INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 4 (2015), No. 1, 27 – 36

On the Wittenbauer Type Parallelograms

SÁNDOR NAGYDOBAI KISS

Abstract. Dividing all sides of a quadrilateral into n equal parts and connecting this points of division in a certain mode, we obtain parallelograms, which we will name Wittenbauer type parallelograms. In this paper we calculate their perimeter, area and we examine some properties of these parallelograms.

1. Introduction

If the convex quadrilateral $PQRS$ is given, let A, B, C, D be the midpoints of the sides PQ, QR, RS, SP. The quadrilateral $ABCD$ is a parallelogram, which is called the *Varignon parallelogram of* $PQRS$ [1]. We obtain this parallelogram by dividing all sides of the quadrilateral $PQRS$ into two equal parts and connecting the points of division. Divide the sides of the quadrilateral $PQRS$ into three equal parts. The figure formed by connecting and extending adjacent points on either side of a polygon vertex is a parallelogram known as *Wittenbauer's parallelogram.* Dividing all sides of the quadrilateral $PQRS$ into n equal parts, where n is natural number ($n \geq 2$), and connecting this points of division in a certain mode, we obtain further parallelograms, which we will name *Wittenbauer type parallelograms.* In this paper we will calculate their perimeter, area and we will also examine some properties of these parallelograms.

2. The Wittenbauer's parallelogram

Note with θ the angle ABC and let $AB = 2q$ and $BC = 2p$, where $p > 0$, $q > 0$. Let O be the intersection of diagonals AC and BD of the parallelogram $ABCD$. We attach to the quadrilateral $PQRS$ an oblique coordinate system xOy so that its origin let be the point O, the x-axis parallel with the line BC and the y-axis with the line AB (Figure 1). In this oblique coordinate system xOy, the coordinate of the points A, B, C, D are:

$$A = (-p, q),\ B = (-p, -q),\ C = (p, -q),\ D = (p, q).$$

2010 *Mathematics Subject Classification.* 51M04, 51M16, 51M25.
Key words and phrases. Wittenbauer type parallelograms, oblique coordinates.
Received: 1.08.2014. In revised form: 13.01.2015. Accepted: 16.03.2015.

Let $X=(\alpha,\beta)$ be the midpoint of the diagonal PR. It is well known that if the point Y is the midpoint of the other diagonal QS, then it is the symmetric of X with respect to O. Consequently, $Y=(-\alpha,-\beta)$. Now we determine the coordinates of the vertices of quadrilateral $PQRS$. The side PQ passes through the point A and is parallel with the line BX. Its equation is

$$-(\beta+q)x+(\alpha+p)y=(\beta+q)p+(\alpha+p)q.$$

We obtain the ordinate of the point P for $x=\alpha$, i.e. $y=\beta+2q$. So the coordinates of the points P,Q,R,S are:

$$P=(\alpha,\beta+2q),\ Q=(-\alpha-2p,-\beta),\ R=(\alpha,\beta-2q),\ S=(-\alpha+2p,-\beta).$$

