INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 3 (2014), No. 2, 74 - 80

# A SYNTHETIC PROOF OF A. MYAKISHEV'S GENERALIZATION OF VAN LAMOEN CIRCLE THEOREM AND AN APPLICATION

OAI THANH DAO

**Abstract.** In this article we give a synthetic proof of A.Myakishev's generalization of van Lamoen's circle theorem and introduce a family six circumcenters lie on circle of a triangle associated with Kiepert's configuration.

## 1. Introduction

The famous van Lamoen circle theorem states that: *If a triangle is divided by its three medians into 6 smaller triangles, then the circumcenters of these smaller triangles lie on a circle.*

The Van Lamoen circle be introduced in Floor van Lamoen Problem 10830, American Mathematical Monthly 107 (2000) 863 (see [5]); solution by the editors, 109 (2002) 396-397 . The proof of van Lamoen circle theorem can be found in many texts, see [3], [4], [6] or [8]. In 2002, A.Myakishev's generalization of van Lamoen circle theorem as follows:

**Theorem 1.1** (A. Myakishev-[7]). *Let two triangles $ABC$ and $A_1B_1C_1$ perpective and its have same the centroid, if the perpector of two triangles is $D$, then circumcenter of six triangles $ADB_1$, $B_1DC$, $CDA_1$, $A_1DB$, $BDC_1$, $C_1DA$ lie on a circle.*

The first proof of A. Myakishev's theorem by Darij Grinberg, see [2]. In the paper we give another synthetic proof A. Myakishev's theorem and in the application we show that exist a family six circumcenters lie on a circle of a triangle associated with Kiepert's configuration.

**Keywords and phrases:** van Lamoen circle, Triangle, six circumcenter lie on a circle
**(2010)Mathematics Subject Classification:** 51P99, 60A99
Received: 25.06.2014. In revised form: 08.09.2014. Accepted: 30.09.2014.

## 2. A PROOF OF A.MYAKISHEV THEOREM

We omit the proof of a easy lemma following:

**Lemma 2.1.** *Let four circles $(O_1), (O_2), (O_3), (O_4)$ concurrent at $D$. The circle $(O_i)$ meets $(O_{i+1})$ again at $D_{i(i+1)}$ with $i = 1, 2, 3, 4$ and $(O_5) \equiv (O_1)$. Then $O_1, O_2, O_3, O_4$ lie on a circle if only if $\angle D_{23}DD_{12} = \angle D_{34}DD_{41}$ (mod$\pi$).*

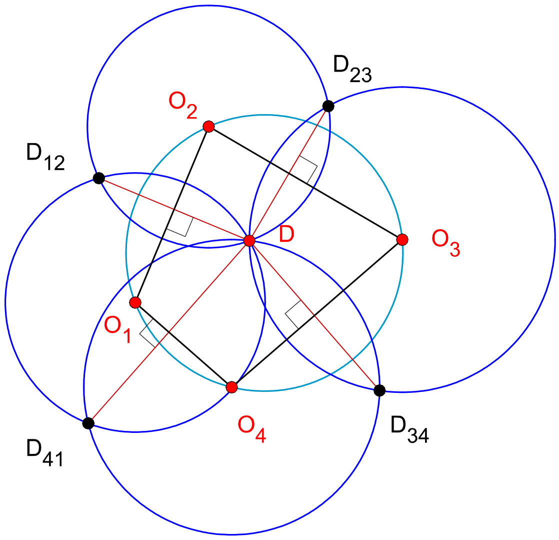


Figure 1

**Lemma 2.2.** *Let $ACA_1C_1$ be a quadrilateral, $AA_1$ meets $CC_1$ at $D$. Let $N, N_1$ be the midpoint of $AC, A_1C_1$ respectively. $NN_1$ meets $AD$ at $F$. The circle $(ADC_1)$ meets the circle $(CDA_1)$ again at $Q$. Then $\angle AFN = \angle QDC$ (mod $\pi$)*

