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RELATION BETWEEN THIRD ORDER FINITE SCREW SYSTEMS AND LINE CONGRUENCE

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Abstract. In this paper, by using Dimentberg's definition of pitch, a screw system which consists of displacement of a line is considered. After forming this third order system, by considering three different systems; the external bisectors which are the elements of these screw systems are obtained. It is demonstrated that these external bisectors belong to the same recticongruence. Then the axis of this recticongruence is found. Finally, it is concluded that the angles between the first, second, third screws and the axis of recticongruence are the same. In this way, it is shown that these screws belong to the same line congruence.

1. Introduction

In the last decade, researchers have paid much attention to line segments and point-lines as opposed to infinite lines, [12]. Having viewed a point-line as an embedded part of a rigid body, some researchers used screws and screw systems to describe a point-line displacement, [6]. Parkin, [10], modeled the finite point-line displacement as a 2-system by assigning a new definition to the pitch. By following Parkin's definition of a pitch, Huang and Roth [5], Hunt and Parkin, [7], and Zhang and Ting, [12, 13], studied different finite displacement of various geometrical elements.

With the conventional pitch definition, it was found that the linear properties, [1], in infinitesimal twists do not exist in finite cases. This problem was resolved by Parkin who redefined the pitch to be the ratio of one-half the translation to the tangent of one-half rotation, [10]. Parkin showed that all possible finite twists to displace a point-line from one position to another could be presented with a 2-system, [7].

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Even though various finite screw systems have been discovered by using Parkin's definition of pitch [10], the displacement of a line was an exceptional case. It was known the finite screws for displacing a line had not form a screw system, [5, 7], since Huang and Wang noticed Dimentberg's definition of pitch, [4]. Upon this, they adopt this definition and show that all possible screws for displacing a line can form a screw system of the third order, [6].

In this paper, by using Dimentberg's definition of pitch three screw systems of the third order are formed. Then the relation between these screw systems and congruence concept are studied.

2. Formation of screw systems

A screw can be defined as 6-dimensional vector composed of a pair of 3-dimensional vectors or as a dual vector as below

$$\widehat{V} = (\overrightarrow{v}, \overrightarrow{v_0}) = \overrightarrow{v} + \varepsilon \overrightarrow{v_0}. \tag{2.1}$$

where $\varepsilon^2 = 0$. And also a screw can be defined, geometrically, as a straight line with which a definite linear magnitude termed the pitch is associated, [1]. The pitch of a screw can be determined by

$$p = \frac{\overrightarrow{v} \cdot \overrightarrow{v_0}}{\overrightarrow{v} \cdot \overrightarrow{v}}\ . \tag{2.2}$$

The Plücker coordinates of a line are as below

$$\widehat{E} = \left(\overrightarrow{a}, \overrightarrow{b} \times \overrightarrow{a}\right) = \overrightarrow{a} + \varepsilon \left(\overrightarrow{b} \times \overrightarrow{a}\right). \tag{2.3}$$

where $\overrightarrow{a}$ denotes the direction cosine vector of the line and $\overrightarrow{b}$ is the position vector of a point on the line.
A screw and its pitch can also be defined as below, respectively,

$$\widehat{V} = \widehat{c}\,\widehat{E} = (c + \varepsilon c_0)\,\widehat{E}, \tag{2.4}$$

$$p = \frac{c_0}{c} \tag{2.5}$$

where $\widehat{c}$ is a dual number and $\widehat{E}$ is a line vector.

In kinematics, screws are used to describe displacements of a rigid body and these displacement screws are named as twists. The pitch of a twist is defined as

$$p = \frac{t_{ij}}{\theta_{ij}}, \tag{2.6}$$

where θ_{ij} and t_{ij} are the rotation and translation parameters of the twist, respectively.

