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MATRIX THEORY OVER THE SPLIT QUATERNIONS

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Abstract: Split quaternions have been expressed in terms of 4×4 matrices by means of Hamilton operators. These matrices can be used to describe rotations in 4-dimensional space E_2^4. In this paper, by De Moivre's formula, we obtain any powers of these matrices. Also, the relation between the powers of matrices of split quaternions is given.

1. Introduction

Split quaternions, H', or coquaternions are elements of a 4-dimensional associative algebra introduced by James Cockle in 1849. Like the quaternions introduced by Hamilton in 1843, they form a four dimensional real vector space equipped with a multiplicative operation. Unlike the quaternion algebra, the split quaternions contain zero divisors, nilpotent elements, and nontrivial idempotents. Manifolds endowed with coquaternion structures are studied in differential geometry and superstring theory. Rotations in Minkowski 3-space can be stated with split quaternions, such as expressing Euclidean rotations using quaternions [3, 11, 12].

Some algebraic properties of Hamilton operators are considered in [2] where real quaternions have been expressed in terms of 4×4 matrices by means of these operators. These matrices have applications in many fields, such as mechanics, quantum physics and computer-aided geometric design [1]. In addition to, Yaylı has considered homothetic motions with aid of the Hamilton operators in four-dimensional Euclidean space E^4 [17]. The eigenvalues, eigenvectors and the others algebraic properties of these matrices are studied by several authors [4, 6, 8]. The Euler's and De-Moivre's formulas for the complex numbers are generalized for quaternions [5]. These formulas are also investigated for the case of dual quaternions in [7, 10].

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Recently, we have derived the De-Moivre's and Euler's formulas for matrices associated with real quaternion and every power of these matrices are immediately obtained [9]. Euler and De Moivre's formulas for split quaternions are expressed in [13] and the roots of a split quaternion with respect to the causal character of the split quaternion are given. Here, after a review of some properties of split quaternions, De Moivre's and Euler's formulas for the matrices associated with these quaternions are studied. In special cases, De Moivre's formula implies that there are uncountably many matrices of unit split quaternions satisfying $A^n = I_4$ for $n \geqslant 3$. Furthermore, the n-th roots of these matrices are derived. We give some examples for more clarification.

2. Preliminaries

In this section, we give a brief summary of the split quaternions. For detailed information about these concepts, we refer the reader to Ref. [3, 11, 12, 13].

Definition 2.1. *The Minkowski space E_1^3 is the Euclidean space E^3 provided with the Lorentzian inner product*

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_l = -u_1v_1 + u_2v_2 + u_3v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)\,, \overrightarrow{v} = (v_1, v_2, v_3) \in E^3$. We say that a vector $\overrightarrow{u}$ in E_1^3 is spacelike, lightlike or timelike if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_l > 0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_l = 0$ or $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_l < 0$ respectively. The norm of the vector $\overrightarrow{u} \in E_1^3$ is defined by $\|\overrightarrow{u}\| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle_l|}$.

The Lorentzian vector product $\overrightarrow{u} \wedge_l \overrightarrow{v}$ of $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as follows:

$$\overrightarrow{u} \wedge_l \overrightarrow{v} = \begin{vmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The hyperbolic and Lorentzian unit spheres are

$$H_\circ^2 = \{ \overrightarrow{a} \in E_1^3 : \langle \overrightarrow{a}, \overrightarrow{a} \rangle_l = -1 \}$$

and

$$S_1^2 = \{ \overrightarrow{a} \in E_1^3 : \langle \overrightarrow{a}, \overrightarrow{a} \rangle_l = 1 \},$$

respectively.

Definition 2.2. *The semi-Euclidean 4-space with 2-index is represented with E_2^4. The inner product of this semi-Euclidean space*

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{E_2^4} = -u_1v_1 - u_2v_2 + u_3v_3 + u_4v_4.$$

We say that $\overrightarrow{u}$ is timelike, spacelike or lightlike if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{E_2^4} < 0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{E_2^4} > 0$ and $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{E_2^4} = 0$ for the vector $\overrightarrow{u}$ in E_2^4 respectively.

Definition 2.3. *A split quaternion is defined as*

$$q = a_\circ + a_1 i + a_2 j + a_3 k,$$

where a_0, a_1, a_2 and a_3 are real numbers and $1, i, j, k$ of q may be interpreted as the four basic vectors of cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= -1, \quad j^2 = k^2 = +1 \\ ij &= k = -ji, \quad jk = -i = -kj, \end{aligned}$$

$$ki = j = -ik$$

and hence $ijk = +1$. A split quaternion may be defined as a pair $\left(S_q, \overrightarrow{V}_q\right)$, where $S_q = a_\circ \in \mathbb{R}$ is scalar part and $\overrightarrow{V}_q = a_1 i + a_2 j + a_3 k$ is the vector part of q. Vector parts of the split quaternions are identified with the Minkowski 3-space. The split quaternion product of two quaternions q and p is defined as

$$qp = S_q S_p + \langle \overrightarrow{V}_q, \overrightarrow{V}_p \rangle_l + S_q \overrightarrow{V}_p + S_p \overrightarrow{V}_q + \overrightarrow{V}_q \wedge_l \overrightarrow{V}_p,$$

here"$\langle , \rangle_l$" and "$\wedge_l$" are Lorentzian inner and vector product, respectively. The quaternion product may be written as

$$qp = \begin{bmatrix} a_\circ & -a_1 & a_2 & a_3 \\ a_1 & a_\circ & a_3 & -a_2 \\ a_2 & a_3 & a_\circ & -a_1 \\ a_3 & -a_2 & a_1 & a_\circ \end{bmatrix} \begin{bmatrix} b_\circ \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Thus, the space H' correspondence with semi-Euclidean four-space E_2^4. The conjugate of a split quaternion, denoted K_q, is defined as $K_q = S_q - \overrightarrow{V}_q$.