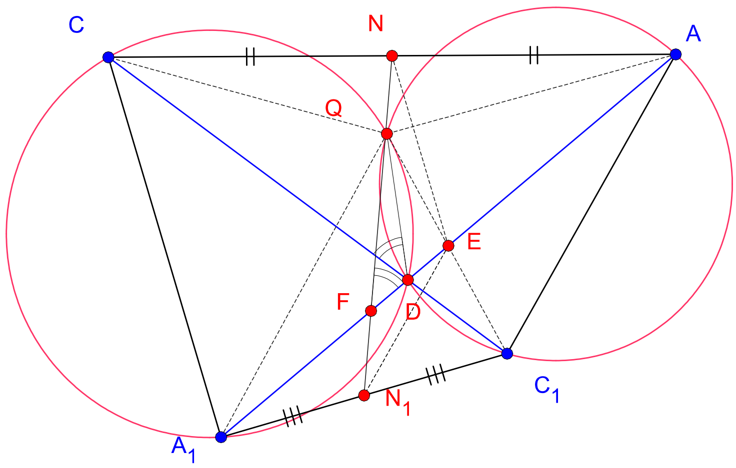


Figure 2

**Proof.** Since $\angle C_1AQ = \angle C_1DQ = \angle CA_1Q$ (mod $\pi$) and $\angle AC_1Q = \angle A_1DQ = \angle A_1CQ$ (mod $\pi$), hence two triangles $AQC_1, A_1QC$ are similar $\Rightarrow \frac{C_1Q}{AC_1} = \frac{CQ}{A_1C}$.

Let $E$ be the midpoint of $AA_1$. We obtain $\overrightarrow{AC_1} = 2\overrightarrow{EN_1}$, $\overrightarrow{A_1C} = 2\overrightarrow{EN} \Rightarrow$

$$\frac{C_1Q}{EN_1} = \frac{CQ}{EN} \tag{1}$$

On the other hand: $\angle C_1QC = \angle C_1QD + \angle DQC \pmod{\pi}$. But $\angle C_1QD = \angle C_1DA \pmod{\pi} = \angle(\overrightarrow{AC_1}, \overrightarrow{DA})$, and $\angle DQC = \angle DA_1C \pmod{\pi} = \angle(\overrightarrow{A_1D}, \overrightarrow{A_1C})$. Thus:

$$\begin{aligned} \angle C_1QC &= \angle(\overrightarrow{AC_1}, \overrightarrow{DA}) + \angle(\overrightarrow{A_1D}, \overrightarrow{A_1C}) = \\ \angle(\overrightarrow{AC_1}, \overrightarrow{A_1C}) &= \angle(\overrightarrow{EN_1}, \overrightarrow{EN}) = \angle N_1EN (\bmod \ \pi) \end{aligned} \tag{2}$$

Since (1) and (2) we get that two triangles $C_1QC, N_1EN$ are similar, since

$$\angle ENN_1 = \angle QCC_1 \tag{3}$$

We obtain: $\angle AFN = \angle FEN + \angle ENF \pmod{\pi}$. On the other hand: $\angle FEN = \angle DA_1C \pmod{\pi}$. And since (3) we obtain $\angle ENF = \angle QCD = \angle QA_1D \pmod{\pi}$. Therefore, $\angle AFN = \angle QA_1D + \angle DA_1C = \angle QA_1C = \angle QDC \pmod{\pi}$. The completes the proof of Lemma 2.2.

*Proof of A. Myakishev theorem:*

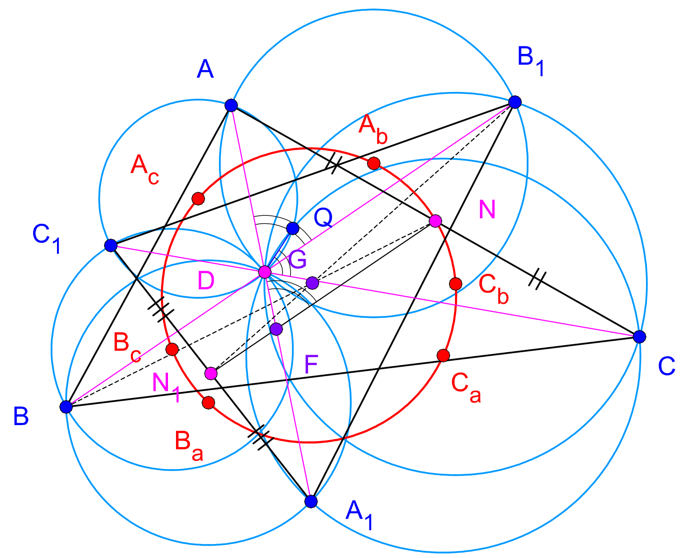


Figure 3

Let $A_c, B_c, B_a, C_a, C_b, A_b$ are circumcenter of six circles $(ADC_1)$, $(C_1DB)$, $(BDA_1)$, $(A_1DC)$, $(CDB_1)$, $(B_1DA)$ respectively. Let $N, N_1$ be the midpoint of $AC, A_1C_1$ respectively, and $NN_1$ meets $AA_1$ at $F$. The circle $(ADC_1)$ meets $(CDA_1)$ again at $Q$. Easily we deduce that $BB_1$, $NN_1$ are parallel, therefor $\angle ADB_1 = \angle AFN \pmod{\pi}$. By Lemma 2.2 we obtain $\angle AFN = \angle QDC$. Thus $\angle ADB_1 = \angle QDC$. By Lemma 2.1 we get that four circumcenters $A_c, A_b, C_b, C_a$ lie on a circle. Similarly, four circumcenters $A_b, C_b, C_a, B_a$ lie on a circle, and $C_b, C_a, B_a, A_b$ lie on a circles. Hence six circumcenters $A_c, B_c, B_a, C_a, C_b, A_b$ lie on a circle. This completes the proof of Myakishev's theorem.

