INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 4 (2015), No. 1, 16 - 26

GENERALISED WARPED PRODUCTS

KRZYSZTOF DRACHAL

Abstract. The concept of warped product is introduced in the category of differential spaces in the sense of Sikorski, i.e. the notion of warped product known from studying classical smooth manifolds is introduced in the wider category of differential spaces. In particular it allows to construct a consistent formalism to describe glued Friedman universes – also if singular points are included. The proposed scheme relies also on some algebraic properties. In order to keep the article self–consistent a brief introduction to differential spaces theory is also presented.

1. Introduction

Warped products on classical smooth manifold were introduced by Bishop and O'Neill in 1969 [2]. They are a nice tool describing e.g. simple models of neighbourhoods of stars and black holes and standard Friedman universe models [10]. However it is assumed that the scaling function is strictly positive, which implies non–degeneracy of the metric. But if considering the gluing – for example two closed FLRW universes – the metric degenerates on the shift. In such a situation one may "stratify" the whole model into two smooth submanifolds and the singular point. However the notion of differential space in the sense of Sikorski allows to consider more general warped products, i.e. the singularity occurs not to be a serious problem against building differential geometry, because glued universes form so called "differential space in the sense of Sikorski". Fortunately differential geometry over such objects is known [15]. (For short expository in English see also, for example, [4].)

Keywords and phrases: differential spaces, differential spaces in the sense of Sikorski, d-spaces, Friedmann-Lemaître-Robertson-Walker metric, generalised warped product, singularity, warped product
(2010)Mathematics Subject Classification: 58A40, 53Z05, 83F05
Received: 1.10.2014. In revised form: 13.01.2015. Accepted: 16.03.2015.

2. Differential spaces

Differential spaces (sometimes called in the literature d–spaces) are one of the ways of generalising the classical manifold concept. (For a review of other, similar concepts see, for example, [1].) The starting point is to properly choose the set and some family of real functions on this set. But actually this family may be arbitrary. However, by choosing these function, one determines the topology and the differential structure on the considered set. The details are given below. It is assumed that $\mathbb{N} = \{1, 2, \dots\}$.

Definition 2.1. *Let M be a set and let $C_0 = \{f_1, \dots, f_n\}$ be a family of some real functions on M, i.e. $f_i : M \to \mathbb{R}$ for all $i = 1, \dots, n$. The weakest topology on M, for which all functions from C_0 are continuous is called a topology induced by C_0 and is denoted by τ_{C_0}.*

Definition 2.2. *Let C be some family of real functions on M. Then the set*

$$\{\omega \circ (f_1, \dots, f_k) \mid \omega \in C^\infty(\mathbb{R}^k), k \in \mathbb{N}, f_i \in C, i = 1, \dots, k\}$$

is called the superposition closure of C and is denoted by $\mathrm{sc}C$. It is the expansion of the initial family of functions, C, by composing them with smooth (in the usual sense) Euclidean functions.

Definition 2.3. *A function g is called the local C–function, if for every point $p \in M$ there exist some function $f \in C$, such that there exists an open neighbourhood $U \ni \tau_{C_0}$ of p, for which $f|_U = g|_U$. (The function f usually varies for different points p.)*

Definition 2.4. *The localisation closure of C on the set M is the family of all local C–functions. It is denoted by $(C)_M$.*

Definition 2.5. *The family of functions C is called the differential structure on a set M, if $C = (\mathrm{sc}C)_M$.*

Lemma 2.1 ([15])**.** *For every family C_0 of real functions on a set M, the corresponding differential structure may be constructed by: first superposition closing and next localisation closing, i.e. by taking $(\mathrm{sc}C_0)_M$.*

Definition 2.6. *The differential structure $C = (\mathrm{sc}C_0)_M$ is called the differential structure generated by C_0. If C_0 is finite, then C is called finitely generated.*

The theory of differential spaces may be studied even if C_0 is not finite. However in this paper (if not stated otherwise) the generating set C_0 is assumed to be finite.

Definition 2.7. *Let M be a set and let C_0 be a set of real functions on M. The pair (M, C), where $C = (\mathrm{sc}C_0)_M$, is called the differential space.*

Differential spaces form a wider category than smooth manifolds.

