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ON THE GEOMETRY OF SUBMERSIONS

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Abstract. In this paper we study the geometry of the level surfaces of functions of certain class. It is proved that the level surfaces of these functions generate a foliation whose leaves are manifolds of constant Gaussian curvature.

1. Introduction

Let M be a smooth Riemannian manifold of dimension n with Riemannian metric g, F-foliation of dimension k on M, where $0 < k < n$ [6]. We denote by L_p - a leaf of the foliation F passing through the point $p \in M$, by T_qF - the tangent space of leaf L_p at the point $q \in L_p$, and by $H(q)$ - orthogonal complement of T_qF. As a result, we get two sub-bundles $TF = \{T_qF\}$, $TH = \{H(q)\}$ of the tangent bundle TM, and we have the orthogonal decomposition $TM = TF \oplus H$. Thus every vector field X expanded in the form $X = X^v + X^h$, where $X^v \in TF$, $X^h \in TH$. If $X^h = 0$ (respectively $X^v = 0$), the field is called the vertical (horizontal) vector field.

The Riemannian metric g on a manifold M induces Riemannian metric $\widetilde{g}$ on the leaf L_p. Canonical injection $i : L_p \to M$ is an isometric immersion with respect to these metrics. A connection ∇ (Levi-Civita connection) defined by the Riemannian metric g induces a connection $\widetilde{\nabla}$ on L_p which coincides with the connection defined by the Riemannian metric $\widetilde{\nabla}$ [1].

Let Z be a horizontal vector field. For each vertical vector field X we define the vertical vector field

$$S(X, Z) = (\nabla_X Z)^v.$$

and we get the tensor field of type (1,1)

$$X, Z \to S_Z X = S(X, Z).$$

This tensor field gives the bilinear form $l_Z(X, Y) = \langle S_Z X, Y\rangle$, where $\langle X, Y\rangle$ - the inner product defined by the Riemannian metric g.

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Tensor field S_Z is called the second fundamental tensor, and the form $l_Z(X,Y)$ is called the second main form with respect to the horizontal field Z.

Mapping $S_Z : T_qF \to T_qF$ defined by the formula $X_q \to S(X,Z)_q$ is self-endomorphism with respect to the scalar product defined by the Riemannian metric $\widetilde{g}$. If the vector field Z is a vector of unit length, the eigenvalues of this endomorphism called the principal curvatures of L_p at the point q , and the corresponding eigenvectors are called the principal directions.

The mean curvature H_Z and the Gauss-Kronecker curvature K_Z of the leaf L_p at the point q are determined by the principal curvatures [1]:

$$H_Z = \frac{1}{k} trS_Z \ , K_Z = detS_Z \ , \text{ where } k = n - m.$$

Let $f : M \to B$ be a differentiable map of maximal rank, where M, B a smooth manifolds of dimensions respectively. Such maps are called submersions. By the theorem on the rank of a differentiable function for each point $p \in B$ inverse image $f^{-1}(p)$ is a submanifold of dimension $k = n - m$. Thus submersion $f : M \to B$ generates a foliation on a manifold of dimension $k = n - m$ whose leaves are submanifolds $L_p = f^{-1}(p), p \in B$.

The geometry and topology of the foliations generated by submersions were subject of numerous studies [2],[4],[5],[6].

If the differential df of mapping $f : M \to B$ preserves the length of horizontal vectors, the submersion $f : M \to B$ is called the Riemannian submersions. Geometry of Riemannian submersions investigated in numerous studies, in particular in [5] derived the fundamental equations of the Riemannian submersion.

2. MAIN RESULT

We consider the case where the manifold B is one-dimensional manifold, to be more precise we consider smooth function $f : M \to R$. If $Crit\{f\}$ - the set of critical points of the function f, then on the manifold $M \setminus Crit\{f\}$ arises foliation F of dimension $n - 1$ (or codimension one foliation), leaves of which are level surfaces of function f.

In [6] it is studied the geometry of the level surfaces of the functions for which $X(|gradf|^2) = 0$ for each vertical vector field X. In particular, it is proved that the level surfaces form the so-called Riemannian foliation.

In this paper, we show that the level surfaces of the function of this class are the surfaces of constant Gaussian curvature.

Theorem 2.1. *Let $f : M \to R$ be a smooth function. Suppose that $Critf = \varnothing$ and $X\left(|gradf|^2\right) = 0$ for each vertical vector field X. Then every leaf of foliation F (level surface of f) is a manifold of constant Gaussian curvature.*

Proof. As is known the Hessian is given by

$$h_f\left(X,Y\right) = l_Z\left(X,Y\right) = \langle \nabla_X Z, Y \rangle$$

where $Z = gradf$, ∇- the Levi-Civita connection defined by Riemannian metric g.

