THE LOCAL DENSITY AND THE LOCAL
WEAK DENSITY OF HYPERSPACES

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Abstract. In the paper the local density and the local weak density of
hyperspaces, are investigated.

1. Introduction

Let $X$ be a $T_1$-space. The set of all nonempty closed subsets of a space $X$
denote by $\exp X$. The family of all sets in the form $O\langle U_1, ..., U_n \rangle = \{ F : F \in \exp X, F \subset \bigcup_{i=1}^{n} U_i, F \cap U_i \neq \emptyset, i = 1, 2, ..., n \}$, where $U_1, ..., U_n$ is a sequence of open sets in $X$, generates a topology on the set
$\exp X$. This topology is called the Vietoris topology. The set $\exp X$ with
the Vietoris topology is called the exponential space or the hyperspace of $X$
[1]. Denote by $\exp_n X$ the family of all nonempty finite subsets of a space
$X$, consisting of at most $n$ elements, i.e. $\exp_n X = \{ F \in \exp X : |F| \leq n \}$. Denote by $\exp_c X$ the family of all finite subsets of $X$. Denote by $\exp_c X$
the family of all closed compact subsets of $X$.

Definition 1.1. The weak density of a topological space $X$ is the smallest
cardinal number $\tau \geq \aleph_0$ such that there is a $\pi$-base in $X$
coinciding with $\tau$ centered systems of open sets, i.e. there is a $\pi$-base $B = \bigcup \{ B_\alpha : \alpha \in A \}$, where $B_\alpha$ is a centered
system of open sets for each $\alpha \in A$ and $|A| = \tau$.

The weak density of a topological space $X$ is denoted by $wd(X)$. If
$wd(X) = \aleph_0$ then we say that a topological space $X$ is weakly separable.

Definition 1.2. We say that a topological space $X$ is locally separable at a
point $x \in X$ if $x$ has a separable neighborhood.

A topological space is locally separable if it is locally separable at each
point $x \in X$.

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**Definition 1.3.** We say that a topological space \( X \) is locally \( \tau \)-dense at a point \( x \in X \) if \( \tau \) is the smallest cardinal number such that \( x \) has a \( \tau \)-dense neighborhood in \( X \).

The local density at a point \( x \) is denoted by \( ld(x) \). The local density of a space \( X \) is defined as the supremum of all numbers \( ld(x) \) for \( x \in X \); this cardinal number is denoted by \( ld(X) \).

**Definition 1.4.** A topological space is locally weakly \( \tau \) dense at a point \( x \in X \) if \( \tau \) is the smallest cardinal number such that \( x \) has a neighborhood of weak density \( \tau \) in \( X \).

The local weak density at a point \( x \) is denoted by \( lwd(x) \).

The local weak density of a topological space \( X \) is defined with following way: \( lwd(X) = \sup \{lwd(x) : x \in X \} \).

**Theorem 1.1.** Let \( X \) be an infinite \( T_1 \)-space. \( O \{V_1, V_2, \ldots, V_k\} \subset \subset \) \( O \{U_1, U_2, \ldots, U_n\} \) iff \( \bigcup_{i=1}^{k} V_i \subset \bigcup_{j=1}^{n} U_j \) and for each \( j \in \{1, 2, \ldots, n\} \) there exists \( i \in \{1, 2, \ldots, k\} \) such that \( V_i \subset U_j \).

**Theorem 1.2.** For any infinite \( T_1 \)-space \( X \)
\[ \text{wd}(X) = \text{wd}(exp_n X) = \text{wd}(exp_{\omega} X) = \text{wd}(exp_\omega X) = \text{wd}(exp X). \]

2. MAIN RESULTS

**Question 2.1.** Is it true that for the local density of a \( T_1 \)-space \( X \) following equalities hold:
\[ ld(X) = ld(exp_n X) = ld(exp_{\omega} X) = ld(exp_\omega X) = ld(exp X). \]

**Theorem 2.1.** Let \( X \) be an infinite \( T_1 \)-space. Then
\[ ld(X) = ld(exp_n X) = ld(exp_{\omega} X) = ld(exp_\omega X). \]