Example 2.1. *Consider $M = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $C = C^\infty(\mathbb{R}^n)$. Then (M, C) is a smooth Euclidean n–dimensional manifold. It is not hard to check that C is generated by projections $\pi_1, \dots, \pi_n$, where $\pi_i(x_1, \dots, x_n) = x_i$, $i = 1, \dots, n$, $(x_1, \dots, x_n) \in \mathbb{R}^n$.*

Example 2.2. *Consider some classical smooth manifold M and all smooth (in the classical sense) real functions on it $C^\infty(M)$. Then $(M, C^\infty(M))$ is a differential space.*

Example 2.3. *The graph of function $|x| : [-1, 1] \to \mathbb{R}$ is not a smooth manifold, but it is a differential space, i.e. let $M = \{(x, |x|) \subset \mathbb{R}^n \mid x \in \mathbb{R}\}$. Then $(M, C^\infty(\mathbb{R}^2)|_M)$ is a differential space.*

Of course the notion of a manifold may be expanded to include manifolds with boundary, etc. But definitely crossed axises are not manifolds.

Example 2.4. *Consider $M = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. From classical point of view $(0, 0)$ is a singular point and M cannot be equipped with a classically smooth differential structure. It is an example of a differential space, which is not a manifold. This differential structure consists of restrictions of smooth function from $\mathbb{R}^2$, i.e. if $C = \{f|_M \mid f \in C^\infty(\mathbb{R}^2)\}$ then (M, C) is a differential space.*

Another example shows how two different points may be identified (glued).

Example 2.5. *Let $M = \mathbb{R}$ and $p, q \in M$, $p \neq q$. Let $C = \{f \in C^\infty(\mathbb{R}) \mid f(p) = f(q)\}$. (M, C) is a differential space.*

Definition 2.8. *Having fixed some differential space (M, C), any function from C is called smooth in the sense of Sikorski.*

The above smoothness is of course different from classical smoothness. But if it does not lead to any misunderstanding it is written shortly "smooth" and the phrase "in the sense of Sikorski" is omitted.

Definition 2.9. *A linear mapping $v_p : C \to \mathbb{R}$ satisfying the Leibniz rule, i.e. $v_p(fg) = f(p)v_p(g) + v_p(f)g(p)$ for all $f, g \in C$ is called a tangent vector to (M, C) at $p \in M$.*

Definition 2.10. *All tangent vectors to (M, C) at $p \in M$ constitute a linear space denoted by T_pM.*

Definition 2.11. *By TM is denoted the disjoint sum $\bigsqcup_{p \in M} T_pM$.*

Definition 2.12. *A mapping $X : M \to TM$, $X : p \mapsto X_p$ is called a tangent vector field to (M, C). It is called smooth, if for all $f \in C$ and F defined as $F : p \mapsto X_p(f)$, $X_p(f)$ belongs to C. The set of all vector fields tangent to (M, C) is denoted by $\mathfrak{X}(M)$.*

Definition 2.13. *A linear mapping $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C$ is called a form on (M, C).*

Definition 2.14. *A skew–symmetric, linear mapping $\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C$ is called a differential form on (M, C).*

Definition 2.15. *A symmetric, linear mapping $g : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C$, is called a metric.*

3. Generalised warped products

At first the (Cartesian) product of differential spaces should be introduced. Let (B, C_B) and (F, C_F) be differential spaces. Let $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be projections. Consider

$$C = \{f \circ \pi \mid f \in C_B\} \cup \{g \circ \sigma \mid g \in C_F\}.$$

Definition 3.1. *$C_B \mathbb{X} C_F := (\mathrm{sc}C)_{B\times F}$ is called the Cartesian product of differential structures C_B and C_F.*

Definition 3.2. *The Cartesian product of two differential spaces (B, C_B) and (F, C_F) is defined as the differential space $(B \times F, C_B \mathbb{X} C_F)$.*

The topology on a Cartesian product is the Tichonov topology: $\tau_{C_B \mathbb{X} C_F} = \tau_{C_B} \times \tau_{C_F}$.

Lemma 3.1. *π and σ are smooth.* □

Some notation will be used, i.e. $f^*(g) := g \circ f$ and $f_{*p}(v) := v \circ f^*$.

