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DAO'S THEOREM ON CONCURRENCE OF THREE EULER LINES

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Abstract. In this article we give a synthetic proof of Dao's theorem on concurrence of three Euler lines.

1. Introduction

Concurrence of three Euler lines is always a nice result in euclidean geometry, see [2][3][4] and [5]. In 2014, Oai Thanh Dao proposed a new remarkable theorem for concurrence of the Euler lines of three triangles.

Consider ABC be a triangle and a line D parallel to the Euler line of ABC. Let A_1, B_1, C_1 be the intersection of D and the sidelines BC,CA,AB respectively. Let A', B', C' be the midpoint of B_1C_1,C_1A_1,A_1B_1 respectively. Let A_2, B_2, C_2 be the reflection of A, B, C in A', B', C' respectively. The Dao theorem refers to the point X_{110} in the Encyclopedia of Triangle Centers [6], as follows:

Theorem 1.1 (Dao-[1])**.** *The Euler lines of triangles $A_2B_1C_1$, $B_2A_1B_1$, $C_2A_1B_1$ concur in a point on the line joining X_{110} and the following point: The orthocenter of the paralogic triangle of ABC whose perspectrix is the Euler line of ABC.*

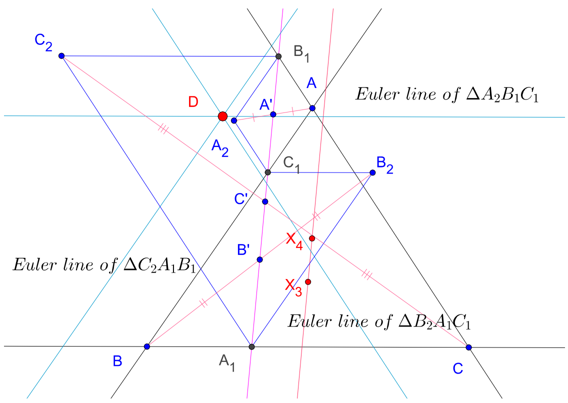

Figure 1

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In this article we give a synthetic proof of this theorem.

2. SYNTHETIC PROOF OF THEOREM 1.1

Lemma 2.1. *Let H, O, G are the orthocenter, the circumcenter and the centroid of the triangle ABC. Let X be a point lie on BC such that $AX \parallel OG$. Let Z be the reflection of B in midpoint of AX. Then the Euler line of triangle ZXA is parallel to AC.*

Proof. Let N be the midpoint of BC, let H' be the point on AH such that $H'Y \parallel AC$, let G' lie on YH' such that $GG' \parallel BC$, denote $Y = OG \cap BC$ and $M = AG' \cap BC$. Since triangles ABX and XZA are symmetric with respect to the midpoint of AX. So we need only prove that the Euler line of ABX is parallel to AC.

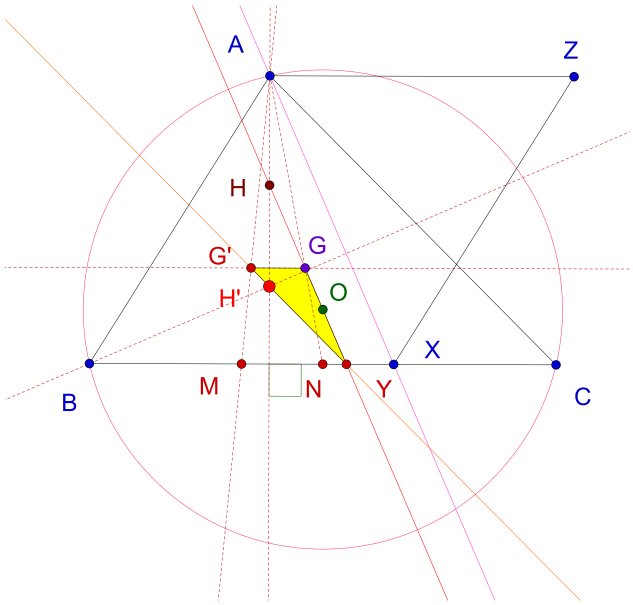

Figure 2

Since $H'Y \parallel AC$ and $BH \perp AC$, we have $BH \perp H'Y$ and $YB \perp HH'$, so that B is the orthocenter of triangle YHH'. Consequently, we have $BH' \perp HY$, so that $BH' \perp AX$.