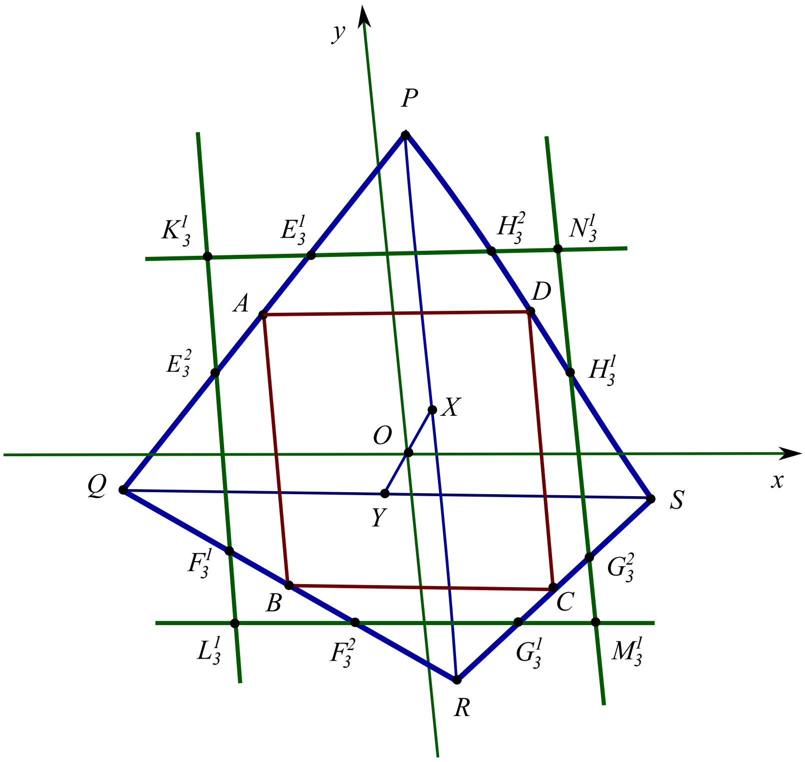

Figure 1

We suppose that the points E_3^1 and E_3^2 divide the side $[PQ]$, the points F_3^1 and F_3^2 divide the side $[QR]$, the points G_3^1 and G_3^2 divide the side $[RS]$ and the points H_3^1 and H_3^2 divide the side $[SP]$ into three equal parts (Figure 1). The order between this points let the following:

$$(P,E_3^1,E_3^2,Q),\ (Q,F_3^1,F_3^2,R),\ (R,G_3^1,G_3^2,S),\ (S,H_3^1,H_3^2,P).$$

We determine the coordinates of the points E_3^1 and E_3^2:

$$\frac{PE_3^1}{E_3^1Q}=\frac{1}{2}\Rightarrow x_{E_3^1}=\frac{x_P+\frac{1}{2}x_Q}{1+\frac{1}{2}}=\frac{\alpha-2p}{3}\ \text{ and }\ y_{E_3^1}=\frac{y_P+\frac{1}{2}y_Q}{1+\frac{1}{2}}=\frac{\beta+4q}{3},$$

$$\frac{PE_3^2}{E_3^2Q}=2\Rightarrow x_{E_3^2}=\frac{x_P+2x_Q}{1+2}=\frac{-\alpha-4p}{3}\ \text{ and }\ y_{E_3^2}=\frac{y_P+2y_Q}{1+2}=\frac{-\beta+2q}{3}.$$

Similarly, we can determine the coordinates of the points F_3^1 and F_3^2, G_3^1 and G_3^2, H_3^1 and H_3^2:

$$E_3^1 = \left(\frac{\alpha - 2p}{3}, \frac{\beta + 4p}{3}\right), \quad E_3^2 = \left(\frac{-\alpha - 4p}{3}, \frac{-\beta + 2q}{3}\right),$$
$$F_3^1 = \left(\frac{-\alpha - 4p}{3}, \frac{-\beta - 2q}{3}\right), \; F_3^2 = \left(\frac{\alpha - 2p}{3}, \frac{\beta - 4q}{3}\right),$$

$$G_3^1 = \left(\frac{\alpha + 2p}{3}, \frac{\beta - 4q}{3}\right), \quad G_3^2 = \left(\frac{-\alpha + 4p}{3}, \frac{-\beta - 2q}{3}\right),$$
$$H_3^1 = \left(\frac{-\alpha + 4p}{3}, \frac{-\beta + 2q}{3}\right), \; H_3^2 = \left(\frac{\alpha + 2p}{3}, \frac{\beta + 4q}{3}\right).$$

Let $K_3^1 = E_3^1H_3^2 \cap F_3^1E_3^2$, $L_3^1 = F_3^1E_3^2 \cap G_3^1F_3^2$, $M_3^1 = G_3^1F_3^2 \cap H_3^1G_3^2$, $N_3^1 = H_3^1G_3^2 \cap E_3^1H_3^2$.
Note with the symbol $u[\cdot]$ the perimeter of a polygon and with $\sigma[\cdot]$ its area. Since $AB = 2q$ and $BC = 2p$, it is evident that $u[ABCD] = 4(p + q) = PR + QS$.

Theorem 2.1. *The quadrilateral $K_3^1L_3^1M_3^1N_3^1$ is a parallelogram. Its perimeter respectively area is*

$$u[K_3^1L_3^1M_3^1N_3^1] = \frac{16}{3}(p + q) = \frac{4}{3}u[ABCD] = \frac{4}{3}(PR + QS), \tag{2.1}$$

respectively

$$\sigma[K_3^1L_3^1M_3^1N_3^1] = \frac{8}{9}\sigma[PQRS]. \tag{2.2}$$

Proof. The sides opposite of the quadrilateral $K_3^1L_3^1M_3^1N_3^1$ are parallel. Indeed,

$$E_3^1F_3^2 \| PR \| F_3^1E_3^2,\; G_3^1H_3^2 \| PR \| H_3^1G_3^2,\; E_3^1H_3^2 \| QS \| H_3^1E_3^2,\; F_3^1G_3^2 \| QS \| G_3^1F_3^2.$$