Definition 3.3. *Mapping $F : B \to F$ is called smooth, if $\forall_{f\in C_F}\ f \circ F \in C_B$.*

Definition 3.4. *F is called a diffeomorphism, if it is bijective and both F and F^{-1} are smooth.*

Theorem 3.1. *$((\pi)_{*(p,q)}, (\sigma)_{*(p,q)}) : T_{(p,q)}(B \times F) \to T_pB \times T_qF$ is an isomorphism.*

Proof. [12] For fixed $p \in B$ and $q \in F$ consider inclusions $i_q : B \to B \times F$ and $i_p : F \to B \times F$ defined as $i_p(q) = (p, q)$ for $q \in F$ and $i_q(p) = (p, q)$ for $p \in B$.

i_p and i_q are smooth. Indeed:

$(f \circ \pi) \circ i_q = f \circ (\pi \circ i_q) = f \in C_B$; $(g \circ \sigma) \circ i_p = g \circ (\sigma \circ i_p) = g \in C_F$; $(g \circ \sigma) \circ i_q = g \circ (\sigma \circ i_q) = g(q) = \mathrm{const} \in C_F$; $(f \circ \pi) \circ i_p = f \circ (\pi \circ i_p) = f(p) = \mathrm{const} \in C_B$.

$i_p : (F, C_F) \to (p \times F, (C_B \mathbb{X} C_F)_{p\times F})$ and $i_q : (B, C_B) \to (B \times q, (C_B \mathbb{X} C_F)_{B\times q})$ are diffeomorphisms.

Consider $T_{(p,q)}(B\times F) \ni w \mapsto ((\pi)_{*(p,q)}(w), (\sigma)_{*(p,q)}(w))$ and $T_pB\times T_qF \ni (u, v) \mapsto (i_q)_{*p}u+(i_p)_{*q}v$ and $(u, v) \mapsto (i_q)_{*p}u+(i_p)_{*q}v \mapsto ((\pi)_{*(p,q)}((i_q)_{*p}u+ (i_p)_{*q}v), (\sigma)_{*(p,q)}((i_q)_{*p}u + (i_p)_{*q}v))$.

$(\pi)_{*(p,q)}((i_q)_{*p}u)+(\pi)_{*(p,q)}((i_p)_{*q}v) = (\pi\circ i_q)_{*p}u+(\pi\circ i_p)_{*q}v = (\mathrm{id}_B)_{*p}u+ 0 = u$. Similarly $(\sigma)_{*(p,q)}((i_q)_{*p}u) + (\pi)_{*(p,q)}((i_p)_{*q}v) = v$.

$w \mapsto ((\pi)_{*(p,q)}(w), (\sigma)_{*(p,q)}(w)) \mapsto (i_q)_{*p}((\pi)_{*(p,q)}w) + (i_p)_{*q}((\sigma)_{*(p,q)}w)$

$(i_q)_{*p}((\pi)_{*(p,q)}w) + (i_p)_{*q}((\sigma)_{*(p,q)}w = (i_q \circ \pi)_{*(p,q)}w + (i_p \circ \sigma)_{*(p,q)}w$

$((i_q)_{*p}((\pi)_{*(p,q)}w) + (i_p)_{*q}((\sigma)_{*(p,q)}w)(f \circ \sigma) = w(f \circ \sigma \circ i_q \circ \pi) + w(f \circ \sigma \circ i_p \circ \sigma) = w(f \circ \sigma)$ Similarly it can be checked for $f \circ \pi$, $g \circ \pi$ and $g \circ \sigma$. Finally it may be concluded that $(i_q)_{*p}((\pi)_{*(p,q)}w) + (i_p)_{*q}((\sigma)_{*(p,q)}w) = w$, because both sides act in the same way on arbitrary function from $C_B \mathbb{X} C_F$. □

Corrolary 3.1. $\dim T_{(p,q)}(B \times F) = \dim T_pB + \dim T_qF$

Consider vector field $X \in \mathfrak{X}(B)$ and $Y \in \mathfrak{X}(F)$. Then the notion of their lifts can be introduced.

Definition 3.5. *The lift of a vector field* X *is defined as* $\widetilde{X}_{(p,q)} = (i_q)_{*p}(X_p)$. *The lift of a vector field* Y *is defined as* $\widetilde{Y}_{(p,q)} = (i_p)_{*q}(Y_q)$.

It can be easily checked that:
$\widetilde{X}$ and $\widetilde{Y}$ are smooth.