Theorem 2.1. *The algebra H' is isomorphic to the algebra $\mathbb{R}_2$.*

Proof. The real (2×2)-matrices are linear combination of the basis matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

whose multiplication rules coincide with the multiplication rules of the basis elements $1, i, j, k$ in the algebra H'. Hence the subalgebra consisting of these matrices and the algebra H' are isomorphism [13].

Theorem 2.2. *The algebra H' is isomorphic to the subalgebra of the algebra $\mathbb{C}_2$ consisting of the (2×2)-matrices*

$$\hat{A} = \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix},$$

and to the subalgebra of the algebra $\mathbb{C}_2'$ cosisting of the (2×2)-matrices

$$\mathring{A} = \begin{bmatrix} A & B \\ -\overline{B} & \overline{A} \end{bmatrix}.$$

Proof. The proof can be found in [13].

Definition 2.4. *We say that a split quaternion q is spacelike, lightlike (null) or timelike if* $I_q < 0$, $I_q = 0$ *or* $I_q > 0$ *respectively where;*

$$I_q = -\langle q, q\rangle_{E_2^4} = a_\circ^2 + a_1^2 - a_2^2 - a_3^2.$$

The set of spacelike quaternions is not a group since it is not closed under multiplication. That is, the product of two spacelike quaternions is timelike. Whereas, the set of timelike quaternions denoted by

$$H_T' = \{q = (a_\circ, a_1, a_2, a_3) : \ a_\circ, a_1, a_2, a_3 \in \mathbb{R}, \ I_q > 0 \ \},$$

forms a group under the split quaternion product. Also, the set of unit timelike quaternions identified with semi-Euclidean sphere

$$S_2^3 = \{\overrightarrow{u} \in E_2^4 : \ \langle \overrightarrow{u}, \overrightarrow{u}\rangle_{E_2^4} = 1\}$$

is subgroup of H_T'. The vector part of any spacelike quaternion is spacelike, but vector part of any timelike quaternion can be spacelike, timelike and null.

Definition 2.5. *The norm of split quaternion* $q = a_\circ + a_1 i + a_2 j + a_3 k$ *is*

$$N_q = \sqrt{\left|a_\circ^2 + a_1^2 - a_2^2 - a_3^2\right|}.$$

If $N_q = 1$ then q is called unit split quaternion and $q_\circ = \frac{q}{N_q}$ is a unit split quaternion for $N_q \neq 0$. Also, spacelike and timelike quaternions have multiplicative inverse and they hold the property $qq^{-1} = q^{-1}q = 1$. Lighelike quaternions have no inverses.

Definition 2.6. *A matrix A is called a semi-orthogonal matrix if* $A\varepsilon A^t\varepsilon = A^t\varepsilon A\varepsilon = I_4$, $\det A = 1$ *where* I_4 *is an identity matrix and*

$$\varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} [11].$$

3. De Moivre's formula for split qauternions

Now, let's express any split quaternion $q = a_\circ + a_1 i + a_2 j + a_3 k$ in polar form similar to quaternions and complex numbers. Polar forms of the split quaternions are as follows:

3.1. Spacelike quaternions

Every spacelike quaternion can be written in the form

$$q = N_q(\sinh\theta + \overrightarrow{v}\cosh\theta)$$

where

$$\sinh\theta = \frac{a_0}{N_q}, \cosh\theta = \frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}}{N_q}$$

and

$$\overrightarrow{v} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{-a_1^2 + a_2^2 + a_3^2}} \in S_1^2$$

is a spacelike unit vector in E_1^3.

The product of two spacelike quaternions is timelike. That is, for a unit spacelike quaternion $q = \sinh\theta + \overrightarrow{v}\cosh\theta$, $q^2 = \cosh\theta + \overrightarrow{v}\sinh\theta$.

Theorem 3.1. *(De Moivre formula) Let* $q = \sinh\theta + \overrightarrow{v}\cosh\theta$ *be a unit spacelike quaternion. Then,*

$$q^n = \sinh n\theta + \overrightarrow{v}\cosh n\theta, \; n \text{ is odd}$$
$$q^n = \cosh n\theta + \overrightarrow{v}\sinh n\theta, \; n \text{ is even.}$$

3.2. Timelike quaternion with spacelike vector part

Every timelike quaternion with spacelike vector part can be written in the form

$$q = N_q(\cosh\theta + \overrightarrow{w}\sinh\theta)$$

where

$$\cosh\theta = \frac{a_0}{N_q}, \sinh\theta = \frac{\sqrt{-a_1^2 + a_2^2 + a_3^2}}{N_q}$$

and

$$\overrightarrow{w} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{-a_1^2 + a_2^2 + a_3^2}} \in S_1^2$$

is a spacelike unit vector in E_1^3 and $\overrightarrow{w}^2 = 1$. A unit timelike quaternion q with spacelike vector part (abbreviated UTS) represents a rotation of a three-dimensional non-lightlike Lorentzian vector by a hyperbolic angle 2θ about the axis of q [13].

For example, the polar form of timelike quaternion $q = \sqrt{2} + (\sqrt{2}, \sqrt{2}, -1)$ is $q = \cosh\theta + \overrightarrow{w}\sinh\theta$ where $\theta = \ln(1+\sqrt{2})$ and $\overrightarrow{w} = (\sqrt{2}, \sqrt{2}, -1)$.