## 3. AN APPLICATION OF A.MYAKISHEV THEOREM

**Theorem 3.1** (Dao-[1]). *Let $ABC$ be a triangle, $G$ is the centroid. Constructed three similar isosceles triangles $AC_0B, BA_0C, CB_0A$ (either all outward, or all inward). Let $A_1, B_1, C_1$ lie on $AA_0, BB_0, CC_0$ respectively, such that:*

$$\frac{\overline{AA_1}}{\overline{AA_0}} = \frac{\overline{BB_1}}{\overline{BB_0}} = \frac{\overline{CC_1}}{\overline{CC_0}} = k_1 \tag{4}$$

*Let $A_2, B_2, C_2$ lie on $GA_1, GB_1, GC_1$ respectively, such that:*

$$\frac{\overline{GA_2}}{\overline{GA_1}} = \frac{\overline{GB_2}}{\overline{GB_1}} = \frac{\overline{GC_2}}{\overline{GC_1}} = k_2 \tag{5}$$

*Then $AA_2, BB_2, CC_2$ are concurrent at a point $K$ lie on Kiepert hyperbola. And circumcenter of six triangles $AKB_2$, $B_2KC$, $CKA_2$, $A_2KB$, $BKC_2$, $C_2KA$ lie on a circle.*

- When $A_0, B_0, C_0$ at midpoint of $BC, CA, AB$ respectively and $k_1 = k_2 = 1$, this circle is called as the van Lamoen circle.
- When $A_0, B_0, C_0$ at midpoint of $BC, CA, AB$ respectively and $k_2 = 0$, and $k_1$ is any real number, this circle is called as the Dao six point circle [4].

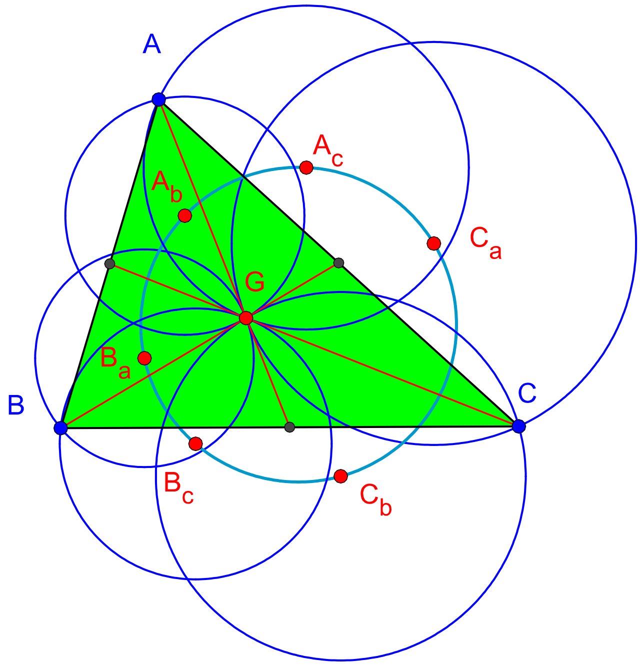


Figure 4

**Proof.** By (4) we get

$$\frac{\overline{AA_0} + \overline{A_0A_1}}{\overline{AA_0}} = \frac{\overline{BB_0} + \overline{B_0B_1}}{\overline{BB_0}} = \frac{\overline{CC_0} + \overline{C_0C_1}}{\overline{CC_0}} \tag{6}$$

$\Leftrightarrow$

$$\frac{\overline{A_0A}}{\overline{A_0A_1}} = \frac{\overline{B_0B}}{\overline{B_0B_1}} = \frac{\overline{C_0C}}{\overline{C_0C_1}} \tag{7}$$

$\Leftrightarrow$

$$\frac{\overline{A_0A_1} + \overline{A_1A}}{\overline{A_0A_1}} = \frac{\overline{B_0B_1} + \overline{B_1B}}{\overline{B_0B_1}} = \frac{\overline{C_0C_1} + \overline{C_1C}}{\overline{C_0C_1}} \tag{8}$$