Definition 3.6. *Let* (B, C_B) *and* (F, C_F) *be differential spaces. Let* $\pi : B \times F \to B$ *and* $\sigma : B \times F \to F$ *be projections. Moreover let* g_B *be a metric tensor on* (B, C_B) *and* g_F *be a metric tensor on* (F, C_F). *Then* $(B \times F, C_B \mathbb{X} C_F)$ *equipped with metric tensor* $g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F)$ *(called warped metric), where* $f \in C_B$, *is called a generalised warped product.*

The case when $f \in C^\infty(B)$ and $f > 0$ was studied by O'Neill [2], [10]. Here it is allowed that $f = 0$ or f diverges to infinity or even $f \notin C^\infty(B)$. In such a situation connection, Riemann curvature and Ricci tensor diverge. But these are not a real obstacles to develop a consistent geometrical formalism, because from the definitions it is easily seen that:

Proposition 3.1. *Every generalised warped product is a differential space.*

As in the classical case for each $q \in F$, $\pi|_{B\times q}$ is an isometry onto B and for each $(p, q) \in B \times F$, $B \times q$ and $p \times F$ are orthogonal at (p, q). In classical case $T_{(p,q)}(p \times F)^\perp = T_{(p,q)}(B \times q)$. But here if $f = 0$ then $T_{(p,q)}(p \times F)^\perp = T_{(p,q)}(B \times F)$. It is a common fact in differential spaces theory, that the tangent space dimension changes in singular points. Here a singularity of f is announced by the dimension change of $T_{(p,q)}(p \times F)^\perp$.

Also a result, similar to the classical case, holds:

Let $X, Y \in \mathfrak{X}(B)$ and $V, W \in \mathfrak{X}(F)$. Then:

(1) $g(\widetilde{X}, \widetilde{V}) = 0$,
(2) $[\widetilde{X}, \widetilde{V}] = 0$,
(3) $\widetilde{V} g(\widetilde{X}, \widetilde{V}) = 0$,
(4) $\widetilde{X} g(\widetilde{V}, \widetilde{W}) = 2 f g_F(V, W) X(f)$.

Proof. The first equality follows easily from definition of g. The second – from the fact that $p \times F$ and $B \times q$ are orthogonal. For third equality it is crucial that $g(\widetilde{X}, \widetilde{V})$ is constant on every $p \times F$ The fourth is also based on easy computations directly from the definition of a vector field. □

Having defined a metric g on a differential space, a connection may be defined. It can be done for example through the Koszul formula, i.e.:

Definition 3.7. *A connection on* (M, C) *is a mapping* $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ *satisfying the Koszul formula in every point of* M, *i.e.* $(\nabla_u v)(w) = \frac{1}{2}(ug(v, w) + vg(u, w) - wg(u, v) + g(w, [u, v]) + g(v, [w, u]) - g(u, [v, w]))$, *where* $u, v, w \in T_p M$.

The notation $(\nabla_u v)(w)$ should be understood as: $(\nabla_u v)$ acts as a 1–form such that

$$(\nabla_u v)(w) = g(\nabla_u v, w) \tag{1}$$

is fulfilled.

Proofs of the existence of the geometric objects introduced by Def. 2.14, Def. 2.15 and Def. 3.7 may be found e.g. in [7].

Remark 3.1. *For a classical semi–Riemannian manifold a connection defined by Koszul formula is unique, because non–degeneracy of the metric imposes that the relation given by Eq. 1 is bijective* [10]*. But if* $g = 0$ *(the metric degenerates) it may be no longer true.*

Lemma 3.2. *For the connection introduced in Def. 3.7 the following relations hold:*

1. $(\nabla_{fu}v)(w) = f(\nabla_u v)(w)$,
2. *additivity and* $\mathbb{R}$*–linearity on each slot,*
3. $(\nabla_u(fv))(w) = u(f)g(v,w) + f(\nabla_u v)(w)$,
4. $ug(v,w) = (\nabla_u v)(w) + (\nabla_u w)(v)$,
5. $(\nabla_u v)(w) - (\nabla_v u)(w) = g(w,[u,v])$.

Proof. By direct computation. □

Classically a connection satisfying the above properties is called the Levi–Civita connection (and in a classical case it is a unique one).

Lemma 3.3. *For the connection introduced in Def. 3.7 also the following relations hold:*

1. $(\nabla_u v)(fw) = f(\nabla_u v)(w)$,
2. $(\nabla_u v)(w) + (\nabla_v w)(u) = vg(u,w) + g(w,[u,v])$.