This parallelogram $K_3^1L_3^1M_3^1N_3^1$ is called the *Wittenbauer's parallelogram.* Note this parallelogram with the symbol $W(3, 1)$. The sides of the $W(3, 1)$ are: $K_3^1N_3^1 = H_3^1E_3^2 = F_3^1G_3^2 = \dfrac{-\alpha + 4p}{3} - \dfrac{-\alpha - 4p}{3} = \dfrac{8p}{3}$ and $K_3^1L_3^1 = E_3^1F_3^2 = G_3^1H_3^2 = \dfrac{\beta + 4q}{3} - \dfrac{\beta - 4q}{3} = \dfrac{8q}{3}$. The perimeter of the Wittenbauer parallelogram is

$$u[K_3^1L_3^1M_3^1N_3^1] = u[W(3,1)] = \frac{16}{3}(p + q) = \frac{4}{3}u[ABCD] = \frac{4}{3}(PR + QS).$$

The area of Varignon parallelogram is $\sigma[ABCD] = 4pq\sin\theta$. The area of quadrilateral $PQRS$ is $\sigma[PQRS] = 2\sigma[ABCD] = 8pq\sin\theta$. The area of Wittenbauer's parallelogram is

$$\sigma[K_3^1L_3^1M_3^1N_3^1] = \sigma[W(3,1)] = \frac{8p}{3}\cdot\frac{8q}{3}\sin\theta = \frac{8}{9}\sigma[PQRS].$$

□

Remark 2.1. In [2] I find the following: "3. The centroid of equal masses at the vertices of a quadrangle is the center of the Varignon parallelogram. 4. The centroid of a quadrangular lamina is the center of the Wittenbauer parallelogram, whose sides join adjacent points of trisection of the sides. This theorem, due to F. Wittenbauer (1857-1922) [Blaschke 2, p. 13], was rediscovered by J. J. Welch and V. W. Foss.” (see [3] and [4]).

3. The Wittenbauer type parallelograms

We suppose that the points $E_n^1, E_n^2, \ldots, E_n^{n-1}$ divide the side $[PQ]$, the points $F_n^1, F_n^2, \ldots, F_n^{n-1}$ divide the side $[QR]$, the points $G_n^1, G_n^2, \ldots, G_n^{n-1}$ divide the side $[RS]$ and the points $H_n^1, H_n^2, \ldots, H_n^{n-1}$ divide the side $[SP]$ into n equal parts (Figure 2). The order between this points let the following: $(P, E_n^1, E_n^2, \ldots, E_n^{n-1}, Q)$, $(Q, F_n^1, F_n^2, \ldots, F_n^{n-1}, R)$, $(R, G_n^1, G_n^2, \ldots, G_n^{n-1}, S)$, $(S, H_n^1, H_n^2, \ldots, H_n^{n-1}, P)$.

We determine the coordinates of the points E_n^i, where $i \in \{1, 2, \ldots, n-1\}$:

$$\frac{PE_n^i}{E_n^i Q} = \frac{i}{n-i} \Rightarrow x_{E_n^i} = \frac{x_P + \frac{i}{n-i}\, x_Q}{1 + \frac{i}{n-i}} = \frac{(n-2i)\alpha - 2ip}{n}$$

and

$$y_{E_n^i} = \frac{y_P + \frac{i}{n-i}\, y_Q}{1 + \frac{i}{n-i}} = \frac{(n-2i)\beta + 2(n-i)q}{n}\,.$$