On the other hand, Dimentberg [4] presented a novel definiton of pitch in formulation the resultant displacement of two successive displacement screws. He showed that if a finite twist of a rigid body from position i to position j is defined as

$$T_{ij} = \tan \frac{\widehat{\theta}_{ij}}{2} \widehat{E}_{ij}, \tag{2.7}$$

where $\widehat{\theta}_{ij} = \theta_{ij} + \varepsilon t_{ij}$ is the dual angle of the twist, then the resultant finite twist (T_{13}) of two successive finite twists (T_{12} and T_{23}) are related by as below, [4].

$$T_{13} = \frac{T_{12} + T_{23} - T_{12} \times T_{23}}{1 - T_{12} \cdot T_{23}} \ . \tag{2.8}$$

The tangent of the dual angle of the twist in Eq. (2.7) is

$$\tan \frac{\widehat{\theta}_{ij}}{2} = \tan \frac{\theta_{ij}}{2} + \varepsilon \frac{t_{ij}}{2} \left(1 + \tan^2 \frac{\theta_{ij}}{2} \right). \tag{2.9}$$

From this point of view with the help of Eq. (2.5), the pitch of the finite twist defined by Dimentberg, [4], is

$$p_{ij} = \frac{\frac{t_{ij}}{2}}{\tan \frac{\theta_{ij}}{2}} \left(1 + \tan^2 \frac{\theta_{ij}}{2} \right) = \frac{t_{ij}}{\sin \theta_{ij}} \ . \tag{2.10}$$

The twist can also be defined as the product of an amplitude and a unit screw, where its pitch is defined by Eq. (2.10), as below

$$T_{ij} = \tan \frac{\theta_{ij}}{2} \widehat{V}_{ij}. \tag{2.11}$$

To determine the position of a rigid body requires six independent parameters, four independent parameters to describe the position of a line and two free parameters to choose a screw for displacing a line from one position to another. That is important if these screws can form a screw system of the third order. To check it we need to know if these screws provide the representation in

$$V = f_1 \widehat{V}_1 + f_2 \widehat{V}_2 + f_3 \widehat{V}_3, \tag{2.12}$$

where f_1, f_2 and f_3 are scalar functions of the free parameters and $\widehat{V}_1, \widehat{V}_2$, $\widehat{V}_3$ are linearly independent screws, [6].

Tsai and Roth, [11], formulated an incompletely specified displacement as computing the resultant of two successive displacement screws. The twist V for displacing a particular element of a body can be thought of as the resultant of a twist K which containing free parameters and a twist L which virtually displaces the element. In the case of displacing a line, the axis of K is taken to be initial position of the line. On the other hand, d and ϕ which are the translation and rotation parameters of K, respectively, do not displace the position of the line because of this they are free parameters. The axis of L is selected to be the line perpendicular to both the initial and final positions of the line and the rotation and translation parameters of L are the angle and distance between the two positions of the line, respectively.

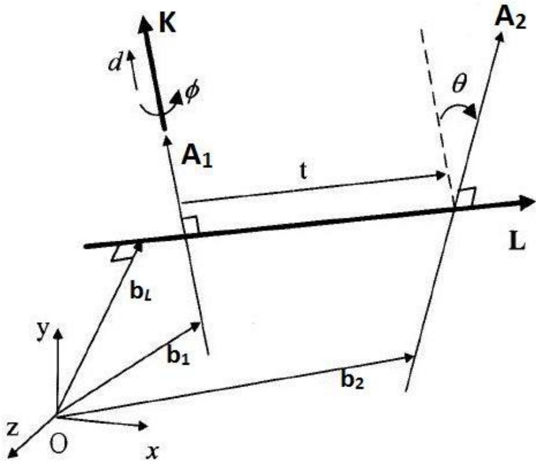

Figure 1

Let the Plücker coordinates of two positions of a line as shown in Fig.1, [6], be

$$\widehat{A}_1 = (\overrightarrow{a_1}, \overrightarrow{a_{01}}) = \left(\overrightarrow{a_1}, \overrightarrow{b_1} \times \overrightarrow{a_1}\right),$$
(2.13)
$$\widehat{A}_2 = (\overrightarrow{a_2}, \overrightarrow{a_{02}}) = \left(\overrightarrow{a_2}, \overrightarrow{b_2} \times \overrightarrow{a_2}\right).$$