It is interesting to see the relation between a connection $\overset{B}{\nabla}$ on (B, C_B) induced by the metric g_B, a connection $\overset{F}{\nabla}$ on (F, C_F) induced by the metric g_F and a connection ∇ induced by the warped metric g (see Def. 3.6). Due to Th. 3.1 each vector $v \in T_{(p,q)}B \times F$ may be decomposed into the sum of vectors from T_pB and T_qF, i.e. $v = v_B + v_F$, where $v_B \in T_pB$ and $v_F \in T_qF$. And this decomposition is unique.

Proposition 3.2. *The following relation holds:*

$$
\begin{aligned}
(\nabla_u v)(w) \quad = \quad & (\overset{B}{\nabla}_{u_B} v_B)(w_B) + f^2(\overset{F}{\nabla}_{u_F} v_F)(w_F) + \\
& 2fg_F(v_F,w_F)u_B(f) + 2fg_F(u_F,w_F)v_B(f) - 2fg_F(u_F,v_F)w_B(f) \quad .
\end{aligned}
$$

Proof. It is the consequence of Lem. 3.3, i.e.:

$$
\begin{aligned}
(\nabla_u v)(w) \quad = \quad & \frac{1}{2}(ug(v,w) + vg(u,w) - wg(u,v) + \\
& g(w,[u,v]) + g(v,[w,u]) - g(u,[v,w])) \\
= \quad & \frac{1}{2}(u(g_B(v,w) + f^2 g_F(v,w)) + v(g_B(u,w) + f^2 g_F(u,w)) - \\
& w(g_B(u,v) + f^2 g_F(u,v)) + g_B(w,[u,v]) + f^2 g_F(w,[u,v]) + \\
& g_B(v,[w,u]) + f^2 g_F(v,[w,u]) - g_B(u,[v,w]) - f^2 g_F(u,[v,w]))
\end{aligned}
$$

$$\begin{aligned}
(\nabla_u v)(w) &= \frac{1}{2}(u(g_B(v,w)) + u(f^2 g_F(v,w)) + v(g_B(u,w)) + v(f^2 g_F(u,w)) - \\
&\quad w(g_B(u,v)) - w(f^2 g_F(u,v)) + g_B(w,[u,v]) + f^2 g_F(w,[u,v]) + \\
&\quad g_B(v,[w,u]) + f^2 g_F(v,[w,u]) - g_B(u,[v,w]) - f^2 g_F(u,[v,w])) \\
&= \frac{1}{2}(u(g_B(v,w)) + 2f g_F(v,w) u(f) + f^2 u g_F(v,w) + v(g_B(u,w)) + \\
&\quad 2f g_F(u,w) v(f) + f^2 v g_F(u,w) - w(g_B(u,v)) - 2f g_F(u,v) w(f) - \\
&\quad f^2 w g_F(u,v) + g_B(w,[u,v]) + f^2 g_F(w,[u,v]) + g_B(v,[w,u]) + \\
&\quad f^2 g_F(v,[w,u]) - g_B(u,[v,w]) - f^2 g_F(u,[v,w])) \\
&= \frac{1}{2}(u_B(g_B(v_B,w_B)) + 2f g_F(v_F,w_F) u_B(f) + f^2 u_F g_F(v_F,w_F) + \\
&\quad v_B(g_B(u_B,w_B)) + 2f g_F(u_F,w_F) v_B(f) + f^2 v_F g_F(u_F,w_F) - \\
&\quad w_B(g_B(u_B,v_B)) - 2f g_F(u_F,v_F) w_B(f) - f^2 w_F g_F(u_F,v_F) + \\
&\quad g_B(w_B,[u_B,v_B]) + f^2 g_F(w_F,[u_F,v_F]) + g_B(v_B,[w_B,u_B]) + \\
&\quad f^2 g_F(v_F,[w_F,u_F]) - g_B(u_B,[v_B,w_B]) - f^2 g_F(u_F,[v_F,w_F])) \\
&= (\overset{B}{\nabla}_{u_B} v_B)(w_B) + f^2(\overset{F}{\nabla}_{u_F} v_F)(w_F) + \\
&\quad 2f g_F(v_F,w_F) u_B(f) + 2f g_F(u_F,w_F) v_B(f) - 2f g_F(u_F,v_F) w_B(f) \quad .
\end{aligned}$$

