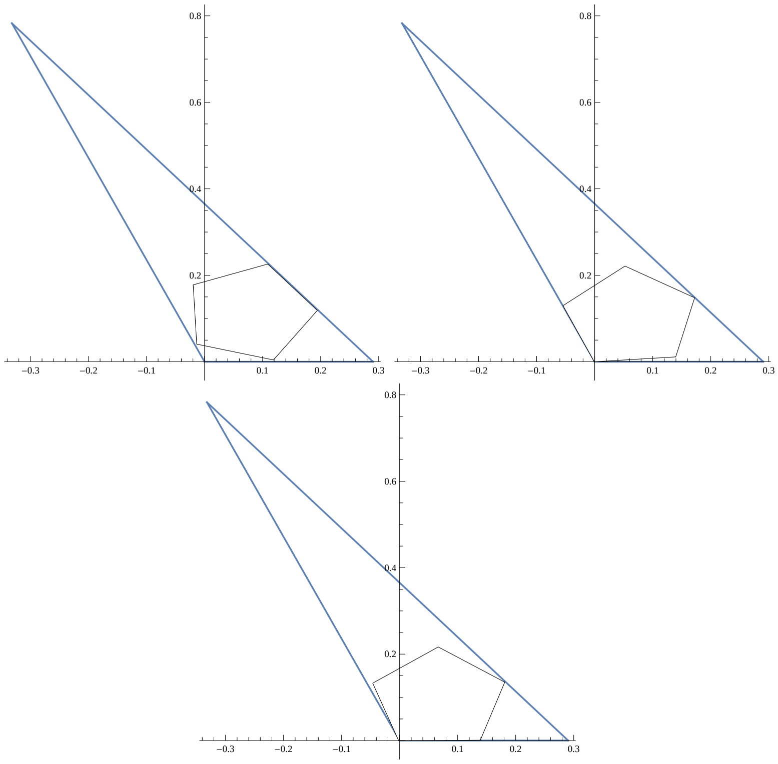

FIGURE 1. Calabi triangle for a regular pentagon

2. Calabi Triangles With Special Angles

A regular polygon with n sides can be subdivided into n isosceles triangles by connecting the vertices of the polygon to the center. The angle in any of these smaller triangles at the center of the polygon is $2\pi/n$ and the two other angles are each $\pi/2 - \pi/n$. Let $\beta = \pi/n$ so the three angles are 2β, $\pi/2 - \beta$ and $\pi/2 - \beta$. Suppose that an inscribed regular polygon has two of its sides along the sides of a triangle. Then the measure of the enclosed angle in the triangle must have a special form, as shown in the next proposition.

Proposition 2.1. *Suppose that a polygon lying inside a triangle ΔABC has one of its sides along AB and another side along AC. Then the angle A must have the form $\pi - 2k\beta$, for some k with $0 < k < n/2$.*

Proof. Let D denote the center of the polygon P and M the midpoint of the side of the polygon contained in the side AB of the triangle. Consider the triangle ΔADM. The angle $\angle DMA$ is $\pi/2$ and the angle $\angle MAD$ is $A/2$. The angle $\angle ADM$ cuts off a "polygonal arc" of P consisting of a number of sides and half-sides of the polygon. (See Figure 2 for an example.) Each half-side of the polygon has central angle β so angle $\angle MDA = k\beta$ for some integer k. The sum of the angles in triangle ΔADM is thus $\pi/2 + A/2 + k\beta = \pi$. From this it follows that the angle A in the original triangle ΔABC is $\pi - 2k\beta$. Note that $A > 0$ only in case $0 < k < n/2$.

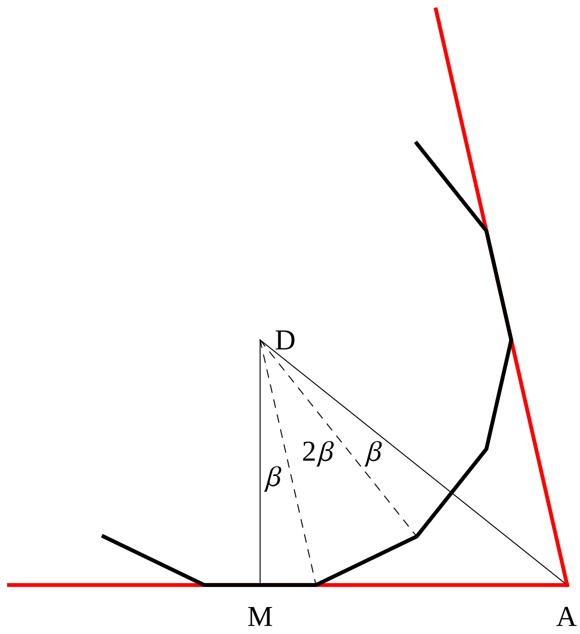

FIGURE 2. Special Angle

We call an angle of the form $\pi - 2k\beta$ a *type 1 angle*. Since $\pi = n\beta$, a type 1 angle is always a multiple of β. We say ΔABC is a *type 1 triangle* in case the three angles A, B, C are all type 1. We will assume that the angles have the values in the special progression

$$A = \pi - 2k\beta, \quad B = \pi - 2l\beta, \quad C = \pi - 2m\beta \tag{1}$$

for integers k, l and m with the following properties:

(a) $0 < m \leq l \leq k < \frac{n}{2}$;
(b) k, l, m are not all the same;
(c) $k + l + m = n$.

In this section we show that for large n, there are many type 1 Calabi triangles. These Calabi triangles are formed by extending three sides of the n-gon (if these extensions form a triangle). See Figure 3. Given a regular n-gon, if a (non-equilateral) triangle is formed in this way, then it must be a Calabi triangle. To see this observe that each side of the triangle contains one of the sides of the polygon. So the n-gon is the only inscribed n-gon for that triangle having the property that each side of the triangle contains one of the sides of the polygon. Therefore, $r_a = r_b = r_c$. In other words, a triangle with angles satisfying (1) and (a)-(c) must be a Calabi triangle. We will call it a **type 1 Calabi triangle**. We first count Calabi triangles that are also isosceles. Let $\lfloor x \rfloor$ denote the floor function (or greatest integer function) so that $\lfloor x \rfloor = m$ where m is the integer satisfying $m \leq x < m + 1$. The next theorem says that type 1 Calabi triangles exist and are numerous if n is large. The second part of the theorem counts more general type 1 Calabi triangles.