**Proof.** 1) Firstly, we shall show that \( ld(X) = ld(exp_n X) \).

a) we shall prove that \( ld(exp_n X) \leq ld(X) \). Suppose \( ld(X) = \tau \geq \aleph_0 \) and \( F \in \text{exp}_n X \) is an arbitrary element of the set \( \text{exp}_n X \). We must show \( ld(F) \leq \tau \). For convenience, assume that a set \( F = \{x_1, x_2, \ldots, x_n\} \) consists of exactly \( n \) distinct points. Then there exist neighborhoods \( O_{x_1}, O_{x_2}, \ldots, O_{x_n} \) of points \( x_1, x_2, \ldots, x_n \) respectively, such that \( d(O_i, x_i) \leq \tau \), \( i = 1, 2, \ldots, n \). Let \( M_1, M_2, \ldots, M_n \) be dense subsets of \( O_{x_1}, O_{x_2}, \ldots, O_{x_n} \) respectively, such that \( |M_i| \leq \tau \) for \( i = 1, 2, \ldots, n \).

Then clearly, the set \( M = \bigcup_{i=1}^{n} M_i \) is dense subset of the union \( \bigcup_{i=1}^{n} O_i x_i \) and \( |M| \leq \tau \). Consider the family \( \mu = \{F \in \text{exp}_n X : F \subset \bigcup_{i=1}^{n} M_i \} \). It is obvious that \( |\mu| \leq \tau \). We shall show that \( \mu \) is dense in \( O_{i=1}^{k} O_{x_1}, O_{x_2}, \ldots, O_{x_n} \).

Let \( O(V_1, V_2, \ldots, V_k) \) \((k \leq n)\) be an arbitrary nonempty open set of \( O_{i=1}^{k} O_{x_1}, O_{x_2}, \ldots, O_{x_n} \). Then it is clear that \( O(V_1, V_2, \ldots, V_k) \) is open in \( \text{exp}_n X \). By theorem 1.1 \([5]\) we have \( \bigcup_{i=1}^{k} V_i \subset \bigcup_{j=1}^{n} O_j x_j \). It implies that \( V_i \subset \bigcup_{j=1}^{n} O_j x_j \) for each \( i = 1, 2, \ldots, k \). Since the set \( M \) is dense in \( \bigcup_{i=1}^{n} O_i x_i \),
each $V_i$ intersects $M$. Let us choose a point $y_i \in V_i \cap M$ for each $i = 1, 2, \ldots, k$. Then $K = \{y_1, y_2, \ldots, y_k\} \in \mu$ and $K \in O\{V_1, V_2, \ldots, V_k\}$. Thus the set $\mu$ is dense in $O\{O_1x_1, O_2x_2, \ldots, O_nx_n\}$. Inequality $\text{ld}(\exp_n X) \leq \text{ld}(X)$ is proved.

b) Now we shall show $\text{ld}(\exp_n X) \geq \text{ld}(X)$. Suppose that $\text{ld}(\exp_n X) = \tau \geq \aleph_0$. We must prove that $\text{ld}(X) \leq \tau$. Consider an arbitrary point $x \in X$. It is clear that $\{x\} \in \exp_n X$. Then there is a neighborhood $O\{U\{x\}\}$ in $\exp_n X$ such that $d(O\{U\{x\}\}) \leq \tau$. Assume that $S = \{F_\alpha : \alpha \in A\}$ is a dense set in $O\{U\{x\}\}$ such that $|S| \leq \tau$. Take a point $x_\alpha \in F_\alpha$ from each set $F_\alpha$. Put $B = \{x_\alpha : x_\alpha \in F_\alpha, F_\alpha \in S\}$. Then it is clear that $|B| \leq \tau$. We shall show that $B$ is dense in $U\{x\}$. Suppose $G \subset U\{x\}$ is any nonempty open subset of $U\{x\}$. Then $O\{G\}$ is an open subset of $O\{U\{x\}\}$. Since $S$ is dense in $O\{U\{x\}\}$, there exists an element $F_\alpha \in S$ such that $F_\alpha \in O\{G\}$. It is easy to see that $F_\alpha \subset G$. According to the choice of the point we have $x_\alpha \in F_\alpha \subset G$. Thus the $B$ is dense in $U\{x\}$. Since the point $x \in X$ is arbitrary, we have $\text{ld}(X) \leq \tau$. This proves the inequality $\text{ld}(\exp_n X) \geq \text{ld}(X)$. From the sections a) and b) we obtain $\text{ld}(X) = \text{ld}(\exp_n X)$.

