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AN n-DIMENSIONAL GENERALIZATION OF A GEOMETRIC INEQUALITY

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Abstract. In this paper, we shall give an n-dimensional generalization of a geometric inequality posed by Jan Ciach [1].

1. Introduction

In 1995 J. Ciach [1] proposed following problem: Let a tetrahedron $A_1A_2A_3A_4$ with centroid G be inscribed in a sphere of radius R. The lines A_1G, A_2G, A_3G, A_4G meet the sphere again at A_1', A_2', A_3', A_4' respectively. Prove or disprove that

$$\frac{4}{R} \leq \frac{1}{GA_1'} + \frac{1}{GA_2'} + \frac{1}{GA_3'} + \frac{1}{GA_4'} \leq \frac{4\sqrt{6}}{9}\left(\sum_{1\leq i<j\leq 4} \frac{1}{A_iA_j}\right). \tag{1}$$

Equality holds if $A_1A_2A_3A_4$ is regular. This inequality is a three-dimensional version of problem 5 of the 1991 Vietnamese Olympiad (see [2]).

In 1996 M. Klamkin [3] proved an n-dimensional generalization of the right hand inequality in (1).

In this paper, we shall give an n-dimensional generalization of the inequality (1) by the different way from [3] in the following section.

2. Main result

In order to prove the main theorem, we need the following two lemmas.

Lemma 2.1. *Let m_1 be the median of $A_1 \dots A_nA_{n+1}, (n \geq 2)$ from vertex A_1. Then*

$$m_1^2 = \frac{1}{n^2}\left(n\sum_{2\leq j\leq n+1} A_1A_j^2 - \sum_{2\leq i<j\leq n+1} A_iA_j^2\right). \tag{2}$$

Proof. Let G_1 and G_2 denote the centroids of two simplexes $A_2A_3 \dots A_{n+1}$ and $A_3 \dots A_nA_{n+1}$ respectively. Let $m_2^{(1)}$ be the median of $A_2 \dots A_nA_{n+1}$ from vertex A_2, and let $m_1^{(2)}$ be the median of $A_1A_3 \dots A_nA_{n+1}$ from A_1.

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We will prove the equality (2) by induction on n, the $n = 2$ being trivial. Now suppose (2) holds for some integer $n-1 \geq 2$. We claim that it also holds form n. By Stewart's theorem, we have

$$A_2G_2 \cdot A_1G_1^2 = G_1G_2 \cdot A_1A_2^2 + A_2G_1 \cdot A_1G_2^2 - A_2G_1 \cdot G_1G_2 \cdot A_2G_2$$

$$\Leftrightarrow m_2^{(1)} \cdot A_1G_1^2 = \frac{1}{n} m_2^{(1)} \cdot A_1A_2^2 + \frac{n-1}{n} \cdot m_2^{(1)} \cdot \left(m_1^{(2)}\right)^2$$

$$-\frac{n-1}{n} \cdot m_2^{(1)} \cdot \frac{1}{n} \cdot m_2^{(1)} \cdot m_2^{(1)}$$

$$\Leftrightarrow A_1G_1^2 = \frac{1}{n} A_1A_2^2 + \frac{n-1}{n} \cdot \left(m_1^{(2)}\right)^2 - \frac{n-1}{n^2} \left(m_2^{(1)}\right)^2.$$

By the induction hypothesis we have

$$\left(m_1^{(2)}\right)^2 = \frac{1}{(n-1)^2} \left((n-1) \sum_{3 \leq j \leq n+1} A_1A_j^2 - \sum_{3 \leq i < j \leq n+1} A_iA_j^2 \right),$$

$$\left(m_2^{(1)}\right)^2 = \frac{1}{(n-1)^2} \left((n-1) \sum_{3 \leq j \leq n+1} A_2A_j^2 - \sum_{3 \leq i < j \leq n+1} A_iA_j^2 \right).$$

From the above equalities, we obtain

$$A_1G_1^2 = \frac{1}{n^2} \left(n \sum_{2 \leq j \leq n+1} A_1A_j^2 - \sum_{2 \leq i < j < n+1} A_iA_j^2 \right).$$

This completes the proof of Lemma 2.1.

