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ARCHIMEDEAN CIRCLES PASSING THROUGH A SPECIAL POINT

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Abstract. We consider infinitely many Archimedean circles of the arbelos passing through a special point.

1. Introduction

Each of the two congruent areas surrounded by three mutually touching circles with collinear centers in the plane is called arbelos. The radical axis of the two inner circles divides each of the arbeloi into two curvilinear triangles with congruent incircles. Circles congruent to those congruent circles are called Archimedean circles of the arbelos, which are one of the main topics on the arbelos.

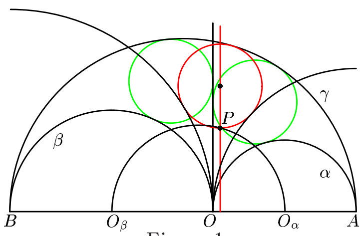

Figure 1.

We denote the circle with a diameter PQ by (PQ) for two points P and Q. The center of a circle δ is denoted by O_δ. Let O be a point on the segment AB, $|AO| = 2a$, $|BO| = 2b$, $\alpha = (AO)$, $\beta = (BO)$ and $\gamma = (AB)$. We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively. The common radius of Archimedean circles is $r_{\mathrm{A}} = ab/(a + b)$. Thomas Schoch has considered the two circles with centers A and B and passing through the point O, and has found that the circles touching the two circles externally and the circle γ internally are Archimedean [1].

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The line joining their centers is called the Schoch line (see Figure 1). We denote the point of intersection of the circle $(O_\alpha O_\beta)$ and the Schoch line lying in the region $y > 0$ by P. In this article we consider infinitely many Archimedean circles passing through this point.

2. A characterization of Archimedean circles passing through P

In this section we characterize the Archimedean circles passing through the point P. Let $s = (b-a)r_{\mathrm{A}}/(a+b)$. The Schoch line is expressed by the equation $x = s$ [6]. The circle $(O_\alpha O_\beta)$ is expressed by the equation $(x-a)(x+b)+y^2 = 0$. Let $p = \sqrt{(3a+b)(a+3b)}$. The point P has coordinates

$$(p_x, p_y) = \left(s, \frac{pr_{\mathrm{A}}}{a+b}\right). \tag{1}$$

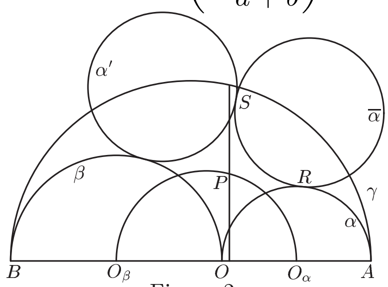

Figure 2.

Let α' be the reflection of the circle α in the point P (see Figure 2). The center of α' has coordinates

$$(2p_x - a, 2p_y).$$

Let $\overline{\alpha}$ be the circle of radius a touching the circles α and α' externally and lying on the side opposite to β of the line $O_\alpha P$. The center of $\overline{\alpha}$ has coordinates

$$(2r_{\mathrm{A}}, r_{\mathrm{A}}p/b). \tag{2}$$

Therefore *the smallest circle passing through the point $O_{\overline{\alpha}}$ and touching the radical axis of α and β is Archimedean.* Let $t = a/(2(a+b))$ and $u = a(a+3b)/(2(a+b)^2)$. The circle $\overline{\alpha}$ touches α and α' at points R and S with coordinates

$$((a+3b)t, pt) \text{ and } ((b-a)u, pu), \tag{3}$$

respectively. The internal center of similitude of $\overline{\alpha}$ and β divides the segment $O_{\overline{\alpha}}O_\beta$ in the ratio $a : b$ internally and has coordinates equal to (1). Hence *the point P is the internal center of similitude of $\overline{\alpha}$ and β.* The next theorem is a generalization of results in [5].

Theorem 2.1. *Let ϵ and ζ be circles being outside each other and having radius e and f, respectively, such that their internal center of similitude is a point I. Let K be the external center of similitude of the circles (IJ) and ζ for a point J. Then the following three statements are equivalent.*
(i) The circle (IJ) has radius $ef/(e+f)$.

(ii) The point K lies on ϵ and the vectors $\overrightarrow{O_\epsilon K}$ and $\overrightarrow{IJ}$ are parallel and have the same direction.
(iii) The point K lies on ϵ.

Proof. We may assume that the centers of ϵ and ζ has coordinates $(ke, 0)$ and $(-kf, 0)$, respectively for a real number k. If (IJ) has radius $r = ef/(e+f)$, then its center has coordinates $(r\cos\theta, r\sin\theta)$ for a real number θ. Then S has coordinates

$$\left(\frac{-rkf + fr\cos\theta}{f-r}, \frac{fr\sin\theta}{f-r}\right) = (ke + e\cos\theta, e\sin\theta).$$

Therefore (i) implies (ii). Obviously (ii) implies (iii). Let us assume (iii) and (IJ) has radius g. Then the center of (IJ) is expressed by $(g\cos\theta, g\sin\theta)$ for a real number θ. But K coincides with the center of similitude of ζ and the circle of radius r with center $(r\cos\theta, r\sin\theta)$ as shown just above. Therefore we get $g = r$. Therefore (iii) implies (i).

