INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 4 (2015), No. 1, 11 - 15

A FEW INEQUALITIES IN QUADRILATERALS

MARTIN JOSEFSSON

Abstract. We prove a few inequalities regarding the diagonals, the angle between them, and the area of convex quadrilaterals.

1. Characterizations of rectangles via quadrilateral inequalities

In [4] we proved in five different ways that a convex quadrilateral with consecutive sides a, b, c, d is a rectangle if and only if its area is given by

$$K = \tfrac{1}{2}\sqrt{(a^2+c^2)(b^2+d^2)}. \tag{1}$$

Here we shall prove a few similar characterizations of rectangles regarding other quantities by first deriving inequalities that hold in all convex quadrilaterals. These were not included in either of the well known books [2] or [5] on geometric inequalities. We start by deriving a double inequality for the product of the diagonals.

Theorem 1.1. *The product of the diagonals p and q in a convex quadrilateral with consecutive sides a, b, c, d satisfies*

$$\left|a^2-b^2+c^2-d^2\right| < 2pq \leq a^2+b^2+c^2+d^2$$

with equality on the right hand side if and only if the quadrilateral is a rectangle.

Proof. It is well known that in all convex quadrilaterals with consecutive sides a, b, c, d, it holds that

$$\left|a^2-b^2+c^2-d^2\right| = 2pq\cos\theta \tag{2}$$

where θ is the acute angle between the diagonals p and q (see [3, p.27]). From this equality, the left hand side of the double inequality follows at once, since $\cos\theta < 1$ except in an uninteresting degenerate case where the quadrilateral has collapsed into a line segment.

Keywords and phrases: Quadrilateral, Area, Inequality, Rectangle
(2010)Mathematics Subject Classification: 51M04, 51M16, 51M25
Received: 25.11.2014. In revised form: 23.12.2014. Accepted: 10.01.2015.

To prove the right hand side, we use Euler's generalization of the parallelogram law. It states that the sides of a convex quadrilateral satisfy

$$a^2+b^2+c^2+d^2=p^2+q^2+4v^2$$

where v is the distance between the midpoints of the diagonals p and q (see [1, p.126]). By adding and subtracting $2pq$ to the right hand side of this equality, we get

$$a^2+b^2+c^2+d^2=2pq+(p-q)^2+4v^2\ge 2pq,$$

where there is equality if and only if $v=0$ and $p=q$. The first equality is a well known characterization of parallelograms, and the only parallelograms with equal diagonals are rectangles. □

Next we prove inequalities for the trigonometric functions of the angle between the diagonals of a quadrilateral in terms of its sides.

Theorem 1.2. *The acute angle θ between the diagonals in a convex quadrilateral with consecutive sides a, b, c, d satisfies the inequalities*

$$\sin\theta\le\frac{2\sqrt{(a^2+c^2)(b^2+d^2)}}{a^2+b^2+c^2+d^2},$$
$$\cos\theta\ge\frac{\left|a^2-b^2+c^2-d^2\right|}{a^2+b^2+c^2+d^2},$$
$$\tan\theta\le\frac{2\sqrt{(a^2+c^2)(b^2+d^2)}}{|a^2-b^2+c^2-d^2|}.$$

Equality hold in either of these if and only if the quadrilateral is a rectangle.

Proof. We prove the second inequality first. Using (2) and Theorem 1.1, we get

$$\left|a^2-b^2+c^2-d^2\right|=2pq\cos\theta\le(a^2+b^2+c^2+d^2)\cos\theta$$

and the inequality follows.

To prove the first one, we have according to the trigonometric version of the Pythagorean theorem that

$$1=\sin^2\theta+\cos^2\theta\ge\sin^2\theta+\left(\frac{a^2-b^2+c^2-d^2}{a^2+b^2+c^2+d^2}\right)^2.$$

Thus

$$\sin^2\theta\le\frac{(a^2+b^2+c^2+d^2)^2-(a^2-b^2+c^2-d^2)^2}{(a^2+b^2+c^2+d^2)^2}.$$

The inequality follows by factoring the numerator.

Finally, the third inequality follows at once from simplifying

$$\tan\theta=\frac{\sin\theta}{\cos\theta}\le\frac{2\sqrt{(a^2+c^2)(b^2+d^2)}}{a^2+b^2+c^2+d^2}\cdot\frac{a^2+b^2+c^2+d^2}{|a^2-b^2+c^2-d^2|}.$$

There is equality in either of the three inequalities if and only if the quadrilateral is a rectangle according to Theorem 1.1. □

Since it is such a nice formula, we emphasize that a part of the last theorem states that a convex quadrilateral with consecutive sides a, b, c, d is a rectangle if and only if the acute angle between the diagonals satisfies

$$\cos\theta = \frac{\left|a^2 - b^2 + c^2 - d^2\right|}{a^2 + b^2 + c^2 + d^2}.$$

From this formula we also get the well known condition that the diagonals of a rectangle (where $a = c$ and $b = d$) are perpendicular if and only if it is a square, a characterization that of course can be proved using more elementary methods.

Using the two theorems above, we get a sixth proof of the area formula (1).

Theorem 1.3. *The area of a convex quadrilateral with consecutive sides a, b, c, d satisfies*

$$K \leq \tfrac{1}{2}\sqrt{(a^2 + c^2)(b^2 + d^2)}$$

with equality if and only if the quadrilateral is a rectangle.