Lemma 2.2. *Let an n-dimensional simplex $A_1 \ldots A_nA_{n+1}$ with centroid G be inscribed in a sphere of radius R. The line A_1G meet the sphere again at A_1'. Then, we have*

$$m_1 \cdot G_1A_1' = \frac{1}{n^2} \sum_{2 \leq i < j \leq n+1} A_iA_j^2. \tag{3}$$

Proof. The line A_2G_1 meet the inscribed sphere of a simplex $A_2 \ldots A_nA_{n+1}$ again at $A_2^{(2)}$. We will prove the equality (3) by induction on n. By the power of a point theorem and Lemma 2.1, we have

$$\begin{aligned} m_1 \cdot G_1A_1' &= A_1G_1 \cdot G_1A_1' \\ &= A_2G_1 \cdot G_1A_2^{(2)} \\ &= \frac{n-1}{n} m_2^{(1)} \cdot \left(\frac{1}{n} m_2^{(1)} + + G_2A_2^{(2)} \right) \\ &= \frac{n-1}{n^2} \cdot \left(m_2^{(1)} \right)^2 + \frac{n-1}{n} \cdot m_2^{(1)} \cdot G_2A_2^{(2)} \\ &\quad \text{(by induction hypothesis)} \\ &= \frac{n-1}{n^2} \cdot \frac{1}{(n-1)^2} \cdot \left((n-1) \sum_{3 \le j \le n+1} A_2A_j^2 - \sum_{3 \le i < j \le n+1} A_iA_j^2 \right) \\ &\quad + \frac{n-1}{n} \cdot \frac{1}{(n-1)^2} \cdot \sum_{3 \le i < j \le n+1} A_iA_j^2 \\ &= \frac{1}{n^2} \sum_{3 \le i < j \le n+1} A_2A_j^2 + \left(\frac{1}{n(n-1)} - \frac{1}{n^2(n-1)} \right) \sum_{3 \le i < j \le n+1} A_iA_j^2 \\ &= \frac{1}{n^2} \sum_{2 \le i < j \le n+1} A_iA_j^2. \end{aligned}$$

Theorem 2.1. *Let an n-dimensional simplex $A_1 \dots A_nA_{n+1}$ with centroid G be inscribed in a sphere of radius R. The lines $A_1G, \dots, A_{n+1}G$ meet the sphere again $A_1', \dots, A_{n+1}'$ respectively. Then, we have*

$$\frac{n+1}{R} \le \sum_{1 \le j \le n+1} \frac{1}{GA_j'} \le \frac{2\sqrt{2n(n+1)}}{n^2} \sum_{1 \le i < j \le n+1} \frac{1}{A_iA_j}. \tag{4}$$

Proof. Let us prove the left-hand side inequality in (4). Using Lemma 2.1 and Lemma 2.2

$$\begin{aligned} GA_1' &= GG_1 + G_1A_1' \\ &= \frac{1}{n+1} m_1 + \frac{1}{m_1} \cdot \frac{1}{n^2} \sum_{2 \le i < j \le n+1} A_iA_j^2 \\ &= \frac{1}{n^2(n+1)m_1} \left(n^2 \cdot m_1^2 + (n+1) \sum_{2 \le i < j \le n+1} A_iA_j^2 \right) \\ &= \frac{1}{n^2(n+1) \cdot m_1} \left(n \cdot \sum_{2 \le j \le n+1} A_1A_j^2 \right. \\ &\quad \left. - \sum_{2 \le i < j \le n+1} A_iA_j^2 + (n+1) \sum_{2 \le i < j \le n+1} A_iA_j^2 \right) \\ &= \frac{1}{n(n+1) \cdot m_1} \sum_{1 \le i < j \le n+1} A_iA_j^2. \end{aligned}$$

Hence,

$$\frac{1}{GA'_1} = \frac{n(n+1)\cdot m_1}{\sum_{1\le i<j\le n+1} A_iA_j^2}.$$

Similarly, we have

$$\frac{1}{GA'_k} = \frac{n(n+1)\cdot m_k}{\sum_{1\le i<j\le n+1} A_iA_j^2} \quad (k = 2,\ldots,n+1).$$