$\Leftrightarrow$

$$\frac{\overline{A_1A_0}}{\overline{A_1A}} = \frac{\overline{B_1B_0}}{\overline{B_1B}} = \frac{\overline{C_1C_0}}{\overline{C_1C}} \tag{9}$$

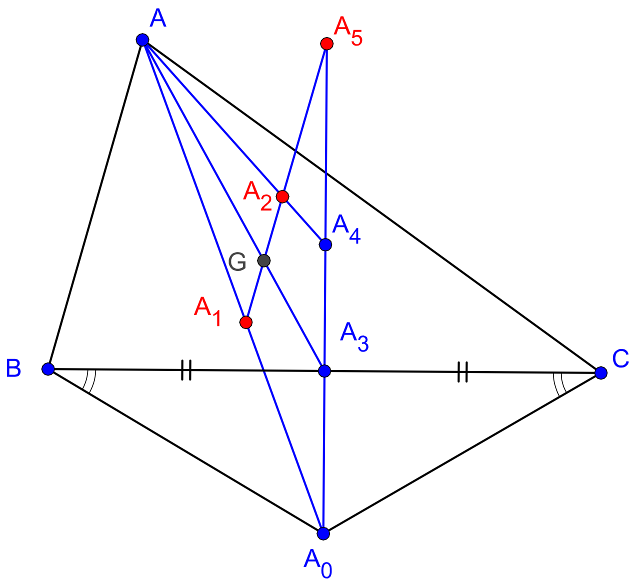


Figure 5

Denote $A_3, B_3, C_3$ are midpoint of $BC, CA, AB$ repectively. Let $AA_2, BB_2, CC_2$ meet $A_3A_0, B_3B_0, C_3C_0$ at $A_4, B_4, C_4$ respectively. Let $GA_1, GB_1, GC_1$ meet $A_3A_0, B_3B_0, C_3C_0$ at $A_5, B_5, C_5$ respectively. By Menelaus' theorem for $\triangle AA_3A_0$ cut by $\overline{A_1A_5G}$, we obtain:

$$\frac{\overline{A_5A_0}}{\overline{A_5A_3}} = \frac{\overline{GA}}{\overline{GA_3}} \cdot \frac{\overline{A_1A_0}}{\overline{A_1A}} = 2\frac{\overline{A_1A_0}}{\overline{A_1A}} \tag{10}$$

Similarly we obtain:

$$\frac{\overline{B_5B_0}}{\overline{B_5B_3}} = 2\frac{\overline{B_1B_0}}{\overline{B_1B}} \tag{11}$$

$$\frac{\overline{C_5C_0}}{\overline{C_5C_3}} = 2\frac{\overline{C_1C_0}}{\overline{C_1C}} \tag{12}$$

Since (10),(11),(12) and (9), we obtain:

$$\frac{\overline{A_5A_0}}{\overline{A_5A_3}} = \frac{\overline{B_5B_0}}{\overline{B_5B_3}} = \frac{\overline{C_5C_0}}{\overline{C_5C_3}} \tag{13}$$

By Menelaus' theorem for $\triangle AGA_1$ cut by $\overline{A_0A_5A_3}$, we obtain:

$$\frac{\overline{A_5G}}{\overline{A_5A_1}} = \frac{\overline{A_3G}}{\overline{A_3A}} \cdot \frac{\overline{A_0A}}{\overline{A_0A_1}} = \frac{1}{3} \cdot \frac{\overline{A_0A}}{\overline{A_0A_1}} \tag{14}$$

Similarly we obtain:

$$\frac{\overline{B_5G}}{\overline{B_5B_1}} = \frac{1}{3} \cdot \frac{\overline{B_0B}}{\overline{B_0B_1}} \tag{15}$$

$$\frac{\overline{C_5G}}{\overline{C_5C_1}} = \frac{1}{3} \cdot \frac{\overline{C_0C}}{\overline{C_0C_1}} \tag{16}$$

Since (14),(15),(16) and (7) we get:

$$\frac{\overline{A_1A_5}}{\overline{GA_5}} = \frac{\overline{B_1B_5}}{\overline{GB_5}} = \frac{\overline{C_1C_5}}{\overline{GC_5}} \tag{17}$$

$\Leftrightarrow$

$$\frac{\overline{A_1G} + \overline{GA_5}}{\overline{GA_5}} = \frac{\overline{B_1G} + \overline{GB_5}}{\overline{GB_5}} = \frac{\overline{C_1G} + \overline{GC_5}}{\overline{GC_5}} \tag{18}$$