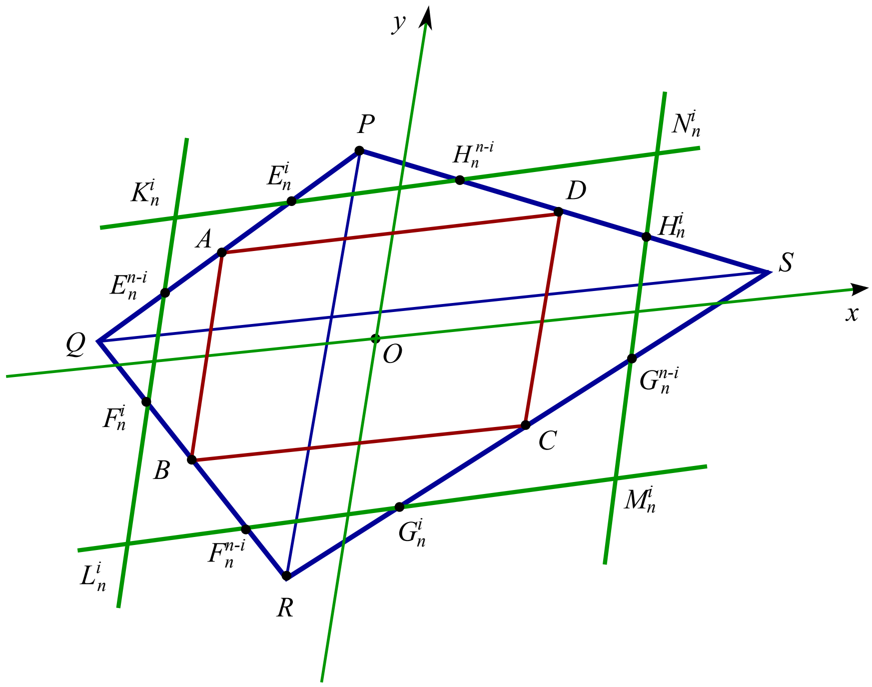

Figure 2

Similarly, we can determine the coordinates of the points F_n^i, G_n^i, H_n^i:

$$E_n^i = \left(\frac{(n-2i)\alpha - 2ip}{n}, \frac{(n-2i)\beta + 2(n-i)q}{n}\right),$$

$$F_n^i = \left(\frac{-(n-2i)\alpha - 2(n-i)p}{n}, \frac{-(n-2i)\beta - 2iq}{n}\right),$$

$$G_n^i = \left(\frac{(n-2i)\alpha + 2ip}{n}, \frac{(n-2i)\beta - 2(n-i)q}{n}\right),$$
$$H_n^i = \left(\frac{-(n-2i)\alpha + 2(n-i)p}{n}, \frac{-(n-2i)\beta + 2iq}{n}\right).$$

Remark 3.1. It is easy to verify that $x_{E_n^{n-i}} = x_{F_n^i}$, $x_{F_n^{n-i}} = x_{E_n^i}$, $x_{G_n^{n-i}} = x_{H_n^i}$, $x_{H_n^{n-i}} = x_{G_n^i}$ and $y_{E_n^{n-i}} = y_{H_n^i}$, $y_{F_n^{n-i}} = y_{G_n^i}$, $y_{G_n^{n-i}} = y_{F_n^i}$, $y_{H_n^{n-i}} = y_{E_n^i}$.

Let $K_n^i = E_n^i H_n^{n-i} \cap F_n^i E_n^{n-i}$, $L_n^i = F_n^i E_n^{n-i} \cap G_n^i F_n^{n-i}$, $M_n^i = G_n^i F_n^{n-i} \cap H_n^i G_n^{n-i}$, $N_n^i = H_n^i G_n^{n-i} \cap E_n^i H_n^{n-i}$ where $i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}$ and $[\cdot]$ is the integer part.

Theorem 3.1. *The quadrilateral $K_n^i L_n^i M_n^i N_n^i$ is a parallelogram and its perimeter is*

$$u[K_n^i L_n^i M_n^i N_n^i] = 8\frac{n-i}{n}(p+q) = 2\frac{n-i}{n}u[ABCD] \tag{3.1}$$
$$= 2\frac{n-i}{n}(PR + QS),$$

where n is natural, $n \geq 2$ and $i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}$.

Proof. The sides opposite of the quadrilateral $K_n^i L_n^i M_n^i N_n^i$ are parallel. Indeed, $E_n^i F_n^{n-i} \| PR \| F_n^i E_n^{n-i}$, $G_n^i H_n^{n-i} \| PR \| H_n^i G_n^{n-i}$, $E_n^i H_n^{n-i} \| QS \| H_n^i E_n^{n-i}$, $F_n^i G_n^{n-i} \| QS \| G_n^i F_n^{n-i}$ (Figure 2). Note this parallelograms with the symbol $W(n, i)$ and call it the *Wittenbauer type parallelograms.* The sides of the $W(n, i)$ are:

$$K_n^i N_n^i = H_n^i E_n^{n-i} = F_n^i G_n^{n-i}$$
$$= \frac{-(n-2i)\alpha + 2(n-i)p}{n} - \frac{-(n-2i)\alpha - 2(n-i)p}{n} = \frac{4(n-i)p}{n}$$

and

$$K_n^i L_n^i = E_n^i F_n^{n-i} = G_n^i H_n^{n-i}$$
$$= \frac{(n-2i)\beta + 2(n-i)q}{n} - \frac{(n-2i)\beta - 2(n-i)q}{n} = \frac{4(n-i)q}{n}.$$

Consequently, the perimeter of the parallelogram $W(n, i)$ is equal to

$$u[W(n,i)] = 8\frac{n-i}{n}(p+q) = 2\frac{n-i}{n}u[W(2,1)] = 2\frac{n-i}{n}(PR+QS),$$

where $u[W(2, 1)] = 4(p + q)$ is the perimeter of the Varignon parallelogram $ABCD$. □

In the following, we calculate the perimeter of Wittenbauer type parallelograms for $n \in \{2, 3, \ldots, 6\}$. Note the perimeter of Varignon parallelogram $ABCD$ briefly with u.