Proof. The area of a convex quadrilateral is given by (see [3, p.27])

$$K = \frac{1}{4}(2pq)\sin\theta \leq \frac{1}{4}(a^2 + b^2 + c^2 + d^2)\frac{2\sqrt{(a^2 + c^2)(b^2 + d^2)}}{a^2 + b^2 + c^2 + d^2}$$

and the inequality follows by simplification. Equality holds if and only if the quadrilateral is a rectangle according to Theorems 1.1 and 1.2. □

2. More inequalities for the area of a convex quadrilateral

Now we will prove three more area inequalities for convex quadrilaterals in terms of the sides. We cannot find these either in [2] or [5].

Theorem 2.1. *The area of a convex quadrilateral with consecutive sides a, b, c, d satisfies*

$$K \leq \tfrac{1}{8}\big((a + c)^2 + 4bd\big)$$

with equality if and only if the quadrilateral is an orthodiagonal isosceles trapezoid with $a = c$.

Proof. For the area of a convex quadrilateral with diagonals p and q, we have

$$K = \tfrac{1}{2}pq\sin\theta \leq \tfrac{1}{2}pq \leq \tfrac{1}{2}(ac + bd) \tag{3}$$

where θ is the angle between the diagonals and we used Ptolemy's inequality (see [1, p.129]) in the last step. Thus we have equality if and only if the diagonals are perpendicular and the quadrilateral is cyclic. The AM-GM-inequality yields

$$(a + c)^2 = a^2 + c^2 + 2ac \geq 2ac + 2ac = 4ac$$

where equality holds if and only if $a = c$. Hence we get

$$8K \leq 4ac + 4bd \leq (a + c)^2 + 4bd \tag{4}$$

from which the inequality in the theorem follows. Equality holds if and only if the quadrilateral is cyclic with two opposite congruent sides and perpendicular diagonals. This is equivalent to an orthodiagonal isosceles trapezoid. $\square$

A more symmetric expression for the area than the one in Theoem 2.1 yields a more regular quadrilateral:

Theorem 2.2. *The area of a convex quadrilateral with consecutive sides a, b, c, d satisfies*

$$K \leq \tfrac{1}{8}\big((a+c)^2 + (b+d)^2\big)$$

with equality if and only if the quadrilateral is a square.

Proof. By using the AM-GM-inequality also for the other pair of opposite sides, we have $(b+d)^2 \geq 4bd$ with equality if and only if $b = d$. Inserting this into (4), we get the inequality in this theorem. Equality holds if and only if the quadrilateral is cyclic, has perpendicular diagonals and both pairs of opposite sides are congruent. This is equivalent to a square. $\square$

A *right kite* is defined to be a kite with two opposite right angles (or a cyclic kite, in which case the two opposite right angles is a direct corollary). Next we derive an inequality for the area of a convex quadrilateral that is related to a right kite. It is very similar to the one in Theorem 1.3.

Theorem 2.3. *The area of a convex quadrilateral with consecutive sides a, b, c, d satisfies*

$$K \leq \tfrac{1}{2}\sqrt{(a^2+b^2)(c^2+d^2)}$$

where equality holds if and only if the quadrilateral is a right kite.

Proof. According to (3), the area of a convex quadrilateral satisfies

$$K \leq \tfrac{1}{2}(ac+bd)$$

where equality holds if and only if it's a cyclic orthodiagonal quadrilateral. Now we use an algebraic identity due to Diophantus of Alexandria

$$(ac+bd)^2 + (ad-bc)^2 = (a^2+b^2)(c^2+d^2),$$

which directly yields the two dimensional Cauchy-Schwarz inequality

$$ac + bd \leq \sqrt{(a^2+b^2)(c^2+d^2)}.$$

Equality holds if and only if $ad = bc$. Hence the area of a convex quadrilateral satisfies

$$K \leq \tfrac{1}{2}\sqrt{(a^2+b^2)(c^2+d^2)}$$

where there is equality if and only if the quadrilateral if cyclic, $a^2 + c^2 = b^2 + d^2$ (the condition for perpendicular diagonals) and $ad = bc$. Combining the two conditions on the sides yields

$$\frac{b^2c^2}{d^2} + c^2 = b^2 + d^2 \quad \Leftrightarrow \quad b^2c^2 + c^2d^2 = b^2d^2 + d^4 \quad \Leftrightarrow \quad (c^2-d^2)(b^2+d^2) = 0$$

so equality holds (regarding the sides) if and only if $c = d$ and $a = b$, that is for a kite. This proves that equality in the theorem holds if and only if the quadrilateral is a cyclic kite, i.e. a right kite. $\square$

By symmetry we can make the change $b \leftrightarrow d$ to get a similar inequality. This corresponds to considering a quadrilateral with the sides in reverse order a, d, c, b compared to the last theorem. Thus we have

Corollary 2.1. *The area of a convex quadrilateral with consecutive sides a, b, c, d satisfies*

$$K \leq \tfrac{1}{2}\sqrt{(a^2+d^2)(b^2+c^2)}$$

where equality holds if and only if the quadrilateral is a right kite.

References

[1] Court, N. A., *College Geometry*, Dover reprint, 2007.

[2] Bottema, O., Djordjevic, R. Z., Janic, R. R., Mitrinović D. S. and Vasić, P. M., *Geometric Inequalities*, Wolters-Noordhoff, The Netherlands, 1969.

[3] Durell, C. V. and Robson, A., *Advanced Trigonometry*, G. Bell and Sons, London, 1930. New edition by Dover Publications, Mineola, 2003.

[4] M. Josefsson, *Five proofs of an area characterization of rectangles*, Forum Geom., **13(2013)**, 17–21.

[5] Mitrinović, D. S., Pečarić, J. E. and Volenec, V., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Groningen, 1989.

SECONDARY SCHOOL "KCM"
MARKARYD, SWEDEN
E-mail address: `martin.markaryd@hotmail.com`