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SLANT PSEUDO-LINES IN THE HYPERBOLIC PLANE

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1. Introduction

We consider the Poincaré disk model D of the hyperbolic plane which is conformally equivalent to the Euclidean plane, so that a circle or a line in the Poincaré disk is also a circle or a line in the Euclidean plane. A geodesic in the Poincaré disk is a Euclidean circle or a line perpendicular to the ideal boundary (i.e., the unit circle). If we adopt geodesics as lines in the Poincaré disk, we have the model of the hyperbolic geometry. We have another class of curves in the Poincaré disk which have an analogous property with lines in the Euclidean plane. A *horocycle* is an Euclidean circle which is tangent to the ideal boundary. We remark that a line in the Euclidean plane can be considered as a limit of circles when the radii tend to infinity. A horocycle is also a curve as a limit of circles when the radii tend to infinity in the Poincaré disk. If we adopt horocycles as lines, what kind of geometry do we obtain? We say that two horocycles are *parallel* if they have the common tangent point at the ideal boundary. Under this definition, the axiom of parallel is satisfied. However, for any two points in the disk, there are always two horocycles passing through the points, so that the axiom 1 of Euclidean Geometry is not satisfied. We call this geometry a *horocyclic geometry* (a *horospherical geometry* for the higher dimensional case [2, 3, 4]). However, we have another kind of curves with the properties similar to those of Euclidean lines. A curve in the Poincar´e disk is called an *equidistant curve* if it is a Euclidean circle or a Euclidean line whose intersection with the ideal boundary consists of two points. We define a one parameter family of Euclidean circles (or lines) depending on $\phi \in [0, \pi/2]$. A geodesic is the special case with $\phi = \pi/2$ and a horocycle is the case with $\phi = 0$. In this paper a geodesic is called a *vertical pseudo-line* and a horocycle is called a *horizontal pseudo-line*. For $\phi \in (0, \pi/2]$, the corresponding pseudo-line is an equidistant curve, which we call a *ϕ-slant pseudo-line*.

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If we consider a ϕ-slant pseudo-line as a line, we have a family of geometry in the hyperbolic plane connecting the hyperbolic geometry and the horocyclic geometry which is called a *slant geometry*(cf., [1]). Here we have the natural question what ϕ is. In this paper we give an answer to this question. Actually, ϕ is the angle between a ϕ-slant pseudo-line and the ideal boundary (cf., Theorem 4.1) at an intersection point.

2. Basic concepts

We use the model of the hyperbolic plane in the Lorentz-Minkowski 3-space. We prepare basic notions on the Lorentz-Minkowski 3-space. Let $\mathbb{R}^3$ be a 3-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, x_2)$, $\boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3$ a *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2$. The space $(\mathbb{R}^3, \langle , \rangle)$ is called a *Lorentz-Minkowski* 3-*space* which is denoted by $\mathbb{R}^3_1$. We assume that $\mathbb{R}^3_1$ is time-oriented and choose $e_0 = (1, 0, 0)$ as the *future timelike vector*. We say that a non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^3_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0, = 0$ or < 0 respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. Given a non-zero vector $\boldsymbol{n} \in \mathbb{R}^3_1$ and a real number c, a *plane* with pseudo normal $\boldsymbol{n}$ is defined by

$$P(\boldsymbol{n}, c) = \{\boldsymbol{x} \in \mathbb{R}^3_1 \mid \langle \boldsymbol{x}, \boldsymbol{n} \rangle = c\}$$

We say that $P(\boldsymbol{n}, c)$ is *spacelike* , *timelike* or *lightlike* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. For any vectors $\boldsymbol{x} = (x_0, x_1, x_2)$, $\boldsymbol{y} = (y_0, y_1, y_2) \in \mathbb{R}^3_1$, *pseudo exterior product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be

$$\boldsymbol{x} \wedge \boldsymbol{y} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{vmatrix} = (-(x_1y_2 - x_2y_1), x_2y_0 - x_0y_2, x_0y_1 - x_1y_0),$$

where $\{\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ is the canonical basis of $\mathbb{R}^3_1$. We also define a *hyperbolic plane* by

$$H^2_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^3_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}.$$

We remark that $H^2_+(-1)$ is a Riemannian manifold if we consider the induced metric from $\mathbb{R}^3_1$.