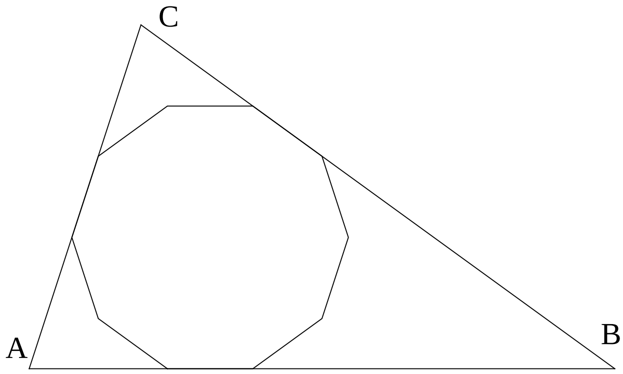

FIGURE 3. Polygon Inscribed in a Triangle with Type 1 Angles

Theorem 2.1. *Let $n \geq 5$ be fixed and consider the collection of triangles with inscribed n-gons. Let $IC1(n)$ be the number of isosceles Calabi triangles with type 1 angles. Let $C1(n)$ be the number of Calabi triangles with type 1 angles (isosceles or not).*

(1) *Then* $IC1(n) \geq \lfloor \frac{n}{4} \rfloor - 1$.
(2) *Let*

$$C^*(n) = \sum_{\frac{n}{3} \leq k < \frac{n}{2}} \left(\left\lfloor \frac{3k-n}{2} \right\rfloor + 1 \right).$$

$$\textit{Then } C1(n) = \begin{cases} C^*(n), & n \text{ not a multiple of } 3 \\ C^*(n) - 1, & n \text{ a multiple of } 3 \end{cases}$$

Proof. (of part 1) Since we wish to count only isosceles Calabi triangles, we consider triples (m, l, k) satisfying (a)-(c) above but with either $m = l$ or $l = k$, but not both. Consider two cases.

(1) If $l = k$ (i.e. $A = B$), then $m = n - 2l$ and the conditions (a)-(c) are reduced to $\frac{n}{3} < l < \frac{n}{2}$.

(2) If $l = m$ (i.e. $B = C$), then $k = n - 2l$ and the conditions (a)-(c) are reduced to $\frac{n}{4} < l < \frac{n}{3}$.

Therefore, $IC1(n)$ is the number of integers $l \neq \frac{n}{3}$ in the interval $\left(\frac{n}{4}, \frac{n}{2}\right)$. If $n \geq 13$, then the length of the interval $\left(\frac{n}{4}, \frac{n}{2}\right)$ is $\frac{n}{4} > 3$. So the interval $\left(\frac{n}{4}, \frac{n}{2}\right)$ contains at least $\lfloor \frac{n}{4} \rfloor$ integers. Therefore, $IC1(n) \geq \lfloor \frac{n}{4} \rfloor - 1$, which approaches ∞ as $n \to \infty$.

Proof of part 2. We need to count the number of triples (k, l, m) such that

(1) $0 < m \leq l \leq k < \frac{n}{2}$
(2) $m + l + k = n$.

First note that as above $\frac{n}{3} \leq k < \frac{n}{2}$. For each such k, find how many pairs (m, l) satisfy the conditions (1) and (2). From condition (2) we get $l = n - k - m$. The condition $l \leq k$ implies that $m \geq n - 2k$. And the condition $0 < m \leq l$ requires that $m \leq \frac{n-k}{2}$. Therefore, m must satisfy

$$n - 2k \leq m \leq \frac{n-k}{2}. \tag{2}$$

The number of possible m satisfying this inequality is $\lfloor \frac{n-k}{2} \rfloor - (n-2k) + 1 = \lfloor \frac{3k-n}{2} \rfloor + 1$. Each such m leads to a triple $(m, l, k) = (m, n-k-m, k)$ that satisfies the equation and the conditions (1) and (2). So the total number of solutions is

$$C^*(n) = \sum_{\frac{n}{3} \leq k < \frac{n}{2}} \left(\left\lfloor \frac{3k-n}{2} \right\rfloor + 1 \right)$$

If n is not multiple of 3, then $C^*(n)$ is the same as $C1(n)$. In case n is a multiple of 3, then $C^*(n)$ also counts the equilateral triangle which, by definition, is not a Calabi triangle so $C1(n) = C^*(n) - 1$ in this case.

Examples of isosceles Calabi triangles are given in Table 1; the entries for cases $n = 3, 4$ are included to indicate that not all Calabi triangles have type 1 angles as already seen in the work of Jerrard and Wetzel [3] and Calabi as reported on p. 206 in [1].

n	$IC1(n)$	m, l, k	A, B, C
3	0		
4	0		
5	1	$1, 2, 2$	$\frac{3\pi}{5}, \frac{\pi}{5}, \frac{\pi}{5}$
6	0		
7	2	$2, 2, 3$; $1, 3, 3$	$\frac{3\pi}{7}, \frac{3\pi}{7}, \frac{\pi}{7}$; $\frac{5\pi}{7}, \frac{\pi}{7}, \frac{\pi}{7}$
8	1	$2, 3, 3$	$\frac{4\pi}{8}, \frac{2\pi}{8}, \frac{2\pi}{8}$
9	1	$1, 4, 4$	$\frac{7\pi}{9}, \frac{\pi}{9}, \frac{\pi}{9}$
10	2	$3, 3, 4; 2, 4, 4$	$\frac{4\pi}{10}, \frac{4\pi}{10}, \frac{2\pi}{10}$; $\frac{6\pi}{10}, \frac{2\pi}{10}, \frac{2\pi}{10}$
11	3	$3, 3, 5$; $3, 4, 4$; $1, 5, 5$	$\frac{5\pi}{11}, \frac{5\pi}{11}, \frac{\pi}{11}$; $\frac{5\pi}{11}, \frac{3\pi}{11}, \frac{3\pi}{11}$; $\frac{9\pi}{11}, \frac{\pi}{11}, \frac{\pi}{11}$
12	1	$2, 5, 5$	$\frac{8\pi}{12}, \frac{2\pi}{12}, \frac{2\pi}{12}$

TABLE 1. List of isosceles Calabi triangles with type 1 angles

The number of scalene Calabi triangles with angles of type 1 also becomes large as the number of sides of the inscribed polygon gets large. Let $SC1(n)$ denote the number of scalene Calabi triangles with angles of type 1 with inscribed regular n-gons. The notation $\lceil x \rceil$ denotes the ceiling function; that is, $\lceil x \rceil = m$ where m is the integer such that $m - 1 < x \leq m$.

Theorem 2.2. *For each $n \geq 3$, the number $SC1(n)$ of scalene Calabi triangles with angles of type 1 is*

$$SC1(n) = \sum_{\frac{n}{3} < k < \frac{n}{2}} \left(\left\lceil \frac{3k-n}{2} \right\rceil - 1 \right).$$

Proof. We count the number of triples (k, l, m) such that

(1) $0 < m < l < k < \frac{n}{2}$

(2) $m + l + k = n$.

Note first we have

$$\frac{n}{3} < k < \frac{n}{2}.$$

Following nearly the same proof as in Theorem 2.1 (part 2), the replacement for inequality (2) is

$$n - 2k < m < \frac{n-k}{2}. \tag{3}$$

The number of possible m satisfying this inequality is $\lceil \frac{n-k}{2} \rceil - 1 - (n-2k) = \lceil \frac{3k-n}{2} \rceil - 1$. Each such m leads to a triple $(m, l, k) = (m, n-k-m, k)$ that satisfies the equation and the conditions (1) and (2). So the total is

$$SC1(n) = \sum_{\frac{n}{3} < k < \frac{n}{2}} \left(\left\lceil \frac{3k-n}{2} \right\rceil - 1 \right).$$

n	SC1(n)
$n \le 8$	0
9	1
10	0
11	1
12	1

Table 2. Numbers of scalene Calabi triangles with type 1 angles

Tables 1 and 2 seem to suggest that the numbers of isosceles type 1 Calabi triangles is about the same as the number of scalene type 1 Calabi triangles when in fact there are many more of the latter for large n. As an example $IC1(50) = 12$ while $SC1(50) = 40$.