Alos, since $AH' \perp BX$, H' is the orthocenter of $\triangle ABX$. Also $GG' \parallel CX$, $GY \parallel XA$, $YG' \parallel AC$ so that triangles YGG' and AXC are homothetic. Moreover, $MN = (3/2)GG' = (3/2)(GY/AX).XC = (1/2)XC$. Since $BN = \frac{1}{2}BC$, we have M is the midpoint of XB, and $AG'/G'M = AG/GN = 2/1$, so that G' is the centroid of ABX. Consequently, $H'G'$ is the Euler line of triangle ABX and is parallel to AC. This completes the proof of Lemma 2.1.

We return now to the proof of Theorem 1.1

Proof. Let L_A, L_B, L_C be the lines through A_1, B_1, C_1 and perpendicular to BC, CA, AB respectively. Let $H_A = L_B \cap L_C$, $H_B = L_C \cap L_A$, and $H_C = L_A \cap L_B$. Let L'_A, L'_B, L'_C be the line through H_A, H_B, H_C parallel to BC, CA, AB, respectively.

By the Lemma 2.1, the Euler lines of three triangles $A_2B_1C_1$, $B_2C_1A_1$, $C_2A_1B_1$ are parallel to BC, CA, AB respectively. Note that H_A, H_B, H_C are the respective orthocenters of the triangles $A_2B_1C_1$, $B_2C_1A_1$, $C_2A_1B_1$, so that L'_A, L'_B, L'_C are the Euler lines of these triangles. And they concur in the orthocenter of triangle $H_AH_BH_C$ (see Figure 3).

In special case, when D is the Euler line of the triangle ABC, by Sondat's theorem, the Euler line bisects the segment whose endpoints are the orthocenters of triangles ABC and $H_AH_BH_C$. On the other hand, the orthocenter of triangle ABC lies on the Euler line of ABC, so that the point of concurrence of five Euler lines is the orthocenter of the paralogic triangle of ABC whose perspectrix is the Euler line of ABC.

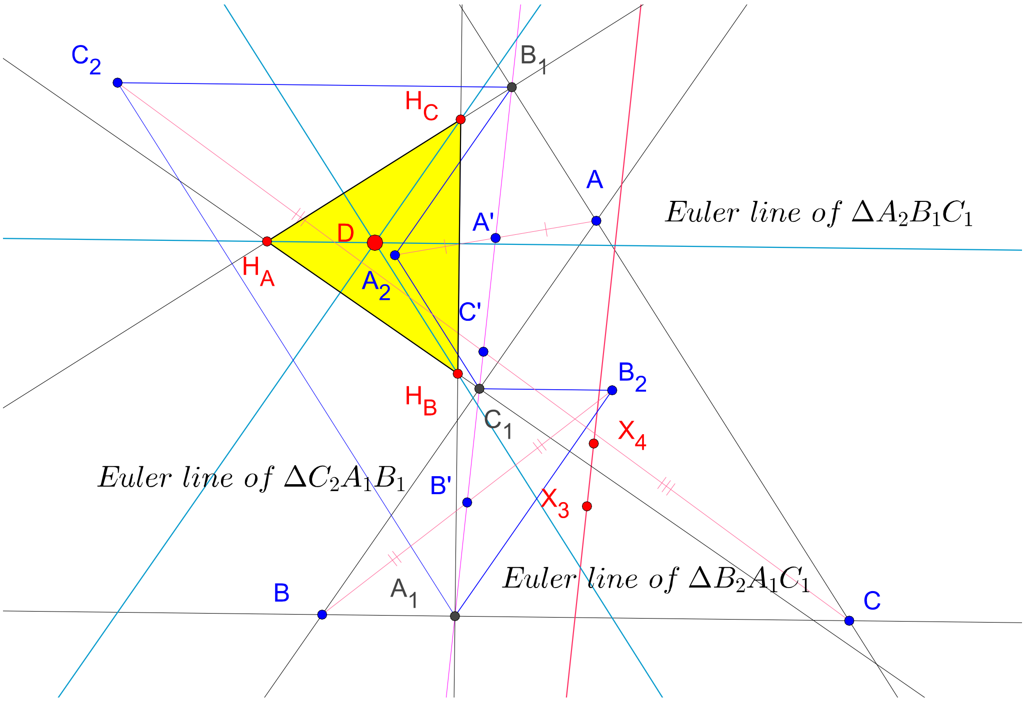

Figure 3

Let A_0, B_0, C_0 be the points of intersection of the Euler line of triangle ABC and the sidelines BC, CA, AB respectively. Denote $H_{A0}H_{B0}H_{C0}$ be the triangle formed by three lines through A_0, B_0, C_0 and perpendicular to BC, CA, AB respectively.