We now consider a plane defined by $\mathbb{R}^2_0 = \{(x_0, x_1, x_2) \in \mathbb{R}^3_1 \mid x_0 = 0\}$. Since $\langle , \rangle|_{\mathbb{R}^2_0}$ is the canonical Euclidean scaler product, we call $\mathbb{R}^2_0$ a *Euclidean plane*. We adopt coordinates (x_1, x_2) of $\mathbb{R}^2_0$ instead of $(0, x_1, x_2)$. On the Euclidean plane $\mathbb{R}^2_0$, we have the Poincaré disc model of the hyperbolic plane. We consider a unit open disc $D = \{x \in \mathbb{R}^2_0 \mid \|\boldsymbol{x}\| < 1\}$ and consider a Riemannian metric

$$ds^2 = \frac{4(dx_1^2 + dx_2^2)}{1 - x_1^2 - x_2^2}.$$

Define a mapping $\Psi : H^2_+ \longrightarrow D$ by

$$\Psi(x_0, x_1, x_2) = \left(\frac{x_1}{x_0 + 1}, \frac{x_2}{x_0 + 1} \right).$$

It is known that Ψ is an isometry. Moreover, the Poincaré disc model is conformal equivalent to the Euclidean plane.

3. Pseudo-lines in the hyperbolic plane

We consider a curve defined by the intersection of Hyperbolic plane with a plane in the Lorentz-Minkowski 3-space. The image of such a curve by the isometry Ψ is a part of a Euclidean circle or a Euclidean line in the Poincaré disc D. Let $P(\boldsymbol{n}, c)$ a plane with a unit pseudo-normal $\boldsymbol{n}$. We call $H^2_+(-1) \cap P(\boldsymbol{n}, c)$ a *circle*, an *equidistant curve* and a *horocycle* if $\boldsymbol{n}$ is timelike, spacelike or lightlike respectively. Moreover, if $\boldsymbol{n}$ is spacelike and $c = 0$, then we call it a *hyperbolic line* (or, a *geodesic*). We consider a hyperbolic line

$$HL(\boldsymbol{n}) = \{\boldsymbol{x} \in H^2_+(-1) \mid \langle \boldsymbol{x}, \boldsymbol{n} \rangle = 0\}$$

and a horocycle

$$HC(\boldsymbol{l}, -1) = \{\boldsymbol{x} \in H^2_+(-1) \mid \langle \boldsymbol{x}, \boldsymbol{l} \rangle = -1\}.$$

In general, a horocycle is defined by $\langle \boldsymbol{x}, \boldsymbol{l} \rangle = c$ for a lightlike vector $\boldsymbol{l}$ and $c \neq 0$. However, if we choose $-\boldsymbol{l}/c$ instead of $\boldsymbol{l}$, then we have the above equation. For any $\boldsymbol{a}_0 \in HC(\boldsymbol{l}, -1)$, let $\boldsymbol{a}_1$ be a unit tangent vector of $HC(\boldsymbol{l}, -1)$ at $\boldsymbol{a}_0$, so that $\langle \boldsymbol{a}_1, \boldsymbol{l} \rangle = 0$. We define $\boldsymbol{a}_2 = \boldsymbol{a}_0 \wedge \boldsymbol{a}_1$. Then we have a pseudo orthonormal basis $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ of $\mathbb{R}^3_1$ such that $\langle \boldsymbol{a}_0, \boldsymbol{a}_0 \rangle = -1$. Since $\langle \boldsymbol{l} - \boldsymbol{a}_0, \boldsymbol{a}_0 \rangle = \langle \boldsymbol{l}, \boldsymbol{a}_1 \rangle = 0$, we have $\pm \boldsymbol{a}_2 = \boldsymbol{l} - \boldsymbol{a}_0$. We choose the direction of $\boldsymbol{a}_1$ such that $\boldsymbol{a}_2 = \boldsymbol{l} - \boldsymbol{a}_0$. It follows that $A = ({}^t\boldsymbol{a}_0 \; {}^t\boldsymbol{a}_1 \; {}^t\boldsymbol{a}_2) \in SO_0(1,2)$, where