Euler's formula for a UTS quaternion holds. Since $\overrightarrow{w}^2 = 1$, we have

$$\begin{aligned} e^{\overrightarrow{w}\theta} &= 1 + \overrightarrow{w}\theta + \frac{\theta^2}{2!} + \overrightarrow{w}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ...(\theta + \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + ...) \\ &= \cosh\theta + \overrightarrow{w}\sinh\theta. \end{aligned}$$

The differential of $e^{\overrightarrow{w}\theta}$ is

$$\frac{d}{d\theta}e^{\overrightarrow{w}\theta} = \sinh\theta + \overrightarrow{w}\cosh\theta = \overrightarrow{w}e^{\overrightarrow{w}\theta} = e^{\overrightarrow{w}\theta}\overrightarrow{w}.$$

Theorem 3.2. *(De Moivre formula) Let* $q = e^{\overrightarrow{w}\theta} = \cosh\theta + \overrightarrow{w}\sinh\theta$ *be a UTS quaternion. Then,*

$$q^n = \cosh n\theta + \overrightarrow{w}\sinh n\theta$$

for $n \in \mathbb{Z}$.
Proof. The proof follows immediately from the induction (see [13]).

3.3. Timelike quaternion with timelike vector part

Every timelike quaternion with timelike vector part can be written in the form

$$q = N_q(\cos\theta + \overrightarrow{u}\sin\theta)$$

where $\cos\theta = \frac{a_0}{N_q}$, $\sin\theta = \frac{\sqrt{a_1^2-a_2^2-a_3^2}}{N_q}$ and $\overrightarrow{u} = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2-a_2^2-a_3^2}} \in H_\circ^2$ is a timelike unit vector in E_1^3 and $\overrightarrow{u}^2 = -1$. Also, a unit timelike quaternion q with timelike vector part (abbreviated UTT) represents a rotation of a three-dimensional non-lightlike Lorentzian vector by an angle 2θ about the axis of q.

For example, the polar form of timelike quaternion $q = 1 + (2, 1, 1)$ is $q = \sqrt{3}(\cos\theta + \overrightarrow{u}\sin\theta) = \sqrt{3}(\frac{1}{\sqrt{3}} + \frac{(2,1,1)}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{3}})$.

Euler's formula for a UTT quaternion also holds. Since $\overrightarrow{u}^2 = -1$, we have

$$\begin{aligned} e^{\overrightarrow{u}\theta} &= 1 + \overrightarrow{u}\theta - \frac{\theta^2}{2!} - \overrightarrow{u}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + ... \\ &= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...) + \overrightarrow{u}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) \\ &= \cos\theta + \overrightarrow{u}\sin\theta. \end{aligned}$$

The differential of $e^{\overrightarrow{u}\theta}$ is

$$\frac{d}{d\theta}e^{\overrightarrow{u}\theta} = -\sin\theta + \overrightarrow{u}\cos\theta = \overrightarrow{u}e^{\overrightarrow{u}\theta} = e^{\overrightarrow{u}\theta}\overrightarrow{u} \quad [11].$$

Theorem 3.3. *(De Moivre formula) Let $q = e^{\overrightarrow{u}\theta} = \cos\theta + \overrightarrow{u}\sin\theta$ be a UTT quaternion. Then,*

$$q^n = \cos n\theta + \overrightarrow{u}\sin n\theta$$

for $n \in \mathbb{Z}$.
Proof. The proof follows immediately from the induction (see [13]).

Corrolary 3.1. *There are uncountably many unit timelike quaternion with timelike vector part satisfying $q^n = 1$ for every integer $n \geq 3$.*

Proof. For every $\overrightarrow{u} \in H_\circ^2$, the quaternion $q = \cos\frac{2\pi}{n} + \overrightarrow{u}\sin\frac{2\pi}{n}$ is of order n. For $n = 1$ or $n = 2$, the quaternion q is independent of $\overrightarrow{u}$.

Note that the corollary 3.1. do not hold for the spacelike quaternions and timelike quaternions with spacelike vector part(TS).

Example: $q = \frac{\sqrt{2}}{2} + (1, 0, \frac{\sqrt{2}}{2}) = \cos\frac{\pi}{4} + \overrightarrow{u}\sin\frac{\pi}{4}$ is of order 8 and $q = -\frac{\sqrt{2}}{2} + (1, \frac{\sqrt{2}}{2}, 0) = \cos\frac{3\pi}{4} + \overrightarrow{u}\sin\frac{3\pi}{4}$ is of order 8, $q = \frac{\sqrt{3}}{2} + (\frac{1}{2}, 0, 0) = \cos\frac{\pi}{6} + \overrightarrow{u}\sin\frac{\pi}{6}$ is of order 12.

3.4. Timelike quaternion with lightlike vector part

Every unit timelike quaternion with null vector part can be written in the form $q = 1 + \overrightarrow{\varepsilon}$ where $\overrightarrow{\varepsilon}$ is a null vector. If $q = 1 + \overrightarrow{\varepsilon}$ is a unit timelike quaternion with null vector part, then $q^n = 1 + n\varepsilon$ and only root of the equation $w^n = q$ is $1 + \frac{\overrightarrow{\varepsilon}}{n}$ [13].