By the theorem we get the following corollary (see Figure 2).

Corrolary 2.1. *Let δ be a circle passing through P, and let K (resp. L) be the external center of similitude of δ and β (resp. $\overline{\alpha}$). If δ is Archimedean, the vectors $\overrightarrow{PO_\delta}$, $\overrightarrow{O_{\overline{\alpha}}K}$ and $\overrightarrow{O_\beta L}$ are parallel and have the same direction. The following three statements are equivalent.*
(i) The circle δ is Archimedean.
(ii) The point K lies on the circle $\overline{\alpha}$.
(iii) The point L lies on the circle β.

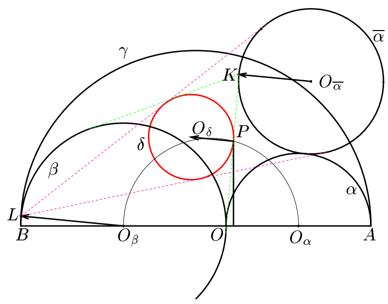

Figure 3.

3. A quadruplet of Archimedean circles

From now on, we consider special Archimedean circles passing through P. If two congruent circles of radius r touch at a point D and also touch a circle δ at points different from D, we say D generates circles of radius r with δ. If the two circles are Archimedean, we say D generates Archimedean circles with δ. Since this kind of Archimedean circles were firstly discovered by Frank Power [7], we call those circles Power type Archimedean circles. In this section we give several Power type Archimedean circles passing through P. The following lemma is needed [2].

Lemma 3.1. *If δ is a circle of radius r, a point D generates circles of radius $|r^2 - |DO_\delta|^2|/(2r)$ with δ.*

By the lemma with (1), we get the following theorem (see Figure 4).

Theorem 3.1. *The point P generates Archimedean circles with each of the circles α and β.*

The line joining the centers of the two Archimedean circles touching α is perpendicular to the line PO_α. Therefore it passes through the point O_β. Similarly the line joining the centers of the two Archimedean circles touching β passes through the point O_α. Hence the two Archimedean circles touching α are orthogonal to the two Archimedean circles touching β. Therefore the centers of the four Archimedean circles form vertices of a square. The four centers also lie on the Archimedean circle with center P.

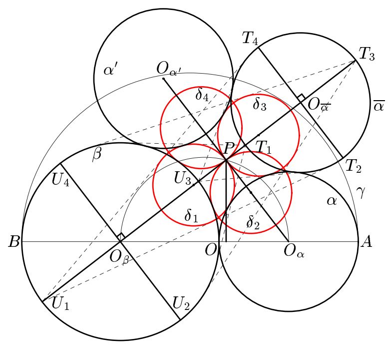

Figure 4.

Let δ_1 (resp. δ_3) be the Power type Archimedean circle touching α and also touching PO_α from the side opposite to $\overline{\alpha}$ (resp. β). Also let δ_2 (resp. δ_4) be the Power type Archimedean circle touching β and also touching PO_β from the side opposite to α' (resp. α). Let T_i (resp. U_i) ($i = 1, 2, 3, 4$) be the external center of similitude of δ_i and β (resp. $\overline{\alpha}$). Then T_1 (resp. U_3) is the closest point on $\overline{\alpha}$ (resp. β) to P, and T_3 (resp. U_1) is the farthest point on $\overline{\alpha}$ (resp. β) from P. The point T_2 and T_4 are the midpoints of the circular arcs of $\overline{\alpha}$ with endpoints T_1 and T_3. Similar fact also holds for the point U_i ($i = 1, 2, 3, 4$).

4. Another quadruplet

We show that there is another quadruplet of Archimedean circles passing through the point P. Unexpectedly the golden ratio appears in this case. Let Q be the point of intersection of OP and γ. The circle $(O_\alpha O_\beta)$ is the image of γ by the homothety with homothety center O and ratio $1/2$. Therefore P is the midpoint of the segment OQ, and Q has coordinates $(2p_x, 2p_y)$. The point Q also lies on the circle α'. Let ϵ_1 and ϵ_2 be circles with center O and radii $\phi|OP|$ and $\phi^{-1}|OP|$, respectively, where ϕ is the value of the golden ratio, i.e., $\phi = (1+\sqrt{5})/2$ (see Figure 5).