I. $n = 2 \Rightarrow u[W(2, 1)] = u[K_2^1 L_2^1 M_2^1 N_2^1] = u[ABCD] = u,$

II. $n = 3 \Rightarrow u[W(3, 1)] = u[K_3^1 L_3^1 M_3^1 N_3^1] = 2\frac{3-1}{3}u = \frac{4}{3}u,$

III. $n = 4 \Rightarrow u[W(4, 1)] = u[K_4^1 L_4^1 M_4^1 N_4^1] = 2\frac{4-1}{4}u = \frac{3}{2}u$

$$u[W(4,2)] = u[K_4^2 L_4^2 M_4^2 N_4^2] = 2\frac{4-2}{4}u = u,$$

IV. $n = 5 \Rightarrow u[W(5,1)] = u[K_5^1 L_5^1 M_5^1 N_5^1] = 2\frac{5-1}{5}u = \frac{8}{5}u,$

$$u[W(5,2)] = u[K_5^2 L_5^2 M_5^2 N_5^2] = 2\frac{5-2}{5}u = \frac{6}{5}u,$$

V. $n = 6 \Rightarrow u[W(6,1)] = u[K_6^1 L_6^1 M_6^1 N_6^1] = 2\frac{6-1}{6}u = \frac{5}{3}u,$

$$u[W(6,2)] = u[K_6^2 L_6^2 M_6^2 N_6^2] = 2\frac{6-2}{6}u = \frac{4}{3}u,$$

$$u[W(6,3)] = u[K_6^3 L_6^3 M_6^3 N_6^3] = 2\frac{6-3}{6}u = u.$$

Theorem 3.2. *The area of the parallelogram $K_n^i L_n^i M_n^i N_n^i$ is*

$$\sigma[K_n^i L_n^i M_n^i N_n^i] = 2\left(\frac{n-i}{n}\right)^2 \cdot \sigma[PQRS], \tag{3.2}$$

where n is natural, $n \geq 2$ and $i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}$.

Proof. The area of Wittenbauer type parallelograms is

$$\sigma[K_n^i L_n^i M_n^i N_n^i] = K_n^i N_n^i \cdot K_n^i L_n^i \sin\theta = \frac{4(n-i)p}{n} \cdot \frac{4(n-i)q}{n} \sin\theta$$
$$= 2\left(\frac{n-i}{n}\right)^2 \cdot \sigma[PQRS].$$

□

In the following we calculate the area of Wittenbauer type parallelograms for $n \in \{2, 3, \ldots, 6\}$. Note the area of quadrilateral $PQRS$ briefly with σ.

I. $n=2 \Rightarrow \sigma[W(2,1)] = \sigma[K_2^1 L_2^1 M_2^1 N_2^1] = \sigma[ABCD] = 2\left(\frac{2-1}{2}\right)^2 \sigma = \frac{1}{2}\sigma,$

II. $n = 3 \Rightarrow \sigma[W(3,1)] = \sigma[K_3^1 L_3^1 M_3^1 N_3^1] = 2\left(\frac{3-1}{3}\right)^2 \sigma = \frac{8}{9}\sigma,$

III. $n = 4 \Rightarrow \sigma[W(4,1)] = \sigma[K_4^1 L_4^1 M_4^1 N_4^1] = 2\left(\frac{4-1}{4}\right)^2 \sigma = \frac{9}{8}\sigma,$

$$\sigma[W(4,2)] = \sigma[K_4^2 L_4^2 M_4^2 N_4^2] = 2\left(\frac{4-2}{4}\right)^2 \sigma = \frac{1}{2}\sigma,$$

IV. $n = 5 \Rightarrow \sigma[W(5,1)] = \sigma[K_5^1 L_5^1 M_5^1 N_5^1] = 2\left(\frac{5-1}{5}\right)^2 \sigma = \frac{32}{25}\sigma,$

$$\sigma[W(5,2)] = \sigma[K_5^2 L_5^2 M_5^2 N_5^2] = 2\left(\frac{5-2}{5}\right)^2 \sigma = \frac{18}{25}\sigma,$$