4. De Moivre's Formula for Matrices of Split Qauternions

In this section, we introduce the $\mathbb{R}$-linear transformations representing left multiplication in H' and look for also the De-Moiver's formula for corresponding matrix representation. Let q be a split quaternion, then $\varphi_l : H' \to H'$ defined as follows:

$$\varphi_l(x) = qx, \quad x \in H'.$$

The Hamilton's operator φ_l, could be represented as the matrices;

$$A_{\varphi_l} = \begin{bmatrix} a_\circ & -a_1 & a_2 & a_3 \\ a_1 & a_\circ & a_3 & -a_2 \\ a_2 & a_3 & a_\circ & -a_1 \\ a_3 & -a_2 & a_1 & a_\circ \end{bmatrix}.$$

If q be a unit split quaternion, then φ_l is semi-orthogonal linear transformation. Properties of these matrices are found in [11].

Theorem 4.1. *The ϕ map defined as*

$$\phi : (H', +, .) \to (M_{(4,R)}, \oplus, \otimes)$$

$$\phi(a_0 + a_1 i + a_2 j + a_3 k) \to \begin{bmatrix} a_\circ & -a_1 & a_2 & a_3 \\ a_1 & a_\circ & a_3 & -a_2 \\ a_2 & a_3 & a_\circ & -a_1 \\ a_3 & -a_2 & a_1 & a_\circ \end{bmatrix}$$

is an isomorphism of algebras.

Proof. See [15] for a similar proof.

We can express the matrix A_{φ_l} in polar form. Let q be a UTT quaternion. Since

$$\begin{aligned} q &= a_0 + a_1 i + a_2 j + a_3 k \\ &= \cos\theta + \overrightarrow{u}\sin\theta \\ &= \cos\theta + (u_1, u_2, u_3)\sin\theta \\ &= \cos\theta + (u_1 \sin\theta, u_2 \sin\theta, u_3 \sin\theta) \end{aligned}$$

we have

$$\begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -u_1\sin\theta & u_2\sin\theta & u_3\sin\theta \\ u_1\sin\theta & \cos\theta & u_3\sin\theta & -u_2\sin\theta \\ u_2\sin\theta & u_3\sin\theta & \cos\theta & -u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix}.$$

Theorem 4.2. *(De-Moivre's formula) Let* $q = e^{\overrightarrow{u}\theta} = \cos\theta + \overrightarrow{u}\sin\theta$ *be a UTT quaternion. For an integer* n

$$A = \begin{bmatrix} \cos\theta & -u_1\sin\theta & u_2\sin\theta & u_3\sin\theta \\ u_1\sin\theta & \cos\theta & u_3\sin\theta & -u_2\sin\theta \\ u_2\sin\theta & u_3\sin\theta & \cos\theta & -u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix} \quad (3.1)$$

the n-th power of the matrix reads

$$A^n = \begin{bmatrix} \cos n\theta & -u_1\sin n\theta & u_2\sin n\theta & u_3\sin n\theta \\ u_1\sin n\theta & \cos n\theta & u_3\sin n\theta & -u_2\sin n\theta \\ u_2\sin n\theta & u_3\sin n\theta & \cos n\theta & -u_1\sin n\theta \\ u_3\sin n\theta & -u_2\sin n\theta & u_1\sin n\theta & \cos n\theta \end{bmatrix}.$$

Proof. The proof follows immediately from the induction.

Note that theorem 4.2. holds for spacelike quaternions and timelike quaternions with spacelike vector part (TS).

Corrolary 4.1. *There are uncountably many matrices associated with UTT quaternions satisfying* $A^n = 1$ *for every integer* $n \geq 3$.

Example: Let $q = \frac{\sqrt{2}}{2} + (1, 0, \frac{\sqrt{2}}{2})$ be a UTT quaternion. The matrix corresponding to this quaternion is

$$A = \begin{bmatrix} \cos\frac{\pi}{4} & -u_1\sin\frac{\pi}{4} & u_2\sin\frac{\pi}{4} & u_3\sin\frac{\pi}{4} \\ u_1\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & u_3\sin\frac{\pi}{4} & -u_2\sin\frac{\pi}{4} \\ u_2\sin\frac{\pi}{4} & u_3\sin\frac{\pi}{4} & \cos\frac{\pi}{4} & -u_1\sin\frac{\pi}{4} \\ u_3\sin\frac{\pi}{4} & -u_2\sin\frac{\pi}{4} & u_1\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & 0 & \frac{\sqrt{2}}{2} \\ 1 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -1 \\ \frac{\sqrt{2}}{2} & 0 & 1 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

every powers of this matix are found to be with the aid of Theorem 4.2. , for example, 15- th power is

$$A^{15} = \begin{bmatrix} \cos\frac{15\pi}{4} & -u_1\sin\frac{15\pi}{4} & u_2\sin\frac{15\pi}{4} & u_3\sin\frac{15\pi}{4} \\ u_1\sin\frac{15\pi}{4} & \cos\frac{15\pi}{4} & u_3\sin\frac{15\pi}{4} & -u_2\sin\frac{15\pi}{4} \\ u_2\sin\frac{15\pi}{4} & u_3\sin\frac{15\pi}{4} & \cos\frac{15\pi}{4} & -u_1\sin\frac{15\pi}{4} \\ u_3\sin\frac{15\pi}{4} & -u_2\sin\frac{15\pi}{4} & u_1\sin\frac{15\pi}{4} & \cos\frac{15\pi}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & 1 & 0 & -\frac{\sqrt{2}}{2} \\ -1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 1 \\ -\frac{\sqrt{2}}{2} & 0 & -1 & \frac{\sqrt{2}}{2} \end{bmatrix}.$$