The map $X \to h_f(X) = \nabla_X Z$ (Hesse tensor) is a linear operator and is given by a symmetric matrix A:

$$h_f(X) = \nabla_X Z = AX.$$

We denote by $\chi(\lambda)$ the characteristic polynomial of the matrix A with a free term $(-1)^n \det A$ and define a new polynomial $\rho(\lambda)$ by the equation

$$\lambda\rho(\lambda) = \det A - (-1)^n \chi(\lambda).$$

Since $\chi(A) = 0$ we have that $A\rho(A) = \det A \cdot E$, where E - is the identity matrix. The elements of the matrix $\rho(A)$ are cofactors of the matrix A. This matrix is denoted by H_f^c.

It is well known that the Gaussian curvature of the surface is calculated by the formula ([1], p.110)

$$K = \det S = \frac{1}{|gradf|^{n+1}} \left\langle H_f^c(gradf), gradf \right\rangle.$$

To prove the theorem it suffices to show that $X(K) = 0$ for each vertical vector field X at any point q of a leaf L_p.

By hypothesis of the theorem we have

$$X\left(|gradf|^2\right) = 0$$

and so

$$X\left(\frac{1}{|gradf|^{n+1}}\right) = 0$$

therefore we need to show that

$$\left\langle \nabla_X H_f^c Z, Z\right\rangle + \left\langle H_f^c Z, \nabla_X Z\right\rangle = 0.$$

We know that if $X(|gradf|^2) = 0$ for each vertical vector field X, each gradient line of f is a geodesic line of Riemannian manifold [4]. By definition, the gradient line is a geodesic if and only if $\nabla_N N = 0$, where $N = \frac{Z}{|Z|}$.

We calculate the covariant differential

$$\nabla_N N = \frac{1}{|Z|}\nabla_Z N = \frac{1}{|Z|}(\frac{1}{|Z|}\nabla_Z Z + Z(\frac{1}{|Z|})Z) = 0$$

and get $\nabla_Z Z = \lambda Z$, where $\lambda = -|Z|Z(\frac{1}{|Z|})$. This means that the gradient vector Z is the eigenvector of matrix A.

Let $X_1^0, X_2^0, \cdots, X_{n-1}^0, Z^0$- be mutually orthogonal eigenvectors of A at the point $q \in L_p$ such that $X_1^0, X_2^0, \cdots, X_{n-1}^0$ the unit vectors, Z^0 - the value of the gradient field at a point q. Locally, they can be extended to the vector fields $X_1, X_2, \cdots, X_{n-1}, Z$ to a neighborhood of (say U) point q so that they formed at each point of an orthogonal basis consisting of eigenvectors. We construct the Riemannian normal system of coordinates $(x_1, x_2, \cdots, x_n)$ in a neighborhood U via vectors $X_1^0, X_2^0, \cdots, X_{n-1}^0, Z^0$ ([1], p.112).

The components g_{ij} of the metric g and the connection components Γ_{ij}^k in the normal coordinate system satisfies the conditions of ([1], p. 132):

$$g_{ij}(q) = \delta_{ij}, \Gamma_{ij}^k(q) = 0.$$

We show that $X(\lambda)=0$ for each vertical field X. From the equality

$$X\left(\lambda\right)=-X\left(|Z|\right)Z\left(\frac{1}{|Z|}\right)-|Z|\,X\left(Z\left(\frac{1}{|Z|}\right)\right)$$

and from the condition $X(|Z|)=0$ follows equality

$$X\left(Z\left(\frac{1}{|Z|}\right)\right)=X\left(Z\left(\phi\right)\right)=[X,Z]\left(\phi\right)-Z\left(X\left(\phi,\right)\right)$$

where $\phi=\frac{1}{|Z|}$, $[X,Z]$-Lie bracket of vector fields X,Z.

From the condition of the theorem follows $X\left(Z\left(\phi\right)\right)=0$. In [6] it is shown that $X\left(|gradf|^2\right)=0$ for each of the vertical vector field X if and only if $[X,Z]$ a vertical field. Therefore $[X,Z]\left(\phi\right)=0$. Thus, λ is a constant function on the leaf L.