2) The proof of the equality $\text{ld}(X) = \text{ld}(\exp_n X)$ is analogous to the proof of section 1). Now we shall show the equality $\text{ld}(X) = \text{ld}(\exp_n X)$.

a) We shall prove $\text{ld}(\exp_n X) \leq \text{ld}(X)$. Assume that $\text{ld}(X) = \tau \geq \aleph_0$. Take an arbitrary element $F \in \exp_n X$. Then $F \subset X$ is compact subset of $X$. Since $\text{ld}(X) \leq \tau$, for each element $x \in F$ there exists a neighborhood $Ox$ such that $d(Ox) \leq \tau$. Suppose that a point $x$ runs over the set $F$, then the system $\{O_x x_\alpha : x_\alpha \in F\}$ covers the set $F$. Since $F$ is compact, there exist finitely many sets $O_1x_1, O_2x_2, \ldots, O_kx_k$ such that $\bigcup_{i=1}^{k} O_i x_i \supseteq F$ and $d(O_i x_i) \leq \tau$ for each $i = 1, 2, \ldots, k$. It is clear that $O\{O_1x_1, O_2x_2, \ldots, O_kx_k\}$ contains the compact set $F$. We shall show that $d(O\{O_1x_1, O_2x_2, \ldots, O_kx_k\}) \leq \tau$. Assume that $M_1 = \{x_\alpha^1 : \alpha_1 \in A_1\}$, $M_2 = \{x_\alpha^2 : \alpha_2 \in A_2\}$, $M_k = \{x_\alpha^k : \alpha_k \in A_k\}$ are dense subsets of sets $O_1x_1, O_2x_2, \ldots, O_kx_k$ respectively and $|M_i| \leq \tau$ for each $i = 1, 2, \ldots, k$. Put $M = \bigcup_{i=1}^{k} M_i$. It is clear that $M$ is dense in $\bigcup_{i=1}^{k} O_i x_i$ and $|M| \leq \tau$. Consider the family $\mu = \{G \subset M : |G| < \aleph_0\}$. We clearly have $|\mu| \leq \tau$. Now we shall show that $\mu$ is dense in $O\{O_1x_1, O_2x_2, \ldots, O_kx_k\}$. Indeed, let us take an arbitrary nonempty open subset $O\{U_1, U_2, \ldots, U_n\} \subset O\{O_1x_1, O_2x_2, \ldots, O_kx_k\}$ in $O\{O_1x_1, O_2x_2, \ldots, O_kx_k\}$. By theorem 1.1[5] we have $\bigcup_{i=1}^{k} U_i \supset \bigcup_{i=1}^{k} O_i x_i$.

This implies that $U_j \subset \bigcup_{i=1}^{k} O_i x_i$ for each $j = 1, 2, \ldots, n$. Since $M$ is dense in $\bigcup_{i=1}^{k} O_i x_i$, we see that $U_1 \cap M \neq \emptyset, U_2 \cap M \neq \emptyset, U_n \cap M \neq \emptyset$. Take a point $x_i$ from each intersection $U_i \cap M$ for $i = 1, 2, \ldots, n$ and we construct the set $K = \{x_1, x_2, \ldots, x_n\}$. We have clearly $K = \{x_1, x_2, \ldots, x_n\} \in O\{U_1, U_2, \ldots, U_n\} \cap \mu$. 


We have proven that $\mu$ is dense in $O(O_1x_1, O_2x_2, \ldots, O_kx_k)$. This implies that $d(O(O_1x_1, O_2x_2, \ldots, O_kx_k)) \leq \tau$. The inequality $ld(\exp_n X) \leq ld(X)$ is proved.