Example: Let $q = \frac{3\sqrt{2}}{4} + (\frac{1}{2}, \frac{1}{2}, -\frac{\sqrt{2}}{4}) = \cosh\theta + \overrightarrow{w}\sinh\theta$ be a UTS quaternion. The matrix corresponding to this quaternion is

$$A = \begin{bmatrix} \cosh\theta & -w_1\sinh\theta & w_2\sinh\theta & w_3\sinh\theta \\ w_1\sinh\theta & \cosh\theta & w_3\sinh\theta & -w_2\sinh\theta \\ w_2\sinh\theta & w_3\sinh\theta & \cosh\theta & -w_1\sinh\theta \\ w_3\sinh\theta & -w_2\sinh\theta & w_1\sinh\theta & \cosh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3\sqrt{2}}{4} & -\frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{4} \\ \frac{1}{2} & \frac{3\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{4} & -\frac{1}{2} \\ -\frac{\sqrt{2}}{4} & -\frac{1}{2} & \frac{1}{2} & \frac{3\sqrt{2}}{4} \end{bmatrix}$$

where $\theta = \ln\sqrt{2}$ and $\overrightarrow{w} = (\sqrt{2}, \sqrt{2}, -1)$. Every powers of this matix are found to be with the aid of De Moivre's theorem, for example, 5- th power is

$$A^5 = \begin{bmatrix} \cosh 5\theta & -w_1\sinh 5\theta & w_2\sinh 5\theta & w_3\sinh 5\theta \\ w_1\sinh 5\theta & \cosh 5\theta & w_3\sinh 5\theta & -w_2\sinh 5\theta \\ w_2\sinh 5\theta & w_3\sinh 5\theta & \cosh 5\theta & -w_1\sinh 5\theta \\ w_3\sinh 5\theta & -w_2\sinh 5\theta & w_1\sinh 5\theta & \cosh 5\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{33}{8\sqrt{2}} & -\frac{31}{8} & \frac{31}{8} & \frac{31}{8\sqrt{2}} \\ \frac{31}{8} & \frac{33}{8\sqrt{2}} & \frac{31}{8\sqrt{2}} & -\frac{31}{8} \\ \frac{31}{8} & \frac{31}{8\sqrt{2}} & \frac{33}{8\sqrt{2}} & -\frac{31}{8} \\ \frac{31}{8\sqrt{2}} & -\frac{31}{8} & \frac{31}{8} & \frac{33}{8\sqrt{2}} \end{bmatrix}$$

5. Euler's Formula for Matrices of Split Quaternions

Let A be a matrix. We choose

$$A = \begin{bmatrix} 0 & -u_1 & u_2 & u_3 \\ u_1 & 0 & u_3 & -u_2 \\ u_2 & u_3 & 0 & u_1 \\ u_3 & -u_2 & u_1 & 0 \end{bmatrix}$$

then one immediately finds $A^2 = -I_4$. We have a netural generalization of Euler's formula for matrix A;

$$\begin{aligned} e^{A\theta} &= I_4 + A\theta + \frac{(A\theta)^2}{2!} + \frac{(A\theta)^3}{3!} + \frac{(A\theta)^4}{4!} + ... \\ &= I_4(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} -)... + A(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...) \\ &= \cos\theta + A\sin\theta, \\ &= \begin{bmatrix} \cos\theta & -u_1\sin\theta & u_2\sin\theta & u_3\sin\theta \\ u_1\sin\theta & \cos\theta & u_3\sin\theta & -u_2\sin\theta \\ u_2\sin\theta & u_3\sin\theta & \cos\theta & -u_1\sin\theta \\ u_3\sin\theta & -u_2\sin\theta & u_1\sin\theta & \cos\theta \end{bmatrix}. \end{aligned}$$

For detalied information about Euler's formula, see [16].

6. $n-th$ Roots of Matrices of Split Quaternions

6.1. The matrix accossiated with the UTT quaternion q is of the form (3.1). In a more general case we assume for the matrix of (3.1)

$$A=\begin{bmatrix} \cos(\theta+2k\pi) & -u_1\sin(\theta+2k\pi) & -u_2\sin(\theta+2k\pi) & -u_3\sin(\theta+2k\pi) \\ u_1\sin(\theta+2k\pi) & \cos(\theta+2k\pi) & -u_3\sin(\theta+2k\pi) & u_2\sin(\theta+2k\pi) \\ u_2\sin(\theta+2k\pi) & -u_3\sin(\theta+2k\pi) & \cos(\theta+2k\pi) & -u_1\sin(\theta+2k\pi) \\ u_3\sin(\theta+2k\pi) & -u_2\sin(\theta+2k\pi) & u_1\sin(\theta+2k\pi) & \cos(\theta+2k\pi) \end{bmatrix}$$

here $k\in\mathbb{Z}$. The equation $x^n=A$ has n roots. Thus

$$A_k^{\frac{1}{n}}=\begin{bmatrix} \cos(\frac{\theta+2k\pi}{n}) & -u_1\sin(\frac{\theta+2k\pi}{n}) & -u_2\sin(\frac{\theta+2k\pi}{n}) & -u_3\sin(\frac{\theta+2k\pi}{n}) \\ u_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -u_3\sin(\frac{\theta+2k\pi}{n}) & u_2\sin(\frac{\theta+2k\pi}{n}) \\ u_2\sin(\frac{\theta+2k\pi}{n}) & -u_3\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) & -u_1\sin(\frac{\theta+2k\pi}{n}) \\ u_3\sin(\frac{\theta+2k\pi}{n}) & -u_2\sin(\frac{\theta+2k\pi}{n}) & u_1\sin(\frac{\theta+2k\pi}{n}) & \cos(\frac{\theta+2k\pi}{n}) \end{bmatrix},$$