Summing up the above equalities, we obtain

$$\sum_{1\le i<j\le n+1} \frac{1}{GA'_j} = n(n+1)\cdot \frac{\sum_{1\le j\le n+1} m_j}{\sum_{1\le i<j\le n+1} A_iA_j^2}. \tag{5}$$

On the other hand, we have

$$2R \ge A_1A'_1 = A_1G_1 + G_1A'_1 = m_1 + \frac{1}{n^2}\cdot\frac{1}{m_1}\cdot \sum_{2\le i<j\le n+1} A_iA_j^2.$$

Thus

$$2nR\cdot m_1 \ge \sum_{2\le j\le n+1} A_1A_j^2.$$

We can prove similar inequalities for the $m_i, i = 2,\ldots,n+1$. Then, summing up these inequalities, we get

$$2nR \sum_{1\le j\le n+1} m_j \ge 2 \sum_{1\le i<j\le n+1} A_iA_j^2,$$

which is equivalent to

$$n(n+1)\cdot \frac{\sum_{1\le j\le n+1} m_j}{\sum_{1\le i<j\le n+1} A_iA_j^2} \ge \frac{n+1}{R}. \tag{6}$$

From the inequalities (5) and (6), we have

$$\sum_{1\le j\le n+1} \frac{1}{GA'_j} \ge \frac{n+1}{R}.$$

Next we prove the right-hand side inequality. Using the Cauchy-Schwarz inequality, Lemma 2.2 and the AM-GM inequality, we have

$$\sum_{1\le j\le n+1} \frac{1}{GA'_j} = n(n+1)\cdot \frac{\sum_{1\le j\le n+1} m_j}{\sum_{1\le i<j\le n+1} A_iA_j^2}$$

$$\leq n(n+1)\cdot\frac{\sqrt{n+1}\cdot\sqrt{\sum_{1\leq j\leq n+1} m_j^2}}{\sum_{1\leq i<j\leq n+1} A_iA_j^2}$$
$$= n(n+1)\sqrt{n+1}\cdot\frac{\sqrt{\frac{n+1}{n^2}\cdot\sum_{1\leq i<j\leq n+1} A_iA_j^2}}{\sum_{1\leq i<j\leq n+1} A_iA_j^2}$$
$$=\frac{(n+1)^2}{\sqrt{\sum_{1\leq i<j\leq n+1} A_iA_j^2}}\leq\frac{(n+1)^2}{\sqrt{\frac{n(n+1)}{2}\cdot\left(\prod_{1\leq i<j\leq n+1} A_iA_j^2\right)^{\frac{1}{n(n+1)}}}}$$
$$=\frac{\sqrt{2}(n+1)^2}{\sqrt{n(n+1)}}\cdot\left(\prod_{1\leq i<j\leq n+1}\frac{1}{A_iA_j}\right)^{\frac{1}{n(n+1)}}$$
$$\leq\frac{\sqrt{2}(n+1)^2}{\sqrt{n(n+1)}}\cdot\frac{1}{n(n+1)}\cdot\sum_{1\leq i<j\leq n+1}\frac{1}{A_iA_j}$$
$$=\frac{\sqrt{2n(n+1)}}{n^2}\cdot\sum_{1\leq i<j\leq n+1}\frac{1}{A_iA_j},$$

which completes the proof.

If $n=3$ in inequality (4), then inequality (1) follows from inequality (4). If we take $n=2$ in inequality (4), we get the following corollary.

Corrolary 2.1. *Let a triangle $A_1A_2A_3$ with the center G be inscribed in a circle of radius R. The lines A_1G, A_2G, A_3G meet the circle again at A_1', A_2', A_3' respectively. Then*

$$\frac{3}{R}\leq\frac{1}{GA_1'}+\frac{1}{GA_2'}+\frac{1}{GA_3'}\leq\sqrt{3}\left(\frac{1}{A_1A_2}+\frac{1}{A_2A_3}+\frac{1}{A_3A_1}\right).$$

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