b) Now we shall prove that $ld(X) \leq ld(\exp_n X)$. Suppose $ld(\exp_n X) = \tau \geq N_0$. We must prove that $ld(X) \leq \tau$. Take an arbitrary point $x \in X$. Since $\{x\}$ is compact, it is clear that $\{x\} \in \exp_n X$. Then there is a neighborhood $O(U\{x\})$ of the point $\{x\}$ in $\exp X$ such that $d(O(U\{x\})) \leq \tau$. Assume that the set $S = \{F_\alpha : \alpha \in A\}$ is dense in $O(U\{x\})$, where $|S| \leq \tau$. Choose a point $x_\alpha \in F_\alpha$ from each set $F_\alpha$. Put $B = \{x_\alpha : x_\alpha \in F_\alpha, F_\alpha \in S\}$. We clearly have $|B| \leq \tau$. We shall show that the set $B$ is dense in $U\{x\}$. Let $G \subseteq U\{x\}$ be any nonempty open subset of $U\{x\}$. Then $O(G)$ is an open subset of $O(U\{x\})$. Since $S$ is dense in $O(U\{x\})$, there exists an element $F_\alpha \in S$ such that $F_\alpha \in O(G)$. Then by the choice of the points $x_\alpha$ we have $x_\alpha \in F_\alpha \subseteq G$. Thus $B$ is dense in $U\{x\}$. Since the point $x \in X$ is arbitrary, we have $ld(X) \leq \tau$. From sections a) and b) we obtain $ld(X) = ld(\exp_n X)$. Theorem 2.1 is proved.

Corollary 2.1. Let $X$ be an infinite compact $T_1$-space. Then

$$ld(X) = ld(\exp_n X) = ld(\exp_{n-1} X) = ld(\exp X).$$

Corollary 2.2. Functors $\exp_n$, $\exp_{n-1}$, $\exp_n$ preserve the local density of infinite $T_1$-spaces. Moreover, the functor $\exp$ preserves the local density in the category of compact spaces.

Proposition 2.1. Let $X$ be an infinite topological space and $U_1, U_2, \ldots, U_n$ are its open subsets such that $wd(U_i) \leq \tau$, $i = 1, 2, \ldots, n$, where $\tau$ is some infinite cardinal number. Then $wd\left(\bigcup_{i=1}^{n} U_i\right) \leq \tau$.

Proof. Assume that the system $\gamma_i = \bigcup_{\alpha \in A} \gamma_i^{(i)}$, where $|A| \leq \tau$, is a $\pi$-base coinciding with $\tau$ centered systems $\gamma_i^{(i)}$ in $U_i$ for $i = 1, 2, \ldots, n$. Then the system $\gamma = \bigcup_{i=1}^{n} \gamma_i$ is a $\pi$-base. Indeed, suppose that $V$ is any nonempty open subset of the space $\bigcup_{i=1}^{n} U_i$. Then there exists $i \in \{1, 2, \ldots, n\}$ such that $V \cap U_i \neq \emptyset$ and this intersection is open in the subspace $U_i$. Since $\gamma_i$ is a $\pi$-base in $U_i$, there exists an element $G$ from $\gamma_i \subseteq \gamma$ such that $G \subseteq V \cap U_i$. Therefore $\gamma$ is a $\pi$-base in $\bigcup_{i=1}^{n} U_i$. Moreover, the system $\gamma$ can be represented as the union of $\tau$ centered systems of open sets. This implies that $wd\left(\bigcup_{i=1}^{n} U_i\right) \leq \tau$.

Proposition 2.1 is proved.

Now for an element $O = O(U_1, U_2, \ldots, U_n)$ of the base of $\exp_n X$ put $S(O) = \{U_1, U_2, \ldots, U_n\}$, where $U_1, U_2, \ldots, U_n$ are open sets in $X$.

Proposition 2.2. Suppose that the system $\Delta = \{O_\beta = O_\beta^{(\beta)} \cup U_1^{\beta}, U_2^{\beta}, \ldots, U_n^{\beta} : \beta \in B\}$, where $U_i^{\beta}$ are open sets in $X$ for $\beta \in B$ and $i = 1, n$, is a centered system of open subsets of $\exp_n X$. Then the family $\mu = \{W_\beta = \bigcup S(O_\beta) : O_\beta \in \Delta, \beta \in B\}$ is a centered system of open sets in $X$. 
Proof. Suppose that proposition 2.1 does not hold, i.e. there exists a finite sequence \( W_{\beta_1}, W_{\beta_2}, ..., W_{\beta_k} \) of elements from \( \mu \) with empty intersection. But, since the system \( \Delta \) is centered in \( \exp_\omega X \), we have
\[
\bigcap_{j=1}^{k} O\left(U_{j}^{\beta_1}, U_{j}^{\beta_2}, ..., U_{j}^{\beta_k}\right) \neq \emptyset.
\]
Then there exists \( F \in \exp_\omega X \) such that
\[
F \subset \bigcup_{i=1}^{n} U_{i}^{\beta_i}
\]
for each \( j = 1, 2, ..., k \). This implies that
\[
F \subset \bigcap_{j=1}^{k} \left( \bigcup_{i=1}^{n} U_{i}^{\beta_j}\right) = \bigcap_{j=1}^{k} W_{\beta_j}.
\]
This contradiction proves that the system \( \mu \) is centered. Proposition 2.2 is proved.