for $k=0$, the first root is

$$A_0^{\frac{1}{n}}=\begin{bmatrix} \cos(\frac{\theta}{n}) & -u_1\sin(\frac{\theta}{n}) & -u_2\sin(\frac{\theta}{n}) & -u_3\sin(\frac{\theta}{n}) \\ u_1\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) & -u_3\sin(\frac{\theta}{n}) & \beta u_2\sin(\frac{\theta}{n}) \\ u_2\sin(\frac{\theta}{n}) & -u_3\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) & -u_1\sin(\frac{\theta}{n}) \\ u_3\sin(\frac{\theta}{n}) & -u_2\sin(\frac{\theta}{n}) & u_1\sin(\frac{\theta}{n}) & \cos(\frac{\theta}{n}) \end{bmatrix}$$

for $k=1$, the second root is

$$A_1^{\frac{1}{n}}=\begin{bmatrix} \cos(\frac{\theta+2\pi}{n}) & -u_1\sin(\frac{\theta+2\pi}{n}) & -u_2\sin(\frac{\theta+2\pi}{n}) & -u_3\sin(\frac{\theta+2\pi}{n}) \\ u_1\sin(\frac{\theta+2\pi}{n}) & \cos(\frac{\theta+2\pi}{n}) & -u_3\sin(\frac{\theta+2\pi}{n}) & u_2\sin(\frac{\theta+2\pi}{n}) \\ u_2\sin(\frac{\theta+2\pi}{n}) & -u_3\sin(\frac{\theta+2\pi}{n}) & \cos(\frac{\theta+2\pi}{n}) & -u_1\sin(\frac{\theta+2\pi}{n}) \\ u_3\sin(\frac{\theta+2\pi}{n}) & -u_2\sin(\frac{\theta+2\pi}{n}) & u_1\sin(\frac{\theta+2\pi}{n}) & \cos(\frac{\theta+2\pi}{n}) \end{bmatrix}$$

Similarly, for $k=n-1$, the n th root could be achieved.

6.2. The matrix accossiated with the UTS quaternion q is of the form

$$A=\begin{bmatrix} \cosh\theta & -w_1\sinh\theta & w_2\sinh\theta & w_3\sinh\theta \\ w_1\sinh\theta & \cosh\theta & w_3\sinh\theta & -w_2\sinh\theta \\ w_2\sinh\theta & w_3\sinh\theta & \cosh\theta & -w_1\sinh\theta \\ w_3\sinh\theta & -w_2\sinh\theta & w_1\sinh\theta & \cosh\theta \end{bmatrix},$$

the equation $x^n=A$ has only one root. Thus

$$A^{\frac{1}{n}}=\begin{bmatrix} \cosh\frac{\theta}{n} & -w_1\sinh\frac{\theta}{n} & w_2\sinh\frac{\theta}{n} & w_3\sinh\frac{\theta}{n} \\ w_1\sinh\frac{\theta}{n} & \cosh\frac{\theta}{n} & w_3\sinh\frac{\theta}{n} & -w_2\sinh\frac{\theta}{n} \\ w_2\sinh\frac{\theta}{n} & w_3\sinh\frac{\theta}{n} & \cosh\frac{\theta}{n} & -w_1\sinh\frac{\theta}{n} \\ w_3\sinh\frac{\theta}{n} & -w_2\sinh\frac{\theta}{n} & w_1\sinh\frac{\theta}{n} & \cosh\frac{\theta}{n} \end{bmatrix}.$$

6.3. The matrix accossiated with the unit spacelike quaternion q is of the form

$$A = \begin{bmatrix} \sinh\theta & -v_1\cosh\theta & v_2\cosh\theta & v_3\cosh\theta \\ v_1\cosh\theta & \sinh\theta & v_3\cosh\theta & -v_2\cosh\theta \\ v_2\cosh\theta & v_3\cosh\theta & \sinh\theta & -v_1\cosh\theta \\ v_3\cosh\theta & -v_2\cosh\theta & v_1\cosh\theta & \sinh\theta \end{bmatrix},$$

the equation $x^n = A$ has only one root if n is an odd number. Thus

$$A^{\frac{1}{n}} = \begin{bmatrix} \sinh\frac{\theta}{n} & -v_1\cosh\frac{\theta}{n} & v_2\cosh\frac{\theta}{n} & v_3\cosh\frac{\theta}{n} \\ v_1\cosh\frac{\theta}{n} & \sinh\frac{\theta}{n} & v_3\cosh\frac{\theta}{n} & -v_2\cosh\frac{\theta}{n} \\ v_2\cosh\frac{\theta}{n} & v_3\cosh\frac{\theta}{n} & \sinh\frac{\theta}{n} & -v_1\cosh\frac{\theta}{n} \\ v_3\cosh\frac{\theta}{n} & -v_2\cosh\frac{\theta}{n} & v_1\cosh\frac{\theta}{n} & \sinh\frac{\theta}{n} \end{bmatrix}.$$

7. Relations Between the Arbitrary Powers of Matrice

The relations between the powers of matrices accossiated with a split quaternion can be realized by the following Theorem.