3. A Necessary And Sufficient Condition For An Isosceles Calabi Triangle And Some Applications

In addition to some examples, in this section we prove a necessary and sufficient condition for an isosceles triangle ΔABC to be a Calabi triangle without regard to the classification of type 1 angles. We use this condition to show that for any odd integer $n \ge 3$ that there are no isosceles Calabi triangles with small middle angle B; i.e, $B \le \beta = \pi/n$. In contrast to this, for even integers n, with $n \ge 4$, we will see that there is always a Calabi triangle with $B < \beta = \pi/n$.

First, a formula is given for the radius, r_a, of an inscribed polygon with one side lying on the side of the triangle with length a opposite the angle A. Its proof is a somewhat lengthy computation, included in the Appendix for the convenience of the reader. The formula is

$$r_a = \frac{a}{f(B) + f(C)} \tag{4}$$

where $f(x)$ is the function defined for $x \in (0, \pi)$ by

$$f(x) = \sin\beta + 2\sin\left(\left\lceil \frac{\pi - x}{2\beta} - 1 \right\rceil \beta \right) \sin\left(\left\lceil \frac{\pi - x}{2\beta} \right\rceil \beta + x \right) \csc x. \tag{5}$$

Similar formulas for r_b and r_c can be obtained by permutation:

$$r_b = \frac{b}{f(A)+f(C)}$$
$$r_c = \frac{c}{f(A)+f(B)}.$$

Our interest is in comparing r_a, r_b and r_c. By of the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

so we will assume without loss of generality that the common ratio is 1 in this paper. It follows that

$$(r_a, r_b, r_c) = \left(\frac{\sin A}{f(B)+f(C)}, \frac{\sin B}{f(A)+f(C)}, \frac{\sin C}{f(B)+f(A)} \right).$$

Suppose $A \geq B \geq C$ and ΔABC is isosceles. We first show that the condition for ΔABC to be Calabi is equivalent to an equation in B alone.

Theorem 3.1. *Suppose $A \geq B \geq C$ and ΔABC is isosceles. Then ΔABC is Calabi if and only if $B \in (0, \frac{\pi}{2})$ and satisfies*

$$\frac{\sin B}{f(B)+f(\pi-2B)} = \frac{\sin 2B}{2f(B)} \tag{6}$$

or equivalently,

$$(\cos B - 1)f(B) + \cos B f(\pi - 2B) = 0. \tag{7}$$

Proof. Since $A \geq B \geq C$ and ΔABC is isosceles, then the angles A and C are completely determined by B:

(1) either $B = C \in (0, \frac{\pi}{3}]$ and $A = \pi - 2B$
(2) or $A = B \in [\frac{\pi}{3}, \frac{\pi}{2})$ and $C = \pi - 2B$.

We then have in case (1), $r_b = r_c$ is automatic, and $r_a = r_b$ is equivalent to

$$\frac{\sin A}{f(B)+f(C)} = \frac{\sin B}{f(A)+f(C)}$$

which becomes

$$\frac{\sin(\pi-2B)}{f(B)+f(B)} = \frac{\sin B}{f(\pi-2B)+f(B)}.$$

Since $\sin(\pi - 2B) = \sin 2B = 2\sin B\cos B$ and $\sin B > 0$, this is equivalent to (6) and (7).

In case (2), $r_a = r_b$ is automatic and condition $r_b = r_c$ is equivalent to

$$\frac{\sin B}{f(A)+f(C)} = \frac{\sin C}{f(A)+f(B)}$$

with $A = B$ and $C = \pi - 2B$. This is also equivalent to (6).

It is instructive to see examples of (6) or (7) for some specific values of n.

Example 3.1. *There are exactly three isosceles (non-equilateral) Calabi triangles in the case of regular hexagons, $n = 6$.*

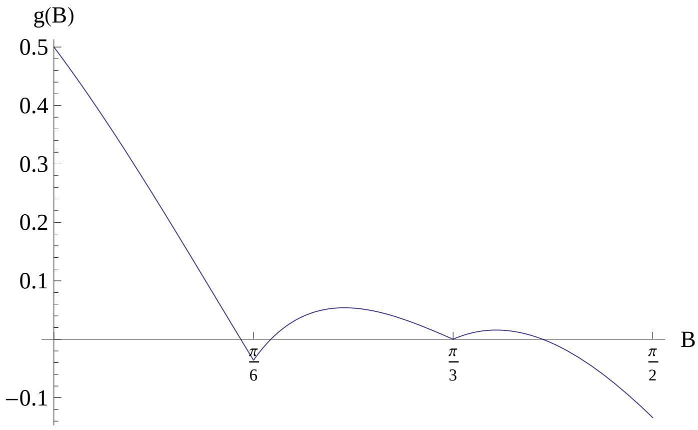

FIGURE 4. A graph of $g(B)$ for $n = 6$

Note that for $n = 6$, we have $\beta = \pi/6$. First calculate $f(B)$.

$$f(B) = \sin\beta + 2\sin\left(\left\lceil\frac{\pi - B}{2\beta} - 1\right\rceil\beta\right)\sin\left(\left\lceil\frac{\pi - B}{2\beta}\right\rceil\beta + B\right)\csc B \tag{8}$$

$$= \frac{1}{2} + 2\sin\left(\left\lceil\frac{\pi - B}{\pi/3} - 1\right\rceil\frac{\pi}{6}\right)\sin\left(\left\lceil\frac{\pi - B}{\pi/3}\right\rceil\frac{\pi}{6} + B\right)\csc B \tag{9}$$

Writing this as a piecewise function we find

$$f(B) = \begin{cases} \frac{1}{2} + \sqrt{3}\cot B, & 0 < B \le \pi/3; \\ 1 + \frac{\sqrt{3}}{2}\cot B, & \pi/3 \le B \le 2\pi/3; \\ \frac{1}{2}, & 2\pi/3 \le B < \pi. \end{cases}$$

From this it follows that

$$f(\pi - 2B) = \begin{cases} \frac{1}{2}, & 0 < B \le \pi/6; \\ 1 + \frac{\sqrt{3}}{2}\cot(\pi - 2B), & \pi/6 \le B \le \pi/3; \\ \frac{1}{2} + \sqrt{3}\cot(\pi - 2B), & \pi/3 \le B < \pi/2. \end{cases}$$

And finally, we write the left side of (7), $g(B) = (\cos(B) - 1)f(B) + \cos(B)f(\pi - 2B)$, in the case $n = 6$:

$$g(B) = \begin{cases} (\cos(B) - 1)(\frac{1}{2} + \sqrt{3}\cot B) + \frac{1}{2}\cos(B), & 0 < B \le \pi/6; \\ (\cos(B) - 1)(1 + \sqrt{3}\cot(B)) \\ \quad + \cos(B)(1 - \frac{\sqrt{3}}{2}\cot(2B)), & \pi/6 \le B \le \pi/3; \\ (\cos(B) - 1)(1 + \frac{\sqrt{3}}{2}\cot(B)) \\ \quad + \cos(B)(\frac{1}{2} - \sqrt{3}\cot(2B)), & \pi/3 \le B < \pi/2. \end{cases}$$

The plot of $g(B)$ in Figure 4 reveals that there are four roots. Since one of them is $B = \pi/3$, the equilateral triangle, we see that there are three isosceles Calabi triangles in case $n = 6$.