$$SO_0(1,2) = \left\{ A = \begin{pmatrix} a_0^0 & a_0^1 & a_0^2 \\ a_1^0 & a_1^1 & a_1^2 \\ a_2^0 & a_2^1 & a_2^2 \end{pmatrix} \;\middle|\; {}^tAI_{1,2}A = I_{1,2},\ a_0^0 \geq 1 \right\}$$

is the *Lorentz group*, where

$$I_{1,2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For any $A = ({}^t\boldsymbol{a}_0 \; {}^t\boldsymbol{a}_1 \; {}^t\boldsymbol{a}_2) \in SO_0(1,2)$, $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ is a pseudo orthonormal basis of $\mathbb{R}^3_1$. Then $\boldsymbol{l} = \boldsymbol{a}_0 + \boldsymbol{a}_2$ is lightlike. It follows that we have $HC(\boldsymbol{l}, -1) = HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$ such that $\boldsymbol{a}_0 \in HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$ and $\boldsymbol{a}_1$ is tangent to $HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$ at $\boldsymbol{a}_0$. Moreover, we have $\boldsymbol{a}_0 \in HL(\boldsymbol{a}_2)$ and $\boldsymbol{a}_1$ is tangent to $HL(\boldsymbol{a}_2)$ at $\boldsymbol{a}_0$. It is known that a horocycle $\Psi(HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1))$ in the Poincaré disc D is a Euclidean circle tangent to the ideal boundary $S^1 = \{\boldsymbol{x} \in \mathbb{R}^2_0 \mid \|\boldsymbol{x}\| = 1\}$. It is also known that a hyperbolic line $\Psi(HL(\boldsymbol{a}_2))$ is a Euclidean circle or a Euclidean line orthogonal to the ideal boundary (cf., [5]). By these reasons, a horocycle is called a *horizontal pseudo-line* and a pseudo-line is called an *orthogonal pseudo-line* respectively. We now define a *ϕ-slant pseudo-line* by

$$EC(\boldsymbol{n}_\phi, -\cos\phi) = \left\{\boldsymbol{x} \in H^2_+ \mid \langle \boldsymbol{x}, \boldsymbol{n}_\phi \rangle = -\cos\phi\right\},$$

where $\boldsymbol{n}_\phi = \cos\phi \boldsymbol{a}_0 + \boldsymbol{a}_2$, $\phi \in [0, \pi/2]$. Since $\langle \boldsymbol{n}_\phi, \boldsymbol{n}_\phi \rangle = \sin^2\phi > 0$, $EC(\boldsymbol{n}_\phi, -\cos\phi)$ is an equidistant curve. Moreover, $\boldsymbol{a}_0 \in EC(\boldsymbol{n}_\phi, -\cos\phi)$ and $\boldsymbol{a}_1$ is tangent to $EC(\boldsymbol{n}_\phi, -\cos\phi)$ at $\boldsymbol{a}_0$. Then $EC(\boldsymbol{n}_{\pi/2}, -\cos(\pi/2)) = HL(\boldsymbol{a}_2)$ and $EC(\boldsymbol{n}_0, -\cos 0) = HC(\boldsymbol{a}_0 + \boldsymbol{a}_2, -1)$.

4. Slant pseudo-lines

In this section we consider what ϕ is for the ϕ-slant line $EC(\boldsymbol{n}_\phi, -\cos\phi)$.