□

From the above proposition it can be concluded that:

Corollary 3.2. *Let $u_B, v_B, w_B \in T_pB$ and $u_F, v_F, w_F \in T_qF$, then:*

$$\begin{aligned}
(\nabla_{u_B} v_B)(w_B) &= (\overset{B}{\nabla}_{u_B} v_B)(w_B), \\
(\nabla_{u_F} v_F)(w_F) &= f^2(\overset{F}{\nabla}_{u_F} v_F)(w_F), \\
(\nabla_{u_B} v_B)(w_F) &= (\nabla_{u_B} v_F)(w_B) = (\nabla_{u_F} v_B)(w_B) = 0, \\
(\nabla_{u_B} v_F)(w_F) &= 2f g_F(v_F,w_F) u_B(f), \\
(\nabla_{u_F} v_B)(w_F) &= 2f g_F(u_F,w_F) v_B(f), \\
(\nabla_{u_F} v_F)(w_B) &= -2f g_F(u_F,v_F) w_B(f).
\end{aligned}$$

It can also be seen that if $f = 0$ then $(\nabla_u v)(w) = (\overset{B}{\nabla}_{u_B} v_B)(w_B)$. Moreover, if both g_B and g_F are non–degenerate, then respective connections are unique. More about the degenerate metric and consequences to the uniqueness of the connection may be found in [13]. What can happen when one assumes only some of the axioms from Def. 2.7 and how it can affect the connection is discussed in [9].

4. Application

To show the direct application of generalised warped product it is better to consider Friedman universes after conformal rescalling its time axises instead of unchanged models. The original Friedman models (see e.g. [10]) are classical warped products with $B \in \{(0, 2\pi k), \mathbb{R}\}$ and $F \in \{\mathbb{R}^3, S^2, \mathbb{R}P^3\}$, equipped with the Friedmann–Lemaître–Robertson–Walker metric, i.e. the one for which a line element is given by the formula $-dt^2 + f^2(t)d\sigma^2$, where $d\sigma^2$ is the line element on F (and k stands for some constant).

Here it will be assumed that $B = (0, 2\pi)$. This is just a conformal rescalling of $\mathbb{R}$ by $t_{\text{conformal}} = \text{arccot } 2t$, or of $(0, 2\pi k)$ by $t_{\text{conformal}} = \frac{t}{k}$. These modifications do not lead to any important differences. They change only numerical values – not a general properties like smoothness, continuity, etc. (which are important for the scope of this paper). For more advanced review of conformal rescalling in the context of pseudo–Riemannian manifolds and General Relativity see e.g. [5] and [8].

The below table presents the important characteristics of the original Friedman universes (here k stands for some constant values) [10].

type	F	$f(t)$
closed	S^2	$t = k(\varphi - \sin\varphi),\ f = k(1 - \cos\varphi),\ \varphi \in (0, 2\pi)$
flat	$\mathbb{R}^3$	$kt^{\frac{2}{3}},\ t \in (0, +\infty)$
open	$\mathbb{R}P^3$	$t = k(\sinh\varphi - \varphi),\ f = k(\cosh\varphi - 1),\ \varphi \in (0, +\infty)$

TABLE 1. Major characteristics of Friedman universes.

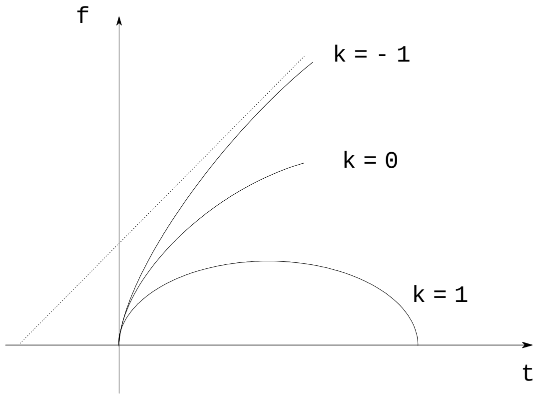

Figure 1 - Scale functions for Friedman universes.

If one would like for example to include the initial ($t = 0$) and the final point ($t = 2\pi$) of closed universe into the warped product then the scale function would vanish in these points, i.e. $f(0) = 0 = f(2\pi)$. So it is impossible in a classical case to treat e.g. two closed universes glued together as a warped product.