The unit screw of K, whose axis coincides with $\widehat{A}_1$, and the pitch of K named as p_K are given as below, respectively,

$$\widehat{K} = \left(\overrightarrow{k}, \overrightarrow{k_0}\right) = (\overrightarrow{a_1}, \overrightarrow{a_{01}} + p_K \overrightarrow{a_1}), \tag{2.14}$$

$$p_K = \frac{d}{\sin\phi} \quad . \tag{2.15}$$

The unit screw of L is

$$\widehat{L} = \left(\overrightarrow{l}, \overrightarrow{l_0}\right) = \left(\overrightarrow{l}, \overrightarrow{b_{\mathbf{L}}} \times \overrightarrow{l} + p_{\mathbf{L}} \overrightarrow{l}\right), \tag{2.16}$$

where

$$b_{\mathbf{L}} = \left(\frac{1}{|\overrightarrow{a_1} \times \overrightarrow{a_2}|}\right)\left[\left(\overrightarrow{l} \cdot \overrightarrow{a_{02}}\right)\overrightarrow{a_1} - \left(\overrightarrow{l} \cdot \overrightarrow{a_{01}}\right)\overrightarrow{a_2}\right], \tag{2.17}$$

$$p_L = \frac{t}{\sin\theta}, \tag{2.18}$$

$$\theta = \tan^{-1}\left(\frac{|\overrightarrow{a_1} \times \overrightarrow{a_2}|}{\overrightarrow{a_1} \cdot \overrightarrow{a_2}}\right), \tag{2.19}$$

$$t = \left(\overrightarrow{b_2} - \overrightarrow{b_1}\right) \cdot \overrightarrow{l}, \tag{2.20}$$

$$\overrightarrow{l} = \frac{(\overrightarrow{a_1} \times \overrightarrow{a_2})}{|\overrightarrow{a_1} \times \overrightarrow{a_2}|}. \tag{2.21}$$

Since the resultant of K and L is V, according to Eqs. (2.8) and (2.11),

$$\tan\frac{\beta}{2}\widehat{V} = \frac{\tan\frac{\phi}{2}\widehat{K} + \tan\frac{\theta}{2}\widehat{L} - \tan\frac{\phi}{2}\tan\frac{\theta}{2}\widehat{K} \times \widehat{L}}{1 - \tan\frac{\phi}{2}\tan\frac{\theta}{2}\widehat{K} \cdot \widehat{L}}, \tag{2.22}$$

where β is the rotation parameter of V. Then because of K and L intersect each other perpendicularly,

$$\widehat{K} \cdot \widehat{L} = 0. \tag{2.23}$$

From Eq. (2.23), Eq. (2.22),

$$\tan\frac{\beta}{2}\widehat{V} = \tan\frac{\phi}{2}\widehat{K} + \tan\frac{\theta}{2}\widehat{L} - \tan\frac{\phi}{2}\tan\frac{\theta}{2}\widehat{K} \times \widehat{L} \tag{2.24}$$

is obtained. So, the cross product of K and L can be simplified as

$$\widehat{K} \times \widehat{L} = \begin{bmatrix} \overrightarrow{k} \times \overrightarrow{l} \\ (\overrightarrow{b_1}.\overrightarrow{l}).\overrightarrow{k} + (\overrightarrow{k} + \overrightarrow{l_0}) + p_K(\overrightarrow{k} \times \overrightarrow{l}) \end{bmatrix}$$

$$= \begin{bmatrix} \overrightarrow{k} \times \overrightarrow{l} \\ (\overrightarrow{b_1}.\overrightarrow{l}).\overrightarrow{k} + (\overrightarrow{k} + \overrightarrow{l_0}) \end{bmatrix} + \begin{bmatrix} 0 \\ p_K(\overrightarrow{k} \times \overrightarrow{l}) \end{bmatrix} \tag{2.25}$$