V. $n = 6 \Rightarrow \sigma[W(6,1)] = \sigma[K_6^1 L_6^1 M_6^1 N_6^1] = 2\left(\frac{6-1}{6}\right)^2 \sigma = \frac{25}{18}\sigma,$

$$\sigma[W(6,2)] = \sigma[K_6^2 L_6^2 M_6^2 N_6^2] = 2\left(\frac{6-2}{6}\right)^2 \sigma = \frac{8}{9}\sigma,$$

$$\sigma[W(6,3)] = \sigma[K_6^3 L_6^3 M_6^3 N_6^3] = 2\left(\frac{6-3}{6}\right)^2 \sigma = \frac{1}{2}\sigma.$$

4. Some properties of the Wittenbauer type parallelograms

Let $O_n^i = K_n^i M_n^i \cap L_n^i N_n^i$, i.e. the center point of symmetry of the parallelogram $K_n^i L_n^i M_n^i N_n^i$ and T the intersection of diagonals PR and QS. The coordinates of the point T are $(\alpha, -\beta)$ (see Figure 3, where $n = 6$).

Theorem 4.1. *All point O_n^i are situated on the line OT, consequently the points O_n^i are collinear, for all natural n, $n \geq 2$ and $i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}$.*

Proof. The coordinates of the points $K_n^i, M_n^i, L_n^i, N_n^i$ are

$$\begin{aligned} K_n^i &= (x_{E_n^{n-i}}, y_{E_n^i}) = (x_{F_n^i}, y_{E_n^i}) \\ &= \left(\frac{-(n-2i)\alpha - 2(n-i)p}{n}, \frac{(n-2i)\beta + 2(n-i)q}{n}\right), \end{aligned}$$

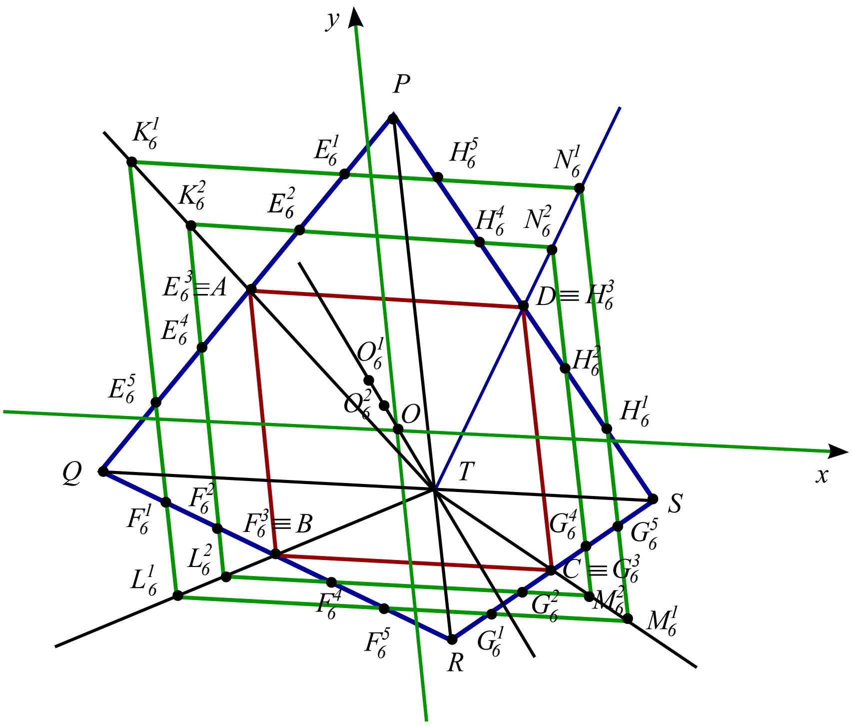

Figure 3

$$\begin{aligned} L_n^i &= (x_{F_n^i}, y_{F_n^{n-i}}) = (x_{F_n^i}, y_{G_n^i}) \\ &= \left(\frac{-(n-2i)\alpha - 2(n-i)p}{n}, \frac{(n-2i)\beta - 2(n-i)q}{n}\right), \\ M_n^i &= (x_{G_n^{n-i}}, y_{G_n^i}) = (x_{H_n^i}, y_{G_n^i}) \\ &= \left(\frac{-(n-2i)\alpha + 2(n-i)p}{n}, \frac{(n-2i)\beta - 2(n-i)q}{n}\right), \end{aligned}$$

$$N_n^i = (x_{H_n^i}, y_{H_n^{n-i}}) = (x_{H_n^i}, y_{E_n^i})$$
$$= \left(\frac{-(n-2i)\alpha + 2(n-i)p}{n}, \frac{(n-2i)\beta + 2(n-i)q}{n}\right).$$