Moreover, if flat or open universes are glued, then $f \to +\infty$ from the left side of the shift and $f \to 0$ from the right side of the shift. The question of continuity and smoothness (in the classical sense) of a function obtained from gluing two such scale functions is a delicate matter (see e.g. [3]). But surely two glued flat or open universes are not a classical warped product.

So in seeking a generalisation of warped product capable of managing with the problem of gluing Friedman universes two major problems occur. The first one is that the new warping function may not be classically smooth on the shift or may be degenerated on the shift. This can be simply overcome by considering wider differential structure (in the sense of Def. 2.5). The problem of smoothness of the new warping function is not a real obstacle in differential spaces category. Classically non–smooth (but still continuous) function may just be incorporated into the differential structure without changing the topology of the space. The second problem – that it may even be discontinuous on the shift – is more subtle, because it may affect the topology.

The first from the above problems occurs when one wants to glue two closed universes. The second – when one wants to glue two open or two flat universes, or open or flat universe with closed universe, or open with flat universes.

Further, it would be also helpful to notice that differential spaces (finitely generated) may be seen as subsets of $\mathbb{R}^n$.

Definition 4.1. *Let (M, C) be the differential space generated by $\{f_1, \ldots, f_n\}$. Consider the mapping $F = (f_1, \ldots, f_n) : (M, C) \to (F(M), C^\infty(\mathbb{R}^n))$. F is called generator embedding.*

It is easy to check that:

Theorem 4.1. *Generator embedding, F, defined as above, is a diffeomorphism.*

Now, the first mentioned problem with gluing of universes can be illustrated by the below example.

Example 4.1. *Consider gluing of two closed, conformally rescalled Friedman universes, i.e. a differential spaces*

$$\begin{aligned} (M_1, C_1) &= \\ ([0, 2\pi] \times \mathbb{S}^2, (\mathrm{sc}\{\{f \circ \mathrm{proj}_{[0,2\pi]} \mid f &\in C^\infty([0, 2\pi])\} \cup \\ \{g \circ \mathrm{proj}_{S^2} \mid g &\in C^\infty(S^2)\}\})_{[0,2\pi]\times\mathbb{S}^2}) \end{aligned}$$

and

$$\begin{aligned} (M_2, C_2) &= \\ ([2\pi, 4\pi] \times \mathbb{S}^2, (\mathrm{sc}\{\{f \circ \mathrm{proj}_{[2\pi,4\pi]} \mid f &\in C^\infty([2\pi, 4\pi])\} \cup \\ \{g \circ \mathrm{proj}_{S^2} \mid g &\in C^\infty(S^2)\}\})_{[2\pi,4\pi]\times\mathbb{S}^2}) \quad . \end{aligned}$$

Each of these universes is itself a classical warped product with scaling function f given by equations: $f(\varphi) = 1 - \cos\varphi$, $t = \varphi - \sin\varphi$, where $\varphi \in (0, 2\pi)$ in case of the first universe and $\varphi \in (2\pi, 4\pi)$ for the sequent universe. (For $\varphi \in \{0, 2\pi, 4\pi\}$ scaling function is 0). After gluing the obtained object is no longer a classical warped product. First of all – because a new (expanded by including limit values) scaling function (defined on $[0, 4\pi]$) is equal to 0 for $\varphi \in \{0, 2\pi, 4\pi\}$. Secondly – because a new scaling function in not (in the classical sense) smooth in $\varphi = 2\pi$. Nevertheless such a function may be included into the differential structure and called smooth in the sense of Def. 2.8. Notice also that the topology induced on the generalised warped product is the same as the initial topology.

The second mentioned problem with gluing of universes can be illustrated by the below example.

Example 4.2. *Consider the differential space (M, C), such that $M = \mathbb{R}$ and $C = (\mathrm{sc}\{f_1, f_2\})_M$, where $f_1 = \mathrm{id}_\mathbb{R}$ and $f_2 = \tan(\frac{\pi}{2}(x - \lfloor x \rfloor))$. This space has a Hausdorff property, because f_1 separates points.*

The generator image $F(M)$ looks like on next figure.