Theorem 7.1. *q is the UTT quaternion with the polar form $q = \cos\varphi + u\sin\varphi$. If $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p \pmod{m}$ is possible if and only if $q^n = q^p$.*

Proof. Let $n \equiv p \pmod{m}$. Then we have $n = a.m + p$, where $a \in \mathbb{Z}$.

$$\begin{aligned} q^n &= \cos n\varphi + \overrightarrow{u}\sin n\varphi \\ &= \cos(am+p)\varphi + \overrightarrow{u}\sin(am+p)\varphi \\ &= \cos(a\frac{2\pi}{\varphi} + p)\varphi + \overrightarrow{u}\sin(a\frac{2\pi}{\varphi} + p)\varphi \\ &= \cos(p\varphi + a2\pi) + \overrightarrow{u}\sin(p\varphi + a2\pi) \\ &= \cos(p\varphi) + \overrightarrow{u}\sin(p\varphi) \\ &= q^p. \end{aligned}$$

Now suppose $q^n = \cos n\varphi + \overrightarrow{u}\sin n\varphi$ and $q^p = \cos p\varphi + \overrightarrow{u}\sin p\varphi$. Since $q^n = q^p$, we have $\cos n\varphi = \cos p\varphi$ and $\sin n\varphi = \sin p\varphi$, which means $n\varphi = p\varphi + 2\pi a$, $a \in \mathbb{Z}$. Thus $n = a\frac{2\pi}{\varphi} + p$, $n \equiv p \pmod{m}$.

Theorem 7.2. *Let q be a timelike quaternion with timelike vector part with the polar form $q = \sqrt{N_q}(\cos\varphi + u\sin\varphi)$. If $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p \pmod{m}$ is possible if and only if $q^n = (\sqrt{N_q})^{n-p}q^p$.*

Theorem 7.3. *q is the UTT quaternion with the polar form $q = \cos\varphi + u\sin\varphi$. Let $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$ and the matrix A corresponds to q. Then $n \equiv p \pmod{m}$ is possible if and only if $A^n = A^p$.*

Proof. Proof is same as above.

Example: Let $q = \frac{\sqrt{2}}{2} + (1, 0, \frac{\sqrt{2}}{2})$ be a UTT quaternion. From the Theorem 4.6, $m = \frac{2\pi}{\pi/4} = 8$, we have

$$\begin{aligned} A &= A^9 = A^{17} = A^{25} = ... \\ A^2 &= A^{10} = A^{18} = A^{26} = ... \\ A^3 &= A^{11} = A^{19} = A^{27} = ... = -I_4 \\ &... \\ A^8 &= A^{16} = A^{24} = ... = I_4. \end{aligned}$$

The square roots of the matrix A can be achieved too

$$A_k^{\frac{1}{2}} = \begin{bmatrix} \cos\left(\frac{2k\pi+45}{2}\right) & -u_1 \sin\left(\frac{2k\pi+45}{2}\right) & u_2 \sin\left(\frac{2k\pi+45}{2}\right) & u_3 \sin\left(\frac{2k\pi+45}{2}\right) \\ u_1 \sin\left(\frac{2k\pi+45}{2}\right) & \cos\left(\frac{2k\pi+45}{2}\right) & u_3 \sin\left(\frac{2k\pi+45}{2}\right) & -u_2 \sin\left(\frac{2k\pi+60}{2}\right) \\ u_2 \sin\left(\frac{2k\pi+45}{2}\right) & u_3 \sin\left(\frac{2k\pi+45}{2}\right) & \cos\left(\frac{2k\pi+45}{2}\right) & -u_1 \sin\left(\frac{2k\pi+45}{2}\right) \\ u_3 \sin\left(\frac{2k\pi+45}{2}\right) & -u_2 \sin\left(\frac{2k\pi+45}{2}\right) & u_1 \sin\left(\frac{2k\pi+45}{2}\right) & \cos\left(\frac{2k\pi+45}{2}\right) \end{bmatrix}$$

The first root for $k = 0$ reads

$$A_0^{\frac{1}{2}} = \begin{bmatrix} \cos\frac{\pi}{8} & -u_1 \sin\frac{\pi}{8} & u_2 \sin\frac{\pi}{8} & u_3 \sin\frac{\pi}{8} \\ u_1 \sin\frac{\pi}{8} & \cos\frac{\pi}{8} & u_3 \sin\frac{\pi}{8} & -u_2 \sin\frac{\pi}{8} \\ u_2 \sin\frac{\pi}{8} & u_3 \sin\frac{\pi}{8} & \cos\frac{\pi}{8} & -u_1 \sin\frac{\pi}{8} \\ u_3 \sin\frac{\pi}{8} & -u_2 \sin\frac{\pi}{8} & u_1 \sin\frac{\pi}{8} & \cos\frac{\pi}{8} \end{bmatrix}$$

and the second one for $k = 1$ becomes

$$A_1^{\frac{1}{2}} = \begin{bmatrix} \cos\frac{9\pi}{8} & -u_1 \sin\frac{9\pi}{8} & u_2 \sin\frac{9\pi}{8} & u_3 \sin\frac{9\pi}{8} \\ u_1 \sin\frac{9\pi}{8} & \cos\frac{\pi}{8} & u_3 \sin\frac{\pi}{8} & -u_2 \sin\frac{9\pi}{8} \\ u_2 \sin\frac{9\pi}{8} & u_3 \sin\frac{9\pi}{8} & \cos\frac{9\pi}{8} & -u_1 \sin\frac{9\pi}{8} \\ u_3 \sin\frac{9\pi}{8} & -u_2 \sin\frac{9\pi}{8} & u_1 \sin\frac{9\pi}{8} & \cos\frac{9\pi}{8} \end{bmatrix}.$$

Also, $A_0^{\frac{1}{2}} + A_1^{\frac{1}{2}} = 0$.