Theorem 4.1. *For $\phi \in [0, \pi/2]$, the angle between $\Psi(EC(\boldsymbol{n}_\phi, -\cos\phi))$ and the ideal boundary S^1 at an intersection point is equal to ϕ.*

Proof. Firstly, we consider the special case when $\boldsymbol{a}_0 = (1,0,0)$, $\boldsymbol{a}_1 = (0,1,0)$ and $\boldsymbol{a}_2 = (0,0,-1)$. By definition, we have $\boldsymbol{n}_\phi = (\cos\phi, 0, -1)$. We now define $\widetilde{\Psi} : \mathbb{R}^3_1 \setminus \{x_0 = -1\} \longrightarrow \mathbb{R}^2_0$ by

$$\widetilde{\Psi}(x_0, x_1, x_2) = \left(\frac{x_1}{x_0+1}, \frac{x_2}{x_0+1}\right),$$

so that $\widetilde{\Psi}|_{H^2_+(-1)} = \Psi$. Then we have

$$\widetilde{\Psi}(\boldsymbol{a}_0) = (0,0),\ \widetilde{\Psi}(\boldsymbol{a}_1) = (1,0),\ \widetilde{\Psi}(\boldsymbol{a}_2) = (0,-1),\ \widetilde{\Psi}(\boldsymbol{n}_\phi) = \left(0, \frac{-1}{\cos\phi + 1}\right).$$

We now consider $\boldsymbol{a}_0 + \cos\phi\boldsymbol{a}_2 = (1, 0, -\cos\phi)$. Then we have $\langle \boldsymbol{n}_\phi, \boldsymbol{a}_0 + \cos\phi\boldsymbol{a}_2\rangle = 0$. Since $\langle \boldsymbol{a}_0, \boldsymbol{n}_\phi\rangle = -\cos\phi$, $\boldsymbol{a}_0 \in EC(\boldsymbol{n}_\phi, -\cos\phi)$. We remark that $\widetilde{\Psi}(\boldsymbol{a}_0 + \cos\phi\boldsymbol{a}_2) = (0, -(\cos\phi)/2)$ We now consider the inversion $\Phi : \mathbb{R}^2_0 \longrightarrow \mathbb{R}^2_0$ with respect to $S^2(1/2) = \{(x_1, x_2) \mid x_1^2 + x_2^2 = 1/4\}$ defined by

$$\Phi(x_1, x_2) = \left(\frac{x_1}{4(x_1^2 + x_2^2)}, \frac{x_2}{4(x_1^2 + x_2^2)}\right).$$

Then we have

$$\Phi\left(0, -\frac{\cos\phi}{2}\right) = \left(0, \frac{-1}{2\cos\phi}\right).$$

We consider a circle whose center is $\boldsymbol{b} = (0, -1/(2\cos\phi))$ and tangent to x_1-axis, which is defined by

$$S^1(\boldsymbol{b}, \phi) :\ x_1^2 + \left(x_2 + \frac{1}{2\cos\phi}\right)^2 = \frac{1}{4\cos^2\phi}.$$

The intersection of $S^1(\boldsymbol{b}, \phi)$ with S^1 are $(\pm\sin\phi, -\cos\phi)$. The tangent vector $\boldsymbol{t}_1$ of S^1 at $\boldsymbol{v}_1 = (\sin\phi, -\cos\phi) \in S^1$ is orthogonal to $\boldsymbol{v}_1$, so that we have $\boldsymbol{t}_1 = (\cos\phi, \sin\phi)$. We also consider the vector

$$\boldsymbol{v}_2 = (\sin\phi, -\cos\phi) - \left(0, \frac{-1}{2\cos\phi}\right) = \left(\sin\phi, \frac{1 - 2\cos^2\phi}{2\cos\phi}\right).$$