Question 2.2. Which of the following equalities hold for the local weak density:
\[
lwd(X) = lwd(\exp_\omega X) = lwd(\exp_\omega X) = lwd(\exp_\omega X) = lwd(\exp_\omega X)?
\]

Theorem 2.2. Let \( X \) be an infinite \( T_1 \)-space. Then
\[
lwd(X) = lwd(\exp_\omega X) = lwd(\exp_\omega X) = lwd(\exp_\omega X).
\]

Proof. Firstly, we shall show that \( lwd(X) = lwd(\exp_\omega X) \).

Suppose \( lwd(X) = \tau \geq \kappa_0 \). We shall show that \( lwd(\exp_\omega X) \leq \tau \).
Take an arbitrary element \( F \in \exp_\omega X \). Assume that \( F = \{x_1, x_2, ..., x_n\} \in \exp_\omega X \), where \( x_1, x_2, ..., x_n \in X \). Since \( lwd(X) = \tau \geq \kappa_0 \), there exist neighborhoods \( Ox_1, Ox_2, ..., Ox_n \) of points \( x_1, x_2, ..., x_n \) respectively, such that \( wd(Ox_i) \leq \tau \) for each \( i = 1, 2, ..., n \). Then by proposition 2.2 we have
\[
wd\left( \bigcup_{i=1}^{n} Ox_i \right) \leq \tau.
\]
We must prove \( wd(O\langle Ox_1, Ox_2, ..., Ox_n \rangle) \leq \tau \). Suppose that \( \mu = \bigcup_{\alpha \in A} \mu_{\alpha} \) is a \( \pi \)-base for \( \bigcup_{i=1}^{n} Ox_i \), coinciding with \( \tau \) centered systems \( \mu_{\alpha_{i}} \), i.e. \( |A| \leq \tau \) and for each \( \alpha \in A \) the system \( \mu_{\alpha} \) is centered. By \( \Sigma \) denote the family of all finite subsets of the index set \( A \). Then it is clear that \( |\Sigma| \leq \tau \). Let \( M \) be the system of all finite subfamilies of the family \( \mu \). Put \( O(M) = \{O\langle W_{\alpha}^{1}, W_{\alpha}^{2}, ..., W_{\alpha}^{m} \rangle \} : \{W_{\alpha}^{1}, W_{\alpha}^{2}, ..., W_{\alpha}^{m} \} \in M\} \).
We shall show that \( O(M) \) can be represented as the union of \( \tau \) centered systems and is a \( \pi \)-base for \( O\langle Ox_1, Ox_2, ..., Ox_n \rangle \). Take an arbitrary open subset \( O\langle U_1, U_2, ..., U_k \rangle \) of \( O\langle Ox_1, Ox_2, ..., Ox_n \rangle \). By theorem 2.1 we have
\[
U_j \subset \bigcup_{i=1}^{n} Ox_i \text{ for } j = 1, 2, ..., k. \text{ Since } \mu \text{ is a } \pi \text{-base in } \bigcup_{i=1}^{n} Ox_i \text{, there exists an element } G_j \text{ from } \mu \text{ such that } G_j \subset U_j \text{ for each } j = 1, 2, ..., k. \text{ Then it is clear that } O\langle G_1, G_2, ..., G_k \rangle \subset O\langle U_1, U_2, ..., U_k \rangle \text{ and } O\langle G_1, G_2, ..., G_k \rangle \subset O(M). \text{ Therefore } O(M) \text{ is a } \pi \text{-base in } O\langle Ox_1, Ox_2, ..., Ox_n \rangle. \text{ Now let us show that } O(M) \text{ can be represented as the union of } \tau \text{ centered systems of open sets in } O\langle Ox_1, Ox_2, ..., Ox_n \rangle. \text{ For each } \psi \in \Sigma \text{ put } O_\psi(M) = \{O\langle W_{\alpha_1}^{1}, W_{\alpha_2}^{2}, ..., W_{\alpha_m}^{m} \rangle \in O(M) : \{\alpha_1, \alpha_2, ..., \alpha_m\} = \psi\}. \text{ Then this system is centered for every } \psi \in \Sigma \text{ and, clearly } \bigcup_{\psi \in \Sigma} O_\psi(M) = O(M). \text{ Indeed, let us take an arbitrary finite sequence of elements from } O_\psi(M): O\langle W_{\alpha_1}^{1}, W_{\alpha_2}^{1}, ..., W_{\alpha_m}^{1} \rangle, O\langle W_{\alpha_1}^{2}, W_{\alpha_2}^{2}, ..., W_{\alpha_m}^{2} \rangle, \ldots \ldots ,
$O \left< W^{(r)}_{\alpha_1}, W^{(r)}_{\alpha_2}, \ldots, W^{(r)}_{\alpha_m} \right>$, where $r$ is some natural number. Since every system $\mu_\alpha$ is centered, we have $\bigcap_{j=1}^{r} W^{(j)}_{\alpha_i} \neq \emptyset$ for $i = 1, 2, \ldots, m$. Choose a point $y_i$ from the intersection for each $i = 1, 2, \ldots, m$ and form the set $E = \{y_1, y_2, \ldots, y_m\}$. For each $j = 1, 2, \ldots, r$ we have $E \subset \bigcup_{i=1}^{m} W^{(j)}_{\alpha_i}$ and $E \cap W^{(j)}_{\alpha_i} \neq \emptyset$, $i = 1, 2, \ldots, m$. This implies that $E \in \bigcap_{j=1}^{r} O \left< W^{(j)}_{\alpha_1}, W^{(j)}_{\alpha_2}, \ldots, W^{(j)}_{\alpha_m} \right>$. We have shown that any finite sequence of elements of $O_\psi(M)$ has nonempty intersection. Therefore $O_\psi(M)$ is centered for each $\psi \in \Sigma$ and, consequently, we obtain $\text{wd}(O(Ox_1, Ox_2, \ldots, Ox_n)) \leq \tau$. The inequality $\text{wd}(\exp_n X) \leq \tau$ is proved.