Now we denote by $\lambda_1,\lambda_2,\ldots,\lambda_{n-1}$ the eigenvalues of the matrix A corresponding to the eigenvectors $X_1,X_2,\ldots,X_{n-1}$. Then in the basis $X_1,X_2,\ldots,$ X_{n-1},Z matrix A has the form:

$$A=\begin{pmatrix}\lambda_1 & 0 & ... & 0\\ 0 & \lambda_2 & ... & 0\\ ... & ... & ... & ...\\ 0 & 0 & ... & \lambda_n\end{pmatrix}$$

By hypothesis of the theorem, the vector field $\nabla_X Z$ is vertical field. It follows Codazzi equations have the form ([3], p.29)

$$(\nabla_X A)Y=(\nabla_Y A)X$$

From this equation we get

$$\nabla_{X_i}AX_j=\nabla_{X_j}AX_i,\nabla_{X_i}AZ=\nabla_Z AX_i \tag{1}$$

at any point of U for each vector field X_i. From first equation of (1) we take following equality

$$X_i(\lambda_j)X_j+\lambda_j\nabla_{X_i}X_j=X_j(\lambda_i)X_j+\lambda_i\nabla_{X_j}X_i. \tag{2}$$

Since $\nabla_{X_i}X_j=\Gamma_{ij}^k X_k=0$ at the point q by properties of normal coordinate system, from (2) follows equality

$$X_i(\lambda_j)X_j=X_j(\lambda_i)X_j. \tag{3}$$

By the linear independence $X_1,X_2,\cdots,X_{n-1}$, we have that

$$X_i(\lambda_j)=0 \text{ for } i\neq j \text{ for all } i.$$

From second equation of (1) we take following

$$X_i(\lambda)Z+\lambda\nabla_{X_i}Z=Z(\lambda_i)X_i+\lambda_i\nabla_Z X_i. \tag{4}$$

Since

$$\nabla_{X_i}Z=\nabla_Z X_i=0$$

at the point q from the linear independence of vectors X_i,Z we have that

$$X_i(\lambda)=0,Z(\lambda_i)=0 \text{ for all } i.$$

On the other hand

$$\nabla_Z AX_i = Z(\lambda_i)X_i + \lambda_i \nabla_Z X_i, \nabla_{X_i} AZ = \nabla_Z AX_i, \nabla_{X_i} Z = \lambda_i X_i. \quad (5)$$

From (5) we get that

$$\lambda_i^2 X_i + Z(\lambda_i)X_i = X_i(\lambda)Z + \lambda\lambda_i X_i. \quad (6)$$

Since $Z(\lambda_i) = 0$,$X_i(\lambda) = 0$ at the point q from the (6) follows that $\lambda_i^2 = \lambda\lambda_i$. In particular, this implies that if $\lambda_i \neq 0$ then $\lambda_i = \lambda$ and $X(\lambda_i) = X(\lambda) = 0$, $Z(\lambda) = Z(\lambda_i) = 0$ for all i. Thus, in the neighborhood U of the point q non-zero eigenvalues of the matrix A are constant and equal λ.

Given this fact we compute $X(K)$. We denote by m the number of zero eigenvalues of A. If $m = 0$, all the eigenvalues are equal to the number λ. In this case, by the definition of the matrix H_f^c we get that $H_f^c Z = \lambda^{n-1} Z$.

Consider the case when $m > 0$. If $m > 1$, then $H_f^c = 0$. If $m = 1$ than $\lambda_i = 0$ for some i and $AX_i = \nabla_{X_i} Z = 0$. This means that the vector field Z is parallel along the integral curve of a vector field X_i (along i - coordinate line). If $i = n$ we have $\lambda = \lambda_i = 0$ for all i and $H_f^c = 0$.

Without loss of generality we assume that $i < n$. In this case vector $H_f^c Z$ have only one nonzero component b_i and $H_f^c Z = b_i \frac{\partial}{\partial x_i}$. In this case we get

$$\nabla_X H_f^c Z = X(b_i)\frac{\partial}{\partial x_i} + b_i \nabla_X \frac{\partial}{\partial x_i}.$$

As we know that $X_i = \frac{\partial}{\partial x_i}$ vertical and $\nabla_X \frac{\partial}{\partial x_i} = 0$. Thus in the case $m = 1$ we have $\left\langle \nabla_X H_f^c(gradf), gradf \right\rangle = 0$

Let us consider the case $m = 0$. In this case we have equalities $H_f^c Z = \lambda^{n-1} Z$ and

$$\nabla_X H_f^c Z = X\left(\lambda^{n-1}\right) Z + \lambda^{n-1} \nabla_X Z.$$

As mentioned above field $\nabla_X gradf$ is a vertical vector field for each vertical vector field X (the field AX is vertical). From this equalities follows $\left\langle \nabla_X H_f^c(gradf), gradf \right\rangle = 0$ at the point q. The theorem is proved.

Examples:

1. $M = R^3 \setminus \{(x, y, z) : x = 0, y = 0\}$,$f(x, y, z) = x^2 + y^2$. Level surfaces of this submersion are manifolds of zero Gaussian curvature.

2. $M = R^3 \setminus \{(0, 0, 0)$,$f(x, y, z) = x^2 + y^2 + z^2$. Level surfaces of this submersion are manifolds of constant positive Gaussian curvature.

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