This is a normal vector of $S^1(\boldsymbol{b}, \phi)$ at $\boldsymbol{v}_2$. Therefore, the tangent vector of $S^1(\boldsymbol{b}, \phi)$ at $\boldsymbol{v}_2$ is

$$\boldsymbol{t}_2 = \left(\frac{2\cos^2\phi - 1}{2\cos\phi}, \sin\phi\right).$$

Thus $\|\boldsymbol{t}_2\| = \|\boldsymbol{v}_2\| = 1/2(\cos\phi)$. Let θ be the angle between $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$. Then

$$\langle \boldsymbol{t}_1, \boldsymbol{t}_2\rangle = \|\boldsymbol{t}_1\|\|\boldsymbol{t}_2\|\cos\theta = \|\boldsymbol{t}_2\|\cos\theta = \frac{1}{2\cos\phi}\cos\theta.$$

Moreover,

$$\langle \boldsymbol{t}_1, \boldsymbol{t}_2\rangle = \cos^2\phi - \frac{1}{2} + \sin^2\phi = \frac{1}{2}.$$

It follows that $\cos\theta = \cos\phi$, so that $\theta = \phi$. On the other hand, the inverse mapping $\Psi^{-1} : D \longrightarrow H^2_+(-1)$ of Ψ is given by

$$\Psi^{-1}(u,v) = \left(\frac{1+u^2+v^2}{1-u^2-v^2}, \frac{2u}{1-u^2-v^2}, \frac{2v}{1-u^2-v^2}\right).$$

Moreover, the circle $S^1(\boldsymbol{b},\phi)$ is parametrized by $\gamma(\psi) = \left(\frac{\cos\psi}{2\cos\phi}, \frac{\sin\psi-1}{2\cos\phi}\right)$. We remark that $\gamma(\psi) \in D$ if and only if $1 - 2\cos^2\phi < \sin\psi$. With this condition, $\Psi^{-1}\circ\gamma(\psi)$ is equal to

$$\left(\frac{2\cos^2\phi + 1 - \sin\psi}{2\cos^2\phi - 1 + \sin\psi}, \frac{2\cos\phi\cos\psi}{2\cos^2\phi - 1 + \sin\psi}, \frac{2\cos\phi(\sin\psi - 1)}{2\cos^2\phi - 1 + \sin\psi}\right).$$

Since $\boldsymbol{n}_\phi = (\cos\phi, 0, -1)$, we have $\langle \boldsymbol{n}_\phi, \Psi^{-1}\circ\gamma(\psi)\rangle = -\cos\phi$. This means that $\Psi^{-1}(S^1(\boldsymbol{b},\phi)\cap D) \subset EC(\boldsymbol{n}_\phi, -\cos\phi)$. Direct computation shows the reverse inclusion $\Psi(EC(\boldsymbol{n}_\phi, -\cos\phi)) \subset S^1(\boldsymbol{b},\phi)\cap D$. This completes the proof for the special case.

We remark that the action of $SO_0(1,2)$ on $H^2_+(-1)$ is transitive and it induces an isometry. Let $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ be a pseudo orthonormal basis of $\mathbb{R}^3_1$ with $\boldsymbol{a}_0 \in H^2_+(-1)$. For a ϕ-slant pseudo-line $EC(\boldsymbol{n}_\phi, -\cos\phi)$, we consider a point $\boldsymbol{v} \in S^1 \cap \overline{\Psi(EC(\boldsymbol{n}_\phi, -\cos\phi))}$, where $\overline{X}$ denotes the closure of X. We also consider the family of parallel horocycles in D tangent to $\boldsymbol{v}$. Since two circles in the Euclidean plane have the same angle at the intersection points, the angle between $EC(\boldsymbol{n}_\phi, -\cos\phi)$ and each member of the above family of parallel horocycles at the intersection point is constant. The angle between $EC(\boldsymbol{n}_\phi, -\cos\phi)$ and the ideal boundary S^1 is the limit of the above angle. Since an isometry sends parallel horocycles to parallel horocycles and preserve the angle between two curves at the intersection, the assertion holds for any pseudo orthonormal basis $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2\}$ of $\mathbb{R}^3_1$ with $\boldsymbol{a}_0 \in H^2_+(-1)$. □

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