So, $O_n^i = \left(-\frac{(n-2i)\alpha}{n}, \frac{(n-2i)\beta}{n}\right) = \frac{n-2i}{n}(-\alpha, \beta)$. The equation of the line OT is $\beta x + \alpha y = 0$. It is evident that $O_n^i \in OT$. □

We will write the coordinates of points $K_n^i, M_n^i, L_n^i, N_n^i$ for $n = 6$ and $i \in \{1, 2, 3\}$ (Figure 3):

$$K_6^1 = \left(\frac{-2\alpha - 5p}{3}, \frac{2\beta + 5q}{3}\right),\ L_6^1 = \left(\frac{-2\alpha - 5p}{3}, \frac{2\beta - 5q}{3}\right)$$
$$M_6^1 = \left(\frac{-2\alpha + 5p}{3}, \frac{2\beta - 5q}{3}\right),\ N_6^1 = \left(\frac{-2\alpha + 5p}{3}, \frac{2\beta + 5q}{3}\right)$$
$$K_6^2 = \left(\frac{-\alpha - 4p}{3}, \frac{\beta + 4q}{3}\right),\quad L_6^2 = \left(\frac{-\alpha - 4p}{3}, \frac{\beta - 4q}{3}\right)$$
$$M_6^2 = \left(\frac{-\alpha + 4p}{3}, \frac{\beta - 4q}{3}\right),\quad N_6^2 = \left(\frac{-\alpha + 4p}{3}, \frac{\beta + 4q}{3}\right)$$

$K_6^3 = (-p, q) = A$, $L_6^3 = (-p, -q) = B$, $M_6^3 = (p, -q) = C$, $N_6^3 = (p, q) = D$.

Remark 4.1. The parallelograms $W(2n, 2i)$ and $W(n, i)$ coincide for $n \geq 2$ and $i \in \left\{1, 2, \dots, \left[\frac{n}{2}\right]\right\}$.

Theorem 4.2. *a) All points K_n^i are situated on the line AT, so the points K_n^i are collinear.*

b) All points L_n^i are situated on the line BT, so the points L_n^i are collinear.

c) All points M_n^i are situated on the line CT, so the points M_n^i are collinear.

d) All points N_n^i are situated on the line DT, so the points N_n^i are collinear, for all natural n, $n \geq 2$, and $i \in \left\{1, 2, \dots, \left[\frac{n}{2}\right]\right\}$.

Proof. The equation of the line AT is

$$\begin{vmatrix} x & y & 1 \\ -p & q & 1 \\ \alpha & -\beta & 1 \end{vmatrix} = 0 \Leftrightarrow (\beta + q)x + (\alpha + p)y + \beta p - \alpha q = 0.$$

We will verify that $K_n^i \in AT$ (Figure 3):
$(\beta+q)[-(n-2i)\alpha - 2(n-i)p] + (\alpha+p)[(n-2i)\beta + 2(n-i)q] + n(\beta p - \alpha q) = 0 \Leftrightarrow (n-2i)(\beta p - \alpha q) - 2(n-i)(\beta p - \alpha q) + n(\beta p - \alpha q) = 0 \Leftrightarrow (\beta p - \alpha q) \cdot 0 = 0$, which is true. Similarly, we obtain the equations of the lines BT, CT and DT:

$$BT : \begin{vmatrix} x & y & 1 \\ -p & -q & 1 \\ \alpha & -\beta & 1 \end{vmatrix} = 0 \Leftrightarrow (\beta - q)x + (\alpha + p)y + \beta p + \alpha q = 0,$$

$$CT : \begin{vmatrix} x & y & 1 \\ p & -q & 1 \\ \alpha & -\beta & 1 \end{vmatrix} = 0 \Leftrightarrow (\beta - q)x + (\alpha - p)y - \beta p + \alpha q = 0,$$

$$DT : \begin{vmatrix} x & y & 1 \\ p & q & 1 \\ \alpha & -\beta & 1 \end{vmatrix} = 0 \Leftrightarrow (\beta + q)x + (\alpha - p)y - \beta p - \alpha q = 0.$$

It is easy to verify that $L_n^i \in BT$, $M_n^i \in CT$ and $N_n^i \in DT$. □

Theorem 4.3. *The following relations hold:*

$$a) \qquad \sigma[W(2i+1,i)]\sigma[W(2i+2,1)] = \sigma^2, \tag{4.1}$$

$$b) \qquad \sigma[W(3i+3,i+1)]\sigma[W(4i,i)] = \sigma^2, \tag{4.2}$$

$$c) \qquad \sigma[W(5i,i)] = \frac{32}{25}\,\sigma, \tag{4.3}$$

$$d) \qquad \sigma[W(5i,2i)] = \frac{18}{25}\,\sigma, \tag{4.4}$$

where $i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}$.