Example: Let $q = \frac{\sqrt{2}}{4} + \frac{3}{2}(1, 1, -\frac{\sqrt{2}}{2}) = \sinh\theta + \overrightarrow{v}\cosh\theta$ be a unit spacelike quaternion. The matrix corresponding to this quaternion is

$$A = \begin{bmatrix} \sinh\theta & -v_1\cosh\theta & v_2\cosh\theta & v_3\cosh\theta \\ v_1\cosh\theta & \sinh\theta & v_3\cosh\theta & -v_2\cosh\theta \\ v_2\cosh\theta & v_3\cosh\theta & \sinh\theta & -v_1\cosh\theta \\ v_3\cosh\theta & -v_2\cosh\theta & v_1\cosh\theta & \sinh\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{3}{2} & \frac{3}{2} & -\frac{3\sqrt{2}}{4} \\ \frac{3}{2} & \frac{\sqrt{2}}{4} & -\frac{3\sqrt{2}}{4} & -\frac{3}{2} \\ \frac{3}{2} & -\frac{3\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{3}{2} \\ -\frac{3\sqrt{2}}{4} & -\frac{3}{2} & \frac{3}{2} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

where $\theta = \ln\sqrt{2}$ and $\overrightarrow{v} = (\sqrt{2}, \sqrt{2}, -1)$. The cube roots of the matrix A can be achieved

$$A^{\frac{1}{3}} = \begin{bmatrix} \sinh\frac{\theta}{3} & -v_1\cosh\frac{\theta}{3} & v_2\cosh\frac{\theta}{3} & v_3\cosh\frac{\theta}{3} \\ v_1\cosh\frac{\theta}{3} & \sinh\frac{\theta}{3} & v_3\cosh\frac{\theta}{3} & -v_2\cosh\frac{\theta}{3} \\ v_2\cosh\frac{\theta}{3} & v_3\cosh\frac{\theta}{3} & \sinh\frac{\theta}{3} & -v_1\cosh\frac{\theta}{3} \\ v_3\cosh\frac{\theta}{3} & -v_2\cosh\frac{\theta}{3} & v_1\cosh\frac{\theta}{3} & \sinh\frac{\theta}{3} \end{bmatrix}$$

$$\simeq \begin{bmatrix} 0.12 & -\sqrt{2} & \sqrt{2} & -1 \\ \sqrt{2} & 0.12 & -1 & -\sqrt{2} \\ \sqrt{2} & -1 & 0.12 & -\sqrt{2} \\ -1 & -\sqrt{2} & \sqrt{2} & 0.12 \end{bmatrix}$$

here $\sinh\frac{\theta}{3} = 0.1153$ and $\cosh\frac{\theta}{3} = 1.0066$.

References

[1] Addler, S.L., *Quternionic Quantum mechanics and Quantum field*, Oxford university press, New York, 1996.

[2] Agrawal, O. P., *Hamilton operators and dual-number-quaternions in spatial kinematics*, Mech. Mach. Theory., **22(6)(1987)**, 569-575.

[3] Ata, E. and Yayli Y., *Split quaternions and semi-Euclidean projective spaces,* Chaos, Solitions and Fractals, **41(2009)**, 1910-1915.

[4] Bharathi, K. and Nagaraj, M., *Geometry of Quaternionic and Pseudo-quaternionic Multiplications*, Indian J. pure appl. Math., **16(7)(1985)**, 741-756.

[5] Cho, E., *De-Moivre Formula for Quaternions*, Appl. Math. Lett., **11(6)(1998)**, 33-35.

[6] Farebrother, R.W., GroB, J. and Troschke, S., *Matrix Representaion of Quaternions,* Linear Algebra and its Appl., **362(2003),** 251-255.

[7] Gungor, M.A. and Sarduvan, M., *A Note on dual Quaternions and Matrices of Dual Quaternions,* Scientia Magna, **7(1)(2011)**, 1-11.

[8] GroB, J., Trenkler, G. and Troschke, S., *Quaternions: Futher Contributions to a Matrix Oriented Approach*, Linear Algebra and its Appl., **326(2001)**, 205-213.

[9] Jafari, M., Mortazaasl, H. and Yayli, Y., *De Moivre's Formula for Matrices of Quaternions,* JP J. of Algebra, Number Theory and Appl., **21(1)(2011)**, 57-67.

[10] Kabadayi, H. and Yayli, Y., *De Moivre's Formula for Dual Quaternions,* Kuwait J. of Sci. & Tech., **38(1)(2011)**, 15-23.

[11] Kula, L. and Yayli, Y., *Split Quaternions and Rotations in Semi-Euclidean space* E_2^4, J. Korean Math. Soc., **44(6)(2007),** 1313-1327.

[12] Ozdemir, M. and Ergin, A., *Rotation with Unit Timelike Quaternions in Minkowski 3-Space,* J. of Geometry and Physics, **56(2006)**, 322-336.

[13] Ozdemir, M., *The Roots of a Split Quaternion,* Applied Math. Lett., **22(2009)**, 258-263.

[14] Rosenfold, B., *Geometry of Lie Groups*, Kluwer Academic Publishers, 1997.

[15] Ward, J. P., *Quaternions and Cayley Numbers Algebra and Applications*, Kluwer Academic Publishers, London, 1997.

[16] Whittlesey, J. and Whittlesey, K., *Some Geometrical Generalizations of Euler's Formula*, Int. J. Of math. Edu. in Sci. & Tech., **21(3)(1990)**, 461-468.

[17] Yayli, Y., *Homothetic Motions at* E^4. Mech. Mach. Theory, **27(3)(1992)**, 303-305.

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