$\Leftrightarrow$

$$\frac{\overline{GA_1}}{\overline{GA_5}} = \frac{\overline{GB_1}}{\overline{GB_5}} = \frac{\overline{GC_1}}{\overline{GC_5}} \tag{19}$$

Since (19) we have: $A_1B_1 \parallel A_5B_5, B_1C_1 \parallel B_5C_5, C_1A_1 \parallel C_5A_5$. And since (5) we have: $A_1B_1 \parallel A_2B_2, B_1C_1 \parallel B_2C_2, C_1A_1 \parallel C_2A_2$. on the other hand $G, A_1, A_2, A_5$ are collinear, $G, B_1, B_2, B_5$ are collinear; $G, C_1, C_2, C_5$ are collinear. Now by Thales'theorem we obtain:

$$\frac{\overline{A_2A_5}}{\overline{A_2A_1}} = \frac{\overline{B_2B_5}}{\overline{B_2B_1}} = \frac{\overline{C_2C_5}}{\overline{C_2C_1}} \tag{20}$$

By Menelaus' theorem for $\triangle A_5A_0A_1$ cut by $\overline{AA_2A_4}$, we obtain:

$$\frac{\overline{A_4A_5}}{\overline{A_4A_0}} = \frac{\overline{AA_1}}{\overline{AA_0}} \cdot \frac{\overline{A_2A_5}}{\overline{A_2A_1}} \tag{21}$$

Similarly we get:

$$\frac{\overline{B_4B_5}}{\overline{B_4B_0}} = \frac{\overline{BB_1}}{\overline{BB_0}} \cdot \frac{\overline{B_2B_5}}{\overline{B_2B_1}} \tag{22}$$

$$\frac{\overline{C_4C_5}}{\overline{C_4C_0}} = \frac{\overline{CC_1}}{\overline{CC_0}} \cdot \frac{\overline{C_2C_5}}{\overline{C_2C_1}} \tag{23}$$

Since (4),(20),(21), (22) and (23) we have:

$$\frac{\overline{A_4A_5}}{\overline{A_4A_0}} = \frac{\overline{B_4B_5}}{\overline{B_4B_0}} = \frac{\overline{C_4C_5}}{\overline{C_4C_0}} \tag{24}$$

Three triangles $BC_0A$, $AB_0C$ and $CA_0B$ are three similar isosceles triangle either all outward, or all inward on the sides $\triangle ABC$. Since (13) we get that three triangles $BA_5C, CB_5A, AC_5B$ are similar isosceles triangle. Since (25) we get that three triangles $BC_4A$, $AB_4C$, $CA_4B$ are similar isosceles triangle on the sides. By famous Kiepert theorem we have the lines $AA_4$, $BB_4$, $CC_4$ concurrent on Kiepert hyperbola, so $AA_2, BB_2, CC_2$ concurrent on Kiepert hyperbola. It is well-known that two triangles $ABC$, $A_0B_0C_0$ have same the centroid. Since (4) we can show that two triangles $A_0B_0C_0$ and $A_1B_1C_1$ have same the centroid. Since (5) we can show that two triangles $A_1B_1C_1$ and $A_2B_2C_2$ have same the centroid. Therefore, two triangles $ABC$ and $A_2B_2C_2$ have same the centroid. By Myakishev theorem we get that the circumcenter of six triangles $AKB_2$, $B_2KC$, $CKA_2$, $A_2KB$, $BKC_2$, $C_2KA$ lie on a circle. This completes the proof of Theorem 3.1.

## References


[1] Dao, T., *Dao Thanh Oai problem*, Art of Problem Solving, Oct. 24, 2013. available at http://www.artofproblemsolving.com/Forum/viewtopic.php?f=47t=559546

[2] Grinberg, D., *Hyacinthos message 7351*, Jul 13, 2003.

[3] Ha, N. M., *Another Proof of van Lamoen's Theorem and Its Converse*, Forum Geom., **5(2005)**, 127-132.

[4] Kimberling, C., *X(5569) Center of the Dao six point circle*, Nov. 3, 2013. Encyclopedia of Triangle Centers, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[5] Lamoen, F. M., *Problem 10830*, Amer. Math. Monthly, **107(2000),** 893.

[6] Li, K., *Concyclic problem*, Mathematical Excalibur, **6(1)(2001)**, 1-2.

[7] Myakishev, A. and Wolk, B., *Hyacinthos message 5612*, May 31, 2002.

[8] Myakishev, A. and Woo, P., *On the Circumcenters of Cevasix Configuration*, Forum Geom., **3(2003),** 57-63.



SON LA HYDROPOWER COMPANY
VIETNAM
*E-mail address*: daothanhoai@hotmail.com