b) Assume that $\text{wd}(\exp_n X) = \tau \geq \aleph_0$. We shall show that $\text{wd}(X) \leq \tau$. Take an arbitrary point $x \in X$. Then $\{x\} \in \exp_n X$. From $\text{wd}(\exp_n X) = \tau$ it follows that there exists a neighborhood $O(U\{x\})$ of the point $\{x\}$ such that $\text{wd}(O(U\{x\})) \leq \tau$, where $U\{x\}$ is an open set in $X$. Let us prove that $\text{wd}(U\{x\}) \leq \tau$. From $\text{wd}(O(U\{x\})) \leq \tau$ it follows that $O(U\{x\})$ has a $\pi$-base $O = \bigcup_{\alpha \in A} O_\alpha$, where the system $O_\alpha = \{O \left< U_1^\beta, U_2^\beta, \ldots, U_n^\beta \right> : \beta \in A_\alpha \}$ is centered for each $\alpha \in A$ and $|A| \leq \tau$. For each $\alpha \in A$ consider the system $\mu_\alpha = \{W_\beta = \bigcup_{i=1}^{n} U_i^\beta : \beta \in A_\alpha \}$ of open sets in $U\{x\}$. Then by proposition 2.2 the system $\mu_\alpha$ is centered for each $\alpha \in A$. Now let us show that the system $\mu = \bigcup_{\alpha \in A} \mu_\alpha$ is a $\pi$-base in $U\{x\}$. Suppose that $G \subset U\{x\}$ is any nonempty open subset of $U\{x\}$. Then $O(G)$ is a nonempty open set in $\exp_n X$ and $O(G) \subset O(U\{x\})$. Since the system $O$ is a $\pi$-base in $O(U\{x\})$, there exists $O \left< U_1^\beta, U_2^\beta, \ldots, U_n^\beta \right> \in O$ such that $O \left< U_1^\beta, U_2^\beta, \ldots, U_n^\beta \right> \subset O(G)$. This implies that $G \supset \bigcup_{i=1}^{n} U_i^\beta = W_\beta$. The set $W_\beta$ is contained in $\mu$. Therefore $\mu$ is a $\pi$-base in $U\{x\}$. We constructed a $\pi$-base coinciding with $\tau$ centered systems in $U\{x\}$. The inequality $\text{wd}(X) \leq \tau$ is proved. From a) and b) it follows that $\text{wd}(X) = \text{wd}(\exp_n X)$. Analogously we can prove equalities $\text{wd}(X) = \text{wd}(\exp_n X)$ and $\text{wd}(X) = \text{wd}(\exp_c X)$. Theorem 2.2 is proved.