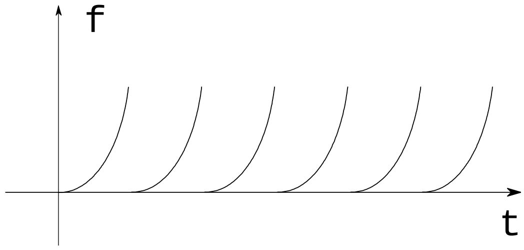

Figure 2 - Glued scale function for a sequence of glued conformally rescalled Friedman open universes.

The line is ripped. Notice, that introducing the new function (a scalar field) to the structure (i.e. f_2) results in changing the geometry of the space. This is somehow consistent with Einstein's ideas.Notice also that the topology is affected. With respect to the initial (Euclidean) topology the glued scale function in right–continuous and lower semi–continuous. As a result, the weakest topology, in which it is continuous, is Scott topology [14].

In fact, the core problem of the gluing of e.g. open universes is illustrated in Ex. 4.2. The full description would need to consider $\mathbb{R} \times \mathbb{R}P^3$ instead of M and more complicated differential structure as in Ex. 4.1. In order to keep the argumentation clear this is omitted. The reader may understand the computational idea from Ex. 4.1, while the constructional idea is clearly presented in simplified Ex. 4.2.

5. Conclusion

In this paper the conceptual new definition of generalised warped product was presented. Its basic properties and possible usefulness for cosmological models were sketched. Also some further researches are planned over curvature, Ricci tensor and geodesics on generalised warped products. Although closed cosmological models have been thoroughly studied in d–spaces formalism (see e.g. [6]), open models have not been studied by such methods. Gluing of them is interesting, for example in the context of the new cosmological idea – Conformal Cyclic Cosmology [11].

Acknowledgments Research funded by the Polish National Science Centre grant under the contract number DEC–2012/06/A/ST1/00256. The author is also very thankful to prof. W. Sasin for his support in the scientific activity.

References

[1] Batubenge, A., Iglesias-Zemmour, P., Karshon, Y. and Watts, J. *Diffeological, Frölicher, and differential spaces*, preprint, 2013.

[2] Bishop, R.L. and O'Neill, B., *Manifolds of negative curvature*, Trans. Am. Math. Soc., **145(1969),** 1–49.

[3] Dimitric, R.M., *Is the function 1/x continuous at 0?*, arXiv:1210.2930 [math.GM], 2012.

[4] Drachal, K., *Introduction to d-spaces theory*, Math. Aeterna, **3(2013),** 753–770.
[5] (eds.) Frauendiener, J. and Friedrich H., *The Conformal Structure of Space-Time*, Springer, Berlin, 2002.
[6] Gruszczak, J. and Heller, M. *Differential structure of space-time and its prolongations to singular boundaries*, Int. J. of Theor. Phys., **32(1993),** 625–648.
[7] Gruszczak, J., Heller, M. and Multarzyński, P., *A generalization of manifolds as space-time models*, J. Math. Phys., **29(1988),** 2576–2580.
[8] Kühnel, W. and Rademacher, H.-B., *Conformal transformations of pseudo-Riemannian manifolds*, in: (eds.) Alekseevsky, D.V. and Baum, H., *Recent Developments in Pseudo-Riemannian Geometry*, European Mathematical Society, Zürich, 2008.
[9] Moreno, G., *On the canonical connection for smooth envelopes*, Demonstratio Math., **47(2014),** 459–464.
[10] O'Neill, B., *Semi-Riemannian Geometry*, Academic Press, San Diego, 1983.
[11] Penrose, R., *Cycles of Time*, Bodley Head, London, 2010.
[12] Sasin, W., *Lectures on Differential Spaces and their Applications*, Lectures on Warsaw University of Technology, 2012.
[13] Sasin, W. and Eledrisi, Y., *On a generalization of the Levi-Civita connections of a degenerate metric*, Acta Cosmologica, **21(1995),** 247–250.
[14] Scott, D., *Continuous lattices*, in (ed.) Lawvere, F.W., *Algebraic Geometry and Logic*, Springer, Berlin, 1972.
[15] Sikorski, R., *Wstęp do geometrii różniczkowej*, Państwowe Wydawnictwo Naukowe, Warszawa, 1972.

FACULTY OF MATHEMATICS AND INFORMATION SCIENCE
WARSAW UNIVERSITY OF TECHNOLOGY
00 – 662 WARSZAWA, KOSZYKOWA 75, POLAND
E-mail address: k.drachal@mini.pw.edu.pl