It's well-known that X_{110} is the Euler reflection point of ABC. X_{110} is point E in Figure 4. Let A'', B'', C'' are the projection of X_{110} on BC, CA, AB respectively. By the Simson line theorem, A'', B'', C'' are collinear and the line $\overline{A''B''C''}$ is parallel to the Euler line of ABC. Since $B''B_0/B_0B_1 = C''C_0/C_0C_1$, the triangles $X_{110}C''B''$, $H_{A0}C_0B_0$, $H_AC_1B_1$ are homothetic, with center A, so that A, H_{A0}, H_A, X_{110} are collinear (see Figure 4). Similarity we have B, H_{B0}, H_B, X_{110} are collinear and C, H_{C0}, H_C, X_{110} are collinear, so that X_{110} is the homothetic center of triangles $H_{A0}H_{B0}H_{C0}$ and triangle $H_AH_BH_C$.

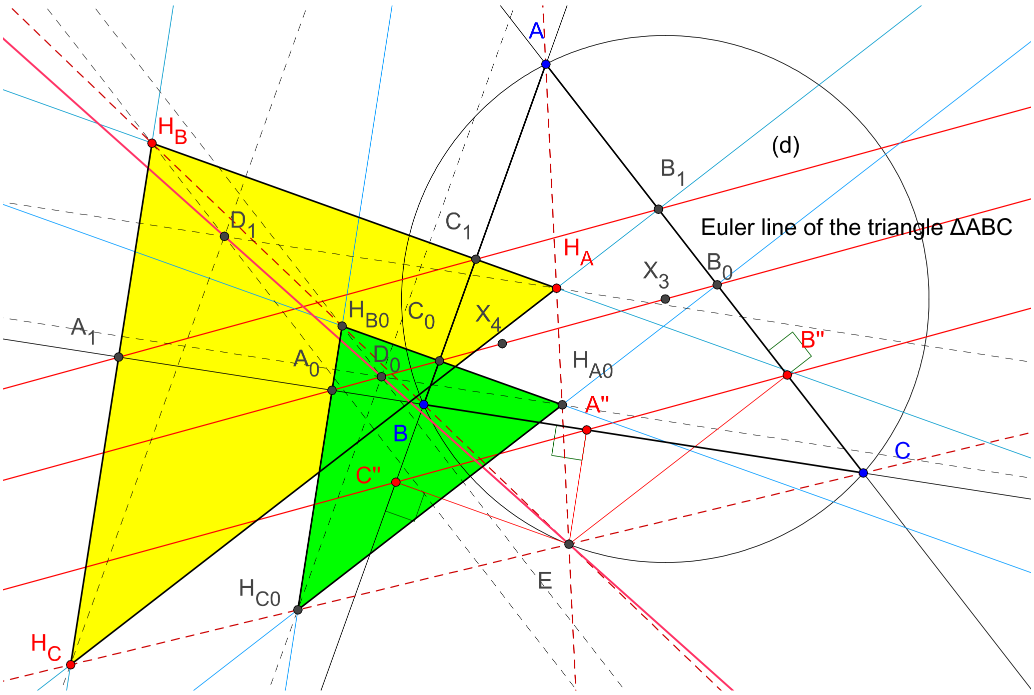

Figure 4

Thus, the orthocenter of $H_AH_BH_C$ lies on the line joining X_{110} and the orthocenter of the paralogic triangle of ABC whose perspectrix is the Euler line of ABC. This complete the proof of Theorem 1.1.

For completeness, we record the coordinates of D given by Peter Moses. If D is parallel to the Euler line through some point $P(p,q,r)$, the concurrence is [1]:

$(S^2-3S_BS_C)(p(S^2-3S_AS_B)(S^2-3S_CS_A)-qS_B(S_A-S_C)(S^2-3S_AS_B)-rS_C(S_A-S_B)(S^2-3S_AS_C)) : ... : ...$

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