Note that by the data gathered in Table 1, these three Calabi triangles do not have Type 1 angles.

As an application of Theorem 3.1, we show the results on odd and even n mentioned in the introduction to this section.

Corollary 3.1. *If $n \geq 3$ is odd, then there is no isosceles Calabi triangle ΔABC with $B < \frac{\pi}{n}$.*

Proof. From (7), the isosceles triangle will be Calabi in case $(\cos B - 1)f(B) + \cos B f(\pi - 2B) = 0$. For the function f we need values of the ceiling function when n is odd: $\lceil \frac{\pi - B}{2\beta} \rceil = \frac{n+1}{2}$ and $\lceil \frac{\pi - B}{2\beta} - 1 \rceil = \frac{n-1}{2}$. Some simplification then shows that $f(B) = \frac{1+\cos B}{\sin B}\cos B$. In a similar manner, the value of $f(\pi - 2B)$ simplifies significantly since $\lceil \frac{\pi - (\pi - 2B)}{2\beta} \rceil = 0$ giving $f(\pi - 2B) = \sin\beta$.

The left side of (7) can now be computed, again for n odd and $0 < B < \beta$:

$$\begin{aligned}
&(\cos B - 1)f(B) + \cos B f(\pi - 2B) \\
&\qquad = (\cos B - 1)\frac{1+\cos B}{\sin B}\cos B + \cos B \sin\beta \\
&\qquad = \cos B\left(\frac{(\cos B - 1)(1+\cos\beta)}{\sin B} + \sin\beta\right) \\
&\qquad = \frac{\cos B}{\sin B}\left(\sin B\sin\beta - (1-\cos B)(1+\cos\beta)\right).
\end{aligned}$$

Consider the term in parentheses: $h(B) = \sin B \sin\beta - (1-\cos B)(1+\cos\beta)$. It is easily seen that $h(0) = h(\beta) = 0$, $h'(0) > 0$, $h'(\beta) < 0$, and $h''(B) < 0$ for $0 < B < \beta$. So h has no roots in the interval $0 < B < \beta$, thus showing that (7) has no roots there and hence there are no isosceles Calabi triangles for n odd and $0 < B < \beta$.

Corollary 3.2. *If $n \geq 4$ is even, then an isosceles triangle ΔABC with $B < \frac{\pi}{n}$ is Calabi if and only if $z = \cos B$ satisfies the cubic equation*

$$(2z-1)^2(1+z)\tan^2\beta = 4z^2(1-z). \tag{10}$$

Moreover, there is a unique root of this equation in the interval $[\cos\beta, 1]$. So for every even $n \geq 4$, there is an isosceles Calabi triangle ΔABC with $B = C < \frac{\pi}{n}$.

Proof. To calculate the function f in the case n is even, we need values $\lceil \frac{\pi - B}{2\beta} \rceil = \frac{n}{2}$ and $\lceil \frac{\pi - B}{2\beta} - 1 \rceil = \frac{n-2}{2}$. Using the trig identity $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$ and the addition formula for the cosine, the function f simplifies to

$$\begin{aligned}
f(B) &= \sin\beta + 2\sin\left(\frac{n-2}{2}\beta\right)\sin\left(\frac{n}{2}\beta + B\right)\csc B \\
&= \sin\beta + 2\cos\beta\frac{\cos B}{\sin B}
\end{aligned}$$

and the additional form of f becomes $f(\pi - 2B) = \sin\beta$ as in the previous proof. Putting this into (7) gives

$$(\cos B - 1)\left(\sin\beta + 2\cos\beta\frac{\cos B}{\sin B}\right) + \cos B\sin\beta = 0. \tag{11}$$

Letting $z = \cos B$, we get

$$(z-1)\left(\sin\beta + 2\cos\beta\frac{z}{\sqrt{1-z^2}}\right) + z\sin\beta = 0. \tag{12}$$

Isolate the square root term, square both sides, and simplify to get a reformulated version of (12):

$$(2z-1)^2(1+z)\tan^2\beta + 4z^2(z-1) = 0 \tag{13}$$

Since we are interested in angles B with $0 < B < \beta$, any z of interest satisfies $\cos\beta < z < 1$. Let $g(z)$ denote the left hand side of (13). Then $g(\cos\beta)$ can be simplified (with some assistance from *Mathematica*) to

$$g(\cos\beta) = -4(1+\cos\beta+2\cos\beta)\sec^2\beta\sin^4(\beta/2).$$

So $g(\cos\beta) < 0$. It is easy to see that $g(1) = 2\tan^2\beta > 0$. Thus g has a root on the interval $\cos\beta < z < 1$. Moreover g is monotone on this interval. To see this, compute and simplify g' to get $g'(z) = 4z(3z-2)+3(4z^2-1)\tan^2\beta$. The first term is positive for $z > 2/3$ and the second term is positive for $z > 1/2$. Since we are only interested in $z > \cos\beta > \cos(\pi/4)$, we see that $g'(z) > 0$. Thus there is a unique root of g on the interval $(\cos\beta, 1)$. This proves the Corollary.

Apply this result to a couple of specific values of n. If $n = 4$, use (10) to get the equation

$$8z^3 - 4z^2 - 3z + 1 = 0$$

one solution of which is $z = 0.775694$ in $(\sqrt{2}/2, 1)$. This corresponds to the Calabi triangle found by Calabi (see Conway and Guy [1]) with $B = \arccos(0.775694) = 0.682983 = 39.132°$.

If $n = 6$, then $\tan\beta = 1/\sqrt{3}$ and equation (10) becomes

$$\frac{16}{3}z^3 - 4z^2 - z + \frac{1}{3} = 0. \tag{14}$$

The relevant solution is $z = 0.882229 \in (\sqrt{3}/2, 1)$. Thus $B = \arccos(0.882229) = 0.49022 = 28.0875°$, giving an isosceles Calabi triangle.

4. Isosceles Calabi Triangles With Type 2 Angles And Applications To Regular Pentagons And Hexagons

Recall that an angle is called type 1 if it has the form $\pi - 2k\beta$ for an integer k. If an angle does not have this form we will call it a *type 2 angle*. We will say that a triangle is a *type 2 triangle* in case at least one of its angles is type 2. In this section we look at the existence of type 2 isosceles Calabi triangles for $n \geq 5$. As remarked in the introduction, the theorem of Sullivan implies a maximal inscribed polygon can always be taken to have at least one of its sides along the side of the triangle. Whether the other two sides of the triangle contain a side of the polygon or merely a vertex of the polygon depends on the type of the angles of the triangle. The first proposition shows that triangles with an angle of the form $\pi - (2m-1)\beta$ (nearly the same form as a type 1 angle) cannot be Calabi. An application of this shows that Calabi triangles with type 2 angles have exactly one side that contains a side of the maximal inscribed polygon so that each of the other two sides of the triangle contains just a vertex of the polygon. Finally we show how to calculate polynomials that can then be used to find Calabi triangles.