Proof. We will demonstrate the premier two relations:

$$a)\ \sigma[W(2i+1,i)]\sigma[W(2i+2,1)] = 4\left(\frac{2i+1-i}{2i+1}\right)^2\left(\frac{2i+2-1}{2i+2}\right)^2\sigma^2 = \sigma^2,$$

$$b)\ \sigma[W(3i+3,i+1)]\sigma[W(4i,i)] = \frac{8}{9}\,\sigma \cdot \frac{9}{8}\,\sigma = \sigma^2.$$

□

In [2] Coxeter pose the following question: "5. For what kind of quadrangle will the centroids described in the two preceding exercises coincide?" The center of the Wittenbauer parallelogram is $O_3^1 = \frac{1}{3}(-\alpha, \beta)$. Consequently, $O_3^1 \equiv O \Leftrightarrow \alpha = 0 = \beta$. In this case

$$P = (0, 2q),\ Q = (-2p, 0),\ R = (0, -2q),\ S = (-2p, 0),$$

so the quadrilateral $PQRS$ is a parallelogram (Figure 4).

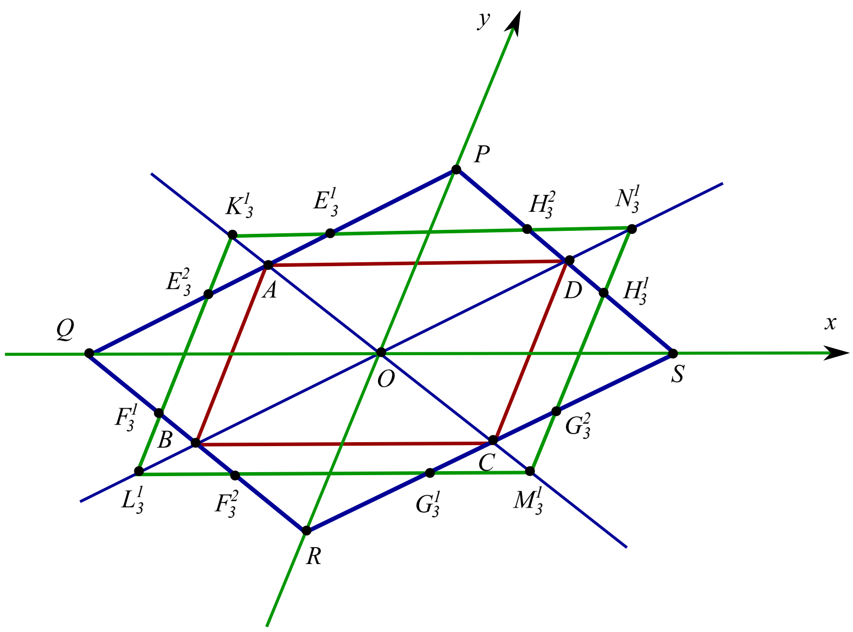

Figure 4

We can say better: in this case all centers $O_n^i = \dfrac{n-2i}{n}(-\alpha,\beta)$ of the Wittenbauer type parallelograms coincide with O, the center of Varignon parallelogram. Since $T \equiv O$, results that for all natural n, $n \geq 2$ and $i \in \left\{1, 2, \ldots, \left[\frac{n}{2}\right]\right\}$ we have: $K_n^i, M_n^i \in AC$ and $L_n^i, N_n^i \in BD$.

In this paper I supposed that the quadrilateral $PQRS$ is convex. Whether this theorems and properties remain valuable, does the quadrilateral $PQRS$ turn into concave or another type of quadrilateral?

References

[1] Coxeter, H.S.M. and Greitzer, S.L., *Geometry Revisited*, Washington, DC: Math. Assoc. Amer., p. 53, 1967.

[2] Coxeter, H.S.M., *Introduction to Geometry*, 2 ed., John Wiley & Sons, Inc., p. 216, 1969.

[3] Welch, J.J., *A centre of gravity*, Mathematical Gazette, **42** (1958), p. 55

[4] Foss, V.W., *Centre of gravity of a quadrilateral*, Mathematical Gazette, **43** (1959), p. 46.

"CONSTANTIN BRÂNCUŞI" TECHNOLOGY LYCEUM
SATU MARE, ROMANIA
e-mail: `d.sandor.kiss@gmail.com`