Corollary 2.3. Let $X$ be an infinite compact $T_1$-space. Then $\text{wd}(X) = \text{wd}(\exp_n X) = \text{wd}(\exp_c X) = \text{wd}(\exp X)$.

Corollary 2.4. Functors $\exp_n$, $\exp_c$, $\exp_n$ preserve the locally weak density of any infinite $T_1$-space. Moreover, the functor $\exp_n$ preserves the local weak density in the category of compact spaces.

Theorem 2.3. (Hewitt-Marczewski-Pondiczery) [3]. If $d(X_s) \leq \tau$ for every $s \in S$ and $|S| \leq 2^\gamma$, then $d\left( \prod_{s \in S} X_s \right) \leq \tau$.

Let $\tau$ be an infinite cardinal number. Consider a family of topological spaces $\{X_s : s \in S\}$, where $|S| \leq 2^\gamma$. 
Proposition 2.3. If every space \( X_s \) is locally \( \tau \)-dense and there exists a finite subset \( S_0 \) of the index set \( S \) such that \( X_s \) is \( \tau \)-dense for all \( s \in S \setminus S_0 \), then the product \( \prod_{s \in S} X_s \) is locally \( \tau \)-dense.

Proof. Take an arbitrary point \( x = \{ x_s : s \in S \} \) from the product \( \prod_{s \in S} X_s \).
Since all the spaces \( X_s \) are locally \( \tau \)-dense, the point \( x_s \in X_s \) has a neighborhood \( U_s \) of density \( \leq \tau \) for every \( s \in S_0 \). The set \( \prod_{s \in S_0} U_s \times \prod_{s \in S \setminus S_0} X_s \) is a neighborhood of the point \( x \) in \( \prod_{s \in S} X_s \) and by theorem 2.3 we have
\[
d \left( \prod_{s \in S_0} U_s \times \prod_{s \in S \setminus S_0} X_s \right) \leq \tau.
\]
So, we have found a \( \tau \)-dense neighborhood of the point \( x \) in \( \prod_{s \in S} X_s \). The point \( x \) was chosen arbitrarily, therefore the product \( \prod_{s \in S} X_s \) is locally \( \tau \)-dense. Proposition 2.3 is proved.

Corollary 2.5. Consider the family of topological spaces \( \{ X_s : s \in S \} \), where \( |S| \leq 2^{\aleph_0} \). If all the spaces \( X_s \) are locally separable and there exists a finite subset \( S_0 \) of the index set \( S \) such that \( X_s \) is separable for \( s \in S \setminus S_0 \), then the product \( \prod_{s \in S} X_s \) is locally separable.

It seems, the inverse statement is also true

Theorem 2.4. Suppose that the product \( \prod_{s \in S} X_s \) is locally \( \tau \)-dense. Then there exists a finite subset \( S_0 \) of the index set \( S \) such that \( X_s \) is \( \tau \)-dense for every \( s \in S \setminus S_0 \).

Note that in the inverse statement the condition \( |S| \leq 2^\tau \) is omitted.

Proof. Take an arbitrary point \( x = \{ x_s : s \in S \} \) from the product \( \prod_{s \in S} X_s \).
Since the product \( \prod_{s \in S} X_s \) is locally \( \tau \)-dense, the point \( x \) has a \( \tau \)-dense neighborhood \( \prod_{s \in S_0} U_s \times \prod_{s \in S \setminus S_0} X_s \) (since the density is hereditary with respect to open subsets, we can assume that the neighborhood is from the canonical base of \( \prod_{s \in S} X_s \)), where \( S_0 \) is a finite subset of \( S \) and \( x_s \in U_s \). Since the density is preserved under open mappings and the projection is an open map, we see that \( X_s \) is \( \tau \)-dense for every \( s \in S \setminus S_0 \). Theorem 2.4 is proved.

Corollary 2.6. Suppose that the product \( \prod_{s \in S} X_s \) is locally separable. Then there exists a finite subset \( S_0 \) of \( S \) such that \( X_s \) is separable for every \( s \in S \setminus S_0 \).
The local density and the local weak density of hyperspaces

REFERENCES


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