Proposition 4.1. *Suppose ΔABC is an isosceles triangle with $A \geq B \geq C$ and $B = \pi - (2m-1)\beta$ for an integer m with $\frac{n+2}{4} < m < \frac{n+1}{2}$. Then ΔABC is not a Calabi triangle.*

Proof. Recall that an isosceles triangle ΔABC, with $A \geq B \geq C$, is Calabi if and only if B satisfies (7). We can get some useful simplifications by multiplying (7) by $2\sin B$ and using the double angle formula for the sine to get

$$2(\cos B - 1)f(B)\sin B + f(\pi - 2B)\sin(2B) = 0 \tag{15}$$

where f is given by (5). Because (15) contains $f(B)\sin B$ and $f(\pi - 2B)\sin(2B) = f(\pi-2B)\sin(\pi-2B)$, it will be useful to rewrite (15) defining the function $g(z) = f(z)\sin z$:

$$2(\cos B - 1)g(B) + g(\pi - 2B) = 0. \tag{16}$$

The arguments of the sine functions in the definition of f (and hence g) involve the ceiling function, so let $k(z) = \lceil \frac{\pi - z}{2\beta} \rceil - 1$. The values of $k(z)$ at $z = B$ and $z = \pi - 2B$ can be simplified significantly (recall $n\beta = \pi$ and $B = \pi - (2m-1)\beta)$:

$$k(B) = \left\lceil \frac{\pi - B}{2\beta} - 1 \right\rceil = m - 1 \tag{17}$$

and

$$k(\pi - 2B) = \left\lceil \frac{\pi - (\pi - 2B)}{2\beta} - 1 \right\rceil = n - 2m. \tag{18}$$

Next, rewrite the function $g(z)$:

$$\begin{aligned} g(z) &= \sin\beta\sin z + 2\sin(k(z)\beta)\sin(k(z)\beta + \beta + z) \\ &= \sin\beta\sin z + \cos(z+\beta) - \cos(2k(z)\beta + \beta + z) \\ &= \cos z\cos\beta - \cos(2k(z)\beta + \beta + z). \end{aligned} \tag{19}$$

Let $z = B$ in (19) and use (17) to get $2k(B)\beta + \beta + B = \pi$ so that $g(B) = \cos B\cos\beta + 1 = 1 - \cos((2m-1)\beta)\cos\beta$.

To reduce some clutter, let $\theta = (2m-1)\beta$ so that $g(B) = 1 - \cos\theta\cos\beta$. Let $z = \pi - 2B$ in (19) and use (18) to get $2k(\pi-2B)\beta + \beta + (\pi - 2B) = \pi - \beta$. After some simplification it can be seen that $g(\pi - 2B) = 2\cos\beta(1 - \cos^2\theta)$. Lastly observe that $\cos(B) - 1 = -(1 + \cos\theta)$. We can now further simplify (16):

$$\begin{aligned} &2(\cos(B) - 1)g(B) + g(\pi - 2B) \\ &\qquad = -2(1+\cos\theta)(1 - \cos\theta\cos\beta) - 2\cos\beta(\cos^2\theta - 1) \\ &\qquad = -2(1+\cos\theta)(1 - \cos\theta\cos\beta + \cos\beta(\cos\theta - 1) \\ &\qquad = -2(1+\cos\theta)(1 - \cos\beta) \\ &\qquad = -2(1 + \cos((2m-1)\beta))(1 - \cos\beta). \end{aligned}$$

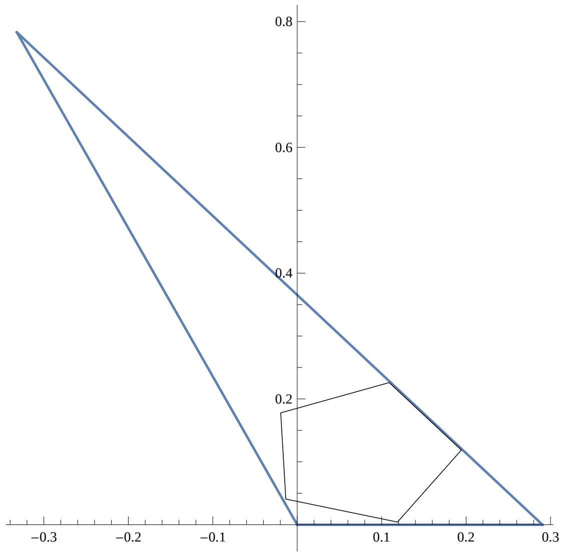

FIGURE 5. Calabi triangle with type 2 angles

Because $\beta = \pi/n$ and $\theta = (2m-1)\beta$ are both in $(0, \pi)$, it follows that this last expression cannot be zero. Thus the criterion (16) cannot hold and so ΔABC is not a Calabi triangle.

A consequence of the preceding result is that a maximal inscribed polygon in an isosceles Calabi triangle cannot have exactly two sides on two sides of the triangle.

Proposition 4.2. *If ΔABC is an isosceles Calabi triangle for a regular polygon (with fixed n), then either exactly one side of the polygon lies on a side of the triangle or exactly three sides of the polygon are contained on the three sides of the triangle.*

Proof. We know that three sides of the triangle contain three sides of the maximal n-gon if and only if the three angles are type 1 angles.

Now assume that exactly two sides of the triangle contain two sides of the n-gon. That is, not all angles are type 1. We also assume that $B = C$ so that the sides AB and AC have equal lengths.

Case (1). Suppose the sides AB and AC lie on two sides of the n-gon. In this case, the angle A must be a type 1 angle by Proposition 2.1. So $A = \pi - 2m\beta$ for some positive integer $m < n/2$. It follows that $B = C = m\beta$. Since $\pi = n\beta$, we can write $m\beta = \pi - (n-m)\beta = \pi - k\beta$. If k is even, then $B = C$ is a type 1 angle and then all three angles are type 1 contrary to the assumption. So k must be odd and then by Proposition 4.1 the triangle cannot be Calabi.

Case (2). Suppose the sides BC and AC contain two sides of the n-gon. In this case, $B = C = \pi - 2m\beta$ are type 1. So $A = \pi - 2B = \pi - 2(n-2m)\beta$ is also type 1. Thus all three sides of the triangle contain a side of the n-gon again contradicting the “exactly two sides” assumption.

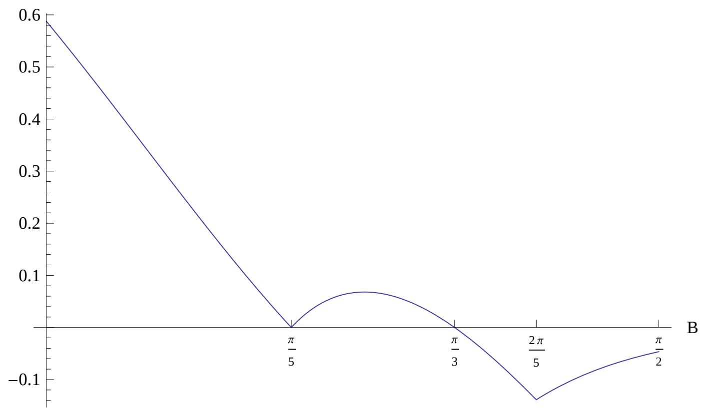

FIGURE 6. Plot of (7) for $n = 5$

It turns out that for $n = 5$ (regular pentagons), there is no isosceles Calabi triangle with type 2 angles, breaking the pattern suggested by results in [3] and Calabi's result cited in [1]. However, there is a scalene Calabi triangle for regular pentagons as will be seen Section 5. For $n = 6$, there are three isosceles (non-equilateral) Calabi triangles as shown in Example 3.1. We also can demonstrate that there are no scalene Calabi triangles in case $n = 6$. Most of this can be proven analytically, with the scalene result relying on some less rigorous graphical calculations.