Substituting the above equation into Eq. (2.24) gives

$$
\begin{aligned}
\tan\frac{\beta}{2}\widehat{V} &= \tan\frac{\theta}{2}\begin{bmatrix} \overrightarrow{l} \\ \overrightarrow{l}_0 \end{bmatrix} \\
&+ \tan\frac{\phi}{2}\begin{bmatrix} \overrightarrow{k} - \tan\frac{\theta}{2}\left(\overrightarrow{k}\times\overrightarrow{l}\right) \\ \overrightarrow{b_1}\times\overrightarrow{k} - \tan\frac{\theta}{2}\left(\overrightarrow{b_1}.\overrightarrow{l}\right)\cdot\overrightarrow{k} - \tan\frac{\theta}{2}\left(\overrightarrow{k}\times\overrightarrow{l_0}\right) \end{bmatrix} \\
&+ \frac{d}{1+\cos\phi}\begin{bmatrix} 0 \\ \overrightarrow{k} - \tan\frac{\theta}{2}\left(\overrightarrow{k}\times\overrightarrow{l}\right) \end{bmatrix} \\
&= \tan\frac{\theta}{2}\widehat{V}_1 + \tan\frac{\phi}{2}\widehat{V}_2 + \frac{d}{1+\cos\phi}\widehat{V}_3. \qquad (2.26)
\end{aligned}
$$

Since that the above equation is in the form of Eq. (2.12), all possible screws for displacing a line form a three system as illustrated in Fig. 2, [6].

As illustrated in Fig 2, $\widehat{V}_1$ is the screw $\widehat{L}$, $\widehat{V}_2$ is a pure rotation screw whose axis is the internal bisector of $\widehat{A}_1$ and $\widehat{A}_2$, $\widehat{V}_3$ is a screw parallel to $\widehat{V}_2$. In order to displace the specified line properly, it can be shown that $\widehat{V}_3$ must intersect $\widehat{D}$, which is external bisector of $\widehat{A}_1$ and $\widehat{A}_2$, perpendicularly, [3].

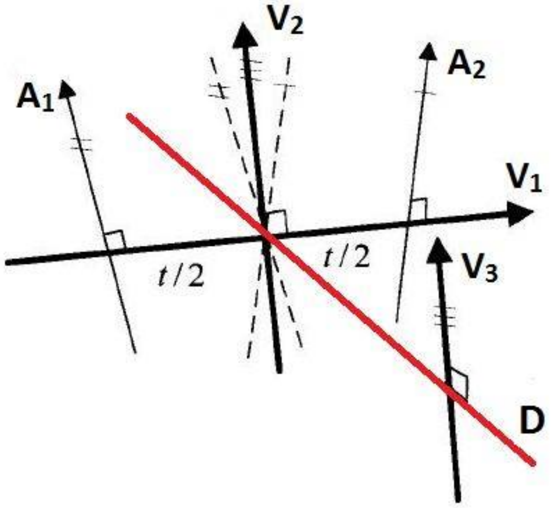

Figure 2

3. The recticongruence which includes external bisectors

Let; lines A_1, A_2 are denoted with $\widehat{A}_1$, $\widehat{A}_2$ dual unit vectors. And $\widehat{V}_2$, $\widehat{D}$ dual unit vectors are bisectors of A_1, A_2, respectively, where the dual angle between $\widehat{A}_1$ and $\widehat{A}_2$ is $\widehat{\theta}_{12} = \theta_{12} + \varepsilon t_{12}$,

$$\widehat{V}_2 = \frac{\widehat{A}_1+\widehat{A}_2}{\sqrt{2\left(1+\cos\widehat{\theta}_{12}\right)}}, \quad \widehat{D} = \frac{\widehat{A}_1-\widehat{A}_2}{\sqrt{2\left(1-\cos\widehat{\theta}_{12}\right)}}. \qquad (3.1)$$

So from equations in following;

$$\widehat{V}_2 \cdot \widehat{D} = 0, \qquad \widehat{