Theorem 4.1. *The point P generates Archimedean circles with each of the circles ϵ_1 and ϵ_2, and the Archimedean circles generated by P with ϵ_1 coincide with the Archimedean circles generated by P with ϵ_2. Also the circles (OP) and (PQ) are Archimedean, which are orthogonal to the Archimedean circles generated by P with ϵ_1.*

Proof. By (1), $|OP| = 2r_{\mathrm{A}}$. By the Lemma 3.1 with this fact, we can see that P generates Archimedean circles with each of the circles ϵ_1 and ϵ_2.

Since O is the common center of ϵ_1 and ϵ_2, the generated circles coincide. The rest of the theorem is obvious.

The centers of the four Archimedean circles also form vertices of a square. The circle (OP) passes through the point of intersection of the Schoch line and the line AB. Recall (3), i.e., the point R has coorinates $((a+3b)t, pt)$. But the external center of similitude of the circles β and (OP) has coordinates

$$\left(\frac{-r_{\rm A}(-b)+bp_x/2}{b-r_{\rm A}}, \frac{bp_y/2}{b-r_{\rm A}}\right) = ((a+3b)t, pt).$$

Therefore *the external center of similitude of β and (OP) coincides with the point R*. This implies that the lines OP and $O_\alpha R$ are parallel by Theorem 2.1. Hence the external center of similitude of β and (PQ) is the point of intersection of the line $O_\alpha R$ and the circle $\overline{\alpha}$ different from R.

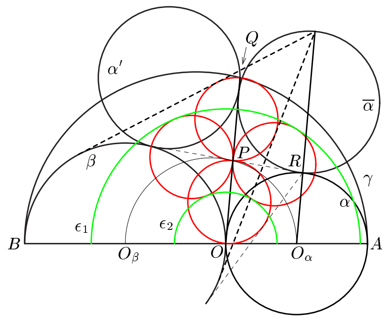

Figure 5.

Since (OP) passes through O, and O is the internal center of similitude of α and β, the external center of similitude of α and (OP) lies on β [5]. We denote this point by T. Let T' be the external center of similitude of $\overline{\alpha}$ and (PQ). Then $\overrightarrow{OP}$ and $\overrightarrow{O_\beta T}$, and $\overrightarrow{PQ}$ and $\overrightarrow{O_\beta T'}$ are parallel and have the same direction, respectively. Therefore the points T and T' coincide.

5. A pair of Archimedean circles

Let U be the point of intersection of the segment AP and the circle α, and let V be the point of intersection of the segment BQ and the circle β (see Figure 6). Then $OUQV$ is a rectangle with center P. Hence $|UV| = |OQ|$. Therefore the circles (PU) and (PV) are Archimedean. These circles are also obtained in a special case considered in [2] and [3]. They touch at P. The points U and V have coordinates

$$(u_x, u_y) = \left(\frac{2r_{\rm A}^2(3a+b)}{(a+b)a}, \frac{2r_{\rm A}^2 p}{(a+b)b}\right) \text{ and } (v_x, v_y) = \left(-\frac{2r_{\rm A}^2(a+3b)}{(a+b)b}, \frac{2r_{\rm A}^2 p}{(a+b)a}\right),$$

respectively.

Recall (3), i.e., the point S has coordinates $((b-a)u, pu)$. Since $pu/((b-a)u) = p/(b-a) = py/px$, it lies on the line OP. On the other hand, the external center of similitude of the circles β and (PV) has coordinates

$$\left(\frac{-r_{\rm A}(-b)+b(p_x+v_x)/2}{b-r_{\rm A}}, \frac{b(p_y+v_y)/2}{b-r_{\rm A}}\right) = ((b-a)u, pu).$$

Hence *the external center of similitude of β and (PV) coincides with S*.

Let us assume $a \neq b$, and let W be the point of intersection of the line OP and β different from O. It has coordinates $(r_{\rm A}(b-a)w, r_{\rm A}pw)$, where $w = (a-b)/(2a(a+b))$. It lies in the region $y > 0$ or $y < 0$ according as $a > b$ or $a < b$. By (2), the external center of similitude of $\overline{\alpha}$ and (PU) has coordinates

$$\left(\frac{-r_{\rm A}2r_{\rm A} + a(p_x+u_x)/2}{a-r_{\rm A}}, \frac{-r_{\rm A}r_{\rm A}p/b + a(p_y+u_y)/2}{a-r_{\rm A}}\right) = (r_{\rm A}(b-a)w, r_{\rm A}pw).$$

Therefore *the external center of similitude of* $\overline{\alpha}$ *and* (PU) *coincides with* W.

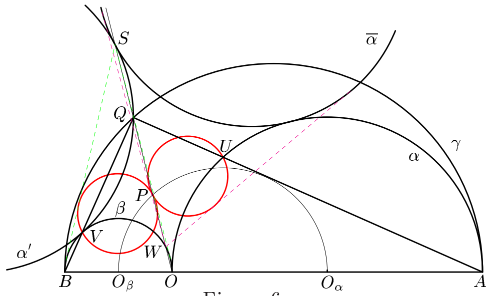

Figure 6.

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