Equation (7) can also be used to show that some families of triangles are not Calabi.

Proposition 4.3. *There is no isosceles Calabi triangle with type 2 angles for regular pentagons.*

This proposition can be proved analytically using (7). Alternatively, if the function on the left side of (7) is graphed, it is easier, if less rigorous, to see the proposition. Recalling Table 1, we know that there is a unique isosceles Calabi triangle with type 1 angles. The plot of the left side of equation (7) in Figure 6 shows only one solution (other than $A = B = C = \frac{\pi}{3}$) and so that solution must be the type 1 isosceles Calabi triangle. It is easily checked that the other root is $\pi/5(= \beta)$ and so indeed it gives the isosceles Calabi triangle with type 1 angles. Hence there are no other isosceles Calabi triangles.

In [10], Wetzel produces a polynomial that can be used to find the largest square in a triangle and also used to find a Calabi triangle. We give an alternative way to see this in the following example. Note that in this example the angle B is "large" in comparison to the angles in the two corollaries in Section 3.

Example 4.1. *If $n = 3$, then an isosceles triangle ΔABC with $A = B \in (\pi/3, \pi/2)$ is Calabi if and only if $z = \cos B < \frac{1}{2}$ satisfies the polynomial equation*

$$1 - 6z + 8z^3 = 0.$$

Proof. When $n = 3$ and $B \in (\pi/3, \pi/2)$, we have $\lceil \frac{\pi - B}{2\pi/3} - 1 \rceil = 0$ so the function f in (5) becomes $f(B) = \sqrt{3}/2$. Similarly, $\lceil \frac{\pi-(\pi-2B)}{2\pi/3} - 1 \rceil = \lceil \frac{3y}{\pi} - 1 \rceil = 1$, and so after some simplification,

$$\begin{aligned} f(\pi - 2B) =& \frac{\sqrt{3}}{2} + 2\sin(2\pi/3)\sin(\pi + \pi/3 - 2B)\csc(2B) \\ =& \frac{3}{2}\left(\frac{1}{2\cos B} - \cos B\right)\sin B. \end{aligned}$$

Since $B \in (\frac{\pi}{3}, \frac{\pi}{2})$, we can substitute into (7) to get a condition for B to be the angle in an isosceles Calabi triangle.

$$(\cos B - 1)\frac{\sqrt{3}}{2} + \cos B \cdot \frac{3}{2}\left(\frac{1}{2\cos B} - \cos B\right)\frac{1}{\sin B} = 0. \tag{20}$$

Let $z = \cos B$ and $\sin B = \sqrt{1-z^2}$, rewrite equation (20) as

$$(z-1)\sqrt{3}/2 + z\frac{3}{2}\left(\frac{1}{2z} - z\right)\frac{1}{\sqrt{1-z^2}} = 0.$$

Isolating the square root term, squaring, and rearranging will result in the polynomial equation

$$3 - 24z + 36z^2 + 24z^3 - 48z^4 = 0$$

which is equivalent to

$$-3(2z-1)(1-6z+8z^3) = 0.$$

One root is $z = 1/2$, but since $B \in (\frac{\pi}{3}, \frac{\pi}{2})$ then $z \in (0, 1/2)$ so the desired z is not $1/2$. The positive roots of the cubic are $z = 0.173648$ and $z = 0.766044$ so the latter root is not of interest. The solution $z = 0.173648$ corresponds to $B = \arccos(z) = 1.39626 = 80°$, which is the Calabi triangle found by Wetzel for inscribed equilateral triangles.

The idea of associating polynomials to isosceles triangle can be extended to other n. In fact the algebraic simplifications are similar to those in the preceding example: substitute $z = \cos B$; isolate $\sin B = \sqrt{1-z^2}$; square both sides and simplify, possibly eliminating some extraneous factors. In the case that $n = 6$, as in the preceding example, letting $z = \cos B$ and using the function $g(B)$ found in Example 3.1 we find, after some severe algebraic manipulations augmented with some use of *Mathematica*,

$$\begin{cases} 16z^3 - 12z^2 - 3z + 1 = 0, & \frac{\sqrt{3}}{2} < z < 1; \text{ i.e., } 0 < B < \pi/6 \\ 48z^4 - 72z^3 + 28z^2 - 1 = 0, & \frac{1}{2} < z < \frac{\sqrt{3}}{2}; \text{ i.e., } \pi/6 < B < \pi/3 \\ 12z^4 - 6z^3 - 8z^2 + 6z - 1 = 0, & 0 < z < \frac{1}{2}; \text{ i.e., } \pi/3 < B < \pi/2. \end{cases}$$

Note that the first of these equations is the same as found in equation (14), and has root $z = 0.882229$ and $B = 28.0875°$. To find the other Calabi triangles for $n = 6$ as indicated in Figure 4, solve the other two equations to get $z = 0.842347$ so that $B = 32.6112°$ and $z = 0.284565$ with $B = 73.4672°$. The cubics have other roots of course, but the ones given here are those satisfying the indicated constraints.

5. Scalene Calabi Triangles

In Section 2 we proved that there exist many scalene Calabi triangles with type 1 angles. Actually finding scalene Calabi triangles is computationally more difficult. Recall that to find an isosceles Calabi triangle, we need to solve the single equation (7) for angle B. However to find a scalene triangle we need to solve a nonlinear system of equations. We need to find sides a, b, and c so that the radii if the polygons are equal, $r_a = r_b = r_c$. Using (4) means we need to solve the system

$$\frac{a}{f(B)+f(C)} = \frac{b}{f(\pi - B - C) + f(C)} = \frac{c}{f(\pi - B - C) + f(B)}. \tag{21}$$

For $n = 5$, we found the following scalene Calabi triangle.

Theorem 5.1. *There exists a unique Calabi triangle for inscribed regular pentagons, which has sides*

$$\{a, b, c\} = \{1.0,\ 0.850187,\ 0.290446\}$$

and angles

$$\{A, B, C\} = \{1.9719,\ 0.899006\ ,\ 0.270687\}.$$

Proof. We may assume that $A \geq B \geq C$, then r_a, r_b, and r_c can be expressed as functions of B and C on the region

$$0 < C \leq \frac{\pi}{3}\ ,\ C \leq B \leq \frac{1}{2}(\pi - C)$$

The system can be written as in (21). We may assume that $a = 1$. So $b = \frac{1}{\sin A} \cdot \sin B = \frac{\sin B}{\sin(B+C)}$ and $c = \frac{\sin C}{\sin(B+C)}$. Thus

$$\begin{aligned}\frac{1}{f(B)+f(C)} &= \frac{\sin B}{\sin(B+C)\,[f(\pi - B - C) + f(C)]}\\ &= \frac{\sin C}{\sin(B+C)\,[f(\pi - B - C) + f(B)]}\end{aligned}$$

We use *Mathematica* to determine the solution stated in the theorem, which can then be verified directly.

6. Questions

We end the paper with a few questions. Wetzel has many questions on this general topic in [9] and [10].

(1) If a selection of functions of the form (7) is plotted using different n, it will be seen that there appear to be an increasing number of roots, indicating an increasing number of Calabi triangles. If $n \geq 7$ and odd and $m = \lfloor n/4 + 1/2 \rfloor$, it looks like there is a solution in each interval of length 2β. That is, is there a collection of Calabi triangles with respective "middle" angles B in each of the intervals in $[\beta, 3\beta], \ldots, [(2m-3)\beta, (2m-1)\beta]$?
(2) Polynomials can be associated to (7). Are there polynomials in two variables that can be derived from (21) that would be useful in finding scalene Calabi triangles?

(3) If a triangle is not Calabi, which side of the triangle contains a side of the maximal inscribed polygon?
(4) Is there an analog of Sullivan's result for the case of maximal polygons that are inscribed in a given polygon. That is, does one side of the outer polygon contain a side of the inner polygon? This is not always the case as is easily seen for a maximal triangle in a square. When is it true?
(5) Rather than inscribing regular polygons, fix another type of planar geometric shape and ask for the maximal such shape in a triangle. There are some results on the maximal rhombus, parallelogram, rectangle, and trapezoid. Similar questions are considered in [2], [4], [5] going back to a more general question by Steinhaus [7].

7. Appendix: Largest Regular Polygon Inscribed In A Triangle

The following result guarantees that a maximal inscribed regular n-gon P exists for a given triangle T. The theorem is essentially proved in Post [6] and Sullivan [8].

Proposition 7.1. *Given a triangle T and an integer $n \geq 3$, there is a regular n-gon P with the following properties.*

(1) *P is inscribed in T;*
(2) *P has the largest radius among all inscribed regular n-gons;*
(3) *P has the largest radius among all n-gons that are contained in T;*
(4) *Any polygon contained in T with property (2) or (3) must have at least one side contained on one of the sides of T.*

In this appendix we fix an integer $n \geq 3$ and derive a formula for the radius r of an inscribed n-gon P in terms of the lengths of the sides a, b, c and the measures of the angles A, B, C. First consider the case where one side of P is on AB as is possible from Proposition 7.1. In this case we can describe the radius of the largest such inscribed polygon as follows.

Theorem 7.1. *The radius of the maximum regular n-gon P that can be inscribed in a triangle $T = \Delta ABC$ with one side on AB is*

$$r = \frac{c}{f(A) + f(B)}$$

where $f(x)$ is the function defined for $x \in (0, \pi)$ by:

$$f(x) = \sin\beta + 2\sin\left(\left\lceil \frac{\pi - x}{2\beta} - 1 \right\rceil \beta\right) \sin\left(\left\lceil \frac{\pi - x}{2\beta} \right\rceil \beta + x\right) \csc x \tag{22}$$

where $\lceil x \rceil$ is the ceiling function; that is, $\lceil x \rceil = n$ (an integer) if and only if $n - 1 < x \leq n$.

Remark 7.1. *An analytic explanation of the expression involving the ceiling function $\left\lceil \frac{\pi - x}{2\beta} \right\rceil$ is given in the proof, but we can also describe the intention in a less technical way. Assume that angles A and B satisfy $A \geq B$. The idea is that the sides of the triangle adjacent to the angle B divide the polygon into two polygonal arcs, one nearer to B and one further from B. The desired expression involving the ceiling function is simply the number of vertices on*

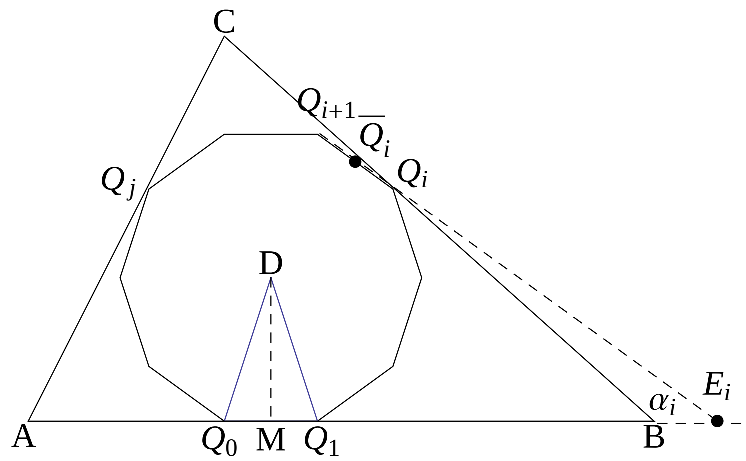

FIGURE 7. Notation used in proof

the polygonal arc that is nearer to B. In case a side of the polygon lies on a side of the triangle, only one of the vertices on that side is counted.

Proof. Place the triangle ABC in a coordinate xy-plane so that $A = (0, 0)$, $B = (c, 0)$, and C is in the upper half plane, as shown in Figure 7.

Denote the midpoint of Q_0Q_1 by M. Because $AM + BM = c$, let us denote AM by c_A and BM by c_B. (We are again abusing notation here by letting the symbol AM refer to both the line segment between A and M and the length of this line segment.) The idea is to find an expression for c_A and c_B in terms of r and then solve for r from the equation $c_A + c_B = c$. Because P is the largest polygon inscribed in the triangle with Q_0Q_1 on AB then, using the floor function $\lfloor\cdot\rfloor$, there is an $i \in \left\{1, \cdots, \left\lfloor\frac{n+1}{2}\right\rfloor\right\}$ such that Q_i is on the side BC and another $j \in \left\{\left\lfloor\frac{n}{2}\right\rfloor, \cdots, n\right\}$ such that Q_j is on AC (recall that $Q_n = Q_0$). For example, if $n = 3$ then $i \in \{1, 2\}$ and $j \in \{2, 3\}$; if $n = 4$ then $i \in \{1, 2\}$ and $j \in \{3, 4\}$. We will focus on finding c_B because its expression will apply to c_A by permuting notation.

The index i can be determined in terms of the angle B as follows. Let E_i be the intersection of the line through Q_0 and Q_1 with the line through Q_i and Q_{i+1} for $i = 0, \ldots, \lfloor\frac{n-1}{2}\rfloor$ and α_i be the measure of the angle $\angle Q_0E_iQ_{i+1}$. (We define $\alpha_0 = \pi$.) For $i = 1, \ldots, \lfloor\frac{n-1}{2}\rfloor$ to say that Q_i lies on BC is equivalent to $\alpha_i \leq B \leq \alpha_{i-1}$. To determine α_i we consider the quadrilateral $DME_i\overline{Q}_i$, where $\overline{Q}_i$ is the midpoint of the segment Q_iQ_{i+1}. It follows that α_i is supplementary to the central angle $\angle MD\overline{Q}_i$, which is congruent to $\angle Q_0DQ_i$, whose measure is $2i\beta$. Thus $\alpha_i = \pi - 2i\beta$ and so

$$\pi - 2i\beta \leq B \leq \pi - 2(i-1)\beta.$$

In case $\frac{\pi-B}{2\beta}$ is an integer, there are two possible choices for i. So the ceiling function is used to identify i,

$$i = \left\lceil\frac{\pi - B}{2\beta}\right\rceil$$

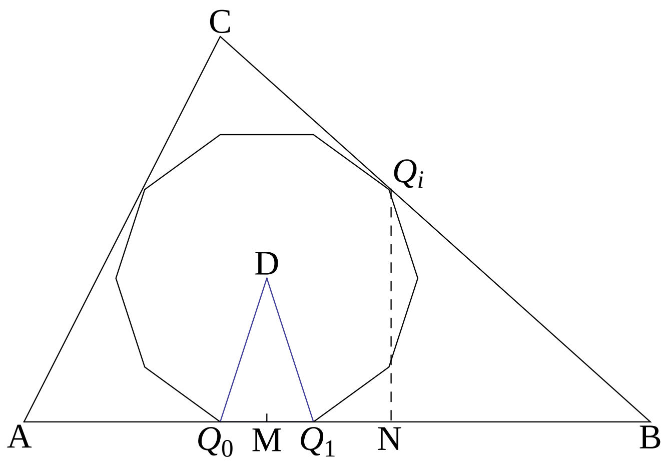

FIGURE 8. Additional notation used in proof

and this gives the desired relation between i and B.

It is useful to pause the proof here to consider a couple of special cases of this description of the index i. For example, if $n = 3$ (the problem of maximal equilateral triangle in a given triangle), then $\beta = \frac{\pi}{3}$ and

$$i = \left\lceil \frac{\pi - B}{2\beta} \right\rceil = \begin{cases} 2, & 0 < B < \frac{\pi}{3} \\ 1, & \frac{\pi}{3} \leq B < \pi \end{cases} \tag{23}$$

The first case only applies to $0 < B < \frac{\pi}{3}$ because B is assumed positive. If $n = 4$ (maximal square in a triangle), then $\beta = \frac{\pi}{4}$ and $i = \left\lceil \frac{\pi - B}{2\beta} \right\rceil = \begin{cases} 2, & 0 < B < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} \leq B < \pi \end{cases}$.

Continuing with the proof of the theorem, let Q_iN be the altitude of the triangle ΔQ_1BQ_i, as in Figure 8. Then $c_B (= MB)$ can be realized as the sum of the lengths of three line segments

$$c_B = MQ_1 + Q_1N + NB.$$

We want to rewrite this equation in terms of r and β. Several lengths can be realized as functions of i:

(1) $MQ_1 = r \sin \beta$
(2) $Q_1Q_i = 2r \sin(\frac{1}{2}\angle Q_1DQ_i) = 2r \sin((i-1)\beta)$. This follows from the fact that the triangle ΔQ_1DQ_i is isosceles with angle $(i-1)2\beta$ between the two sides of length r. As a result, the length of the base is $2r \sin((i-1)\beta)$.
(3) $Q_1N = Q_1Q_i \cos \angle NQ_1Q_i = 2r \sin((i-1)\beta) \cos(i\beta)$. The first equality is just the definition of cosine. The other equality will follow from item #2 if we show $\angle NQ_1Q_i = i\beta$. For this we compute two angles whose sum is complementary to $\angle NQ_1Q_i$. First note that the angle $\angle Q_0DQ_1 = 2\beta$ so the angle at the base of the isosceles triangle ΔQ_0DQ_1 is $\frac{1}{2}(\pi - 2\beta)$. Then item #2 shows that the angle

$\angle DQ_1Q_i = \frac{1}{2}(\pi - (i-1)2\beta) = \frac{\pi}{2} - (i-1)\beta$. Adding these two angles together and subtracting from π gives the desired angle:

$$\angle NQ_1Q_i = \pi - \left(\frac{1}{2}(\pi - 2\beta) + \frac{\pi}{2} - (i-1)\beta\right) = i\beta,$$

as claimed.

(4) $Q_iN = Q_1Q_i \sin \angle NQ_1Q_i = 2r \sin((i-1)\beta)\sin(i\beta)$. This follows from item #3.

(5) $NB = Q_iN \cot B = 2r \sin((i-1)\beta)\sin(i\beta)\cot B$. Consider the right triangle ΔBNQ_i. Then $NB/Q_iN = \cot B$, so we get the first equality. The second equality follows directly from item #4.

We can now compute c_B:

$$\begin{aligned} c_B &= r\sin\beta + 2r\sin((i-1)\beta)\cos(i\beta) + 2r\sin((i-1)\beta)\sin(i\beta)\cot B \\ &= r\sin\beta + 2r\sin((i-1)\beta)\sin(i\beta + B)\csc B. \end{aligned}$$

Recall that i is given by the function $i = \lceil \frac{\pi - B}{2\beta} \rceil$. Defining the function $f(B)$ as in (22), we can write c_B as

$$c_B = rf(B) = r\left(\sin\beta + 2\sin((i-1)\beta)\cos(i\beta) + 2\sin((i-1)\beta)\sin(i\beta)\cot B\right).$$

This relation also applies to c_A: $c_A = rf(A)$ with the same $f(\cdot)$. Consequently, we have

$$c = r(f(A) + f(B)).$$

Therefore we have the radius of the inscribed polygon with one side on AB as $r = \frac{c}{f(A)+f(B)}$.

In a similar manner, the radius of the maximal regular inscribed n-gon with one side on BC or AC is $r = \frac{a}{f(B)+f(C)}$ or $r = \frac{b}{f(A)+f(C)}$, respectively. So we get

Theorem 7.2. *The radius of the maximum regular n-gon that can be inscribed in a triangle $T = \Delta ABC$ is*

$$r = \max\left\{\frac{c}{f(A)+f(B)}, \frac{a}{f(B)+f(C)}, \frac{b}{f(C)+f(A)}\right\} \quad (24)$$

where $f(x)$ is the function defined as in (22).

References

[1] Conway, J.H. and Guy, R.K., *The Book of Numbers*, Springer-Verlag, New York, 1996.

[2] Gardner, M., *Some Surprising Theorems about Rectangles in Triangles*, Math Horizons **5 (1997)**, 18–22.

[3] Jerrard,R.P. and Wetzel, J. E., *Equilateral Triangles and Triangles*, Amer. Math. **10 (2002)**, 909–915.

[4] Jepsen, H. and Vulpe, V. *Fitting One Right Triangle in Another*, Math. Mag.**80 (2007)**, 203–207.

[5] Lange, L.H., *What Is the Biggest Rectangle You Can Put Inside a Given Triangle?* College Math. J. **24 (1993)**, 237–240.

[6] Post, K. A., *Triangle in a Triangle: On a Problem of Steinhaus*, Geom. Dedicata **45 (1993)**, 115–120.

[7] Steinhaus, H., *One Hundred Problems in Elementary Mathematics*, Pergamon, Oxford, 1964, p. 98.

[8] Sullivan, J. M., *Polygon in Triangle: Generalizing a Theorem of Post.* Preprint.

[9] Wetzel, J. E., *Fits and Covers*, Math. Mag. **76 (2003)**, 349–363
[10] Wetzel, J. E., *Squares in Triangles*, Math. Gaz. **86 (2002)**, 28–34.

DEPARTMENT OF MATHEMATICS
BRADLEY UNIVERSITY
PEORIA, IL 61625 USA
E-mail address: mcasey@bradley.edu

DEPARTMENT OF MATHEMATICS
BRADLEY UNIVERSITY
PEORIA, IL 61625 USA
E-mail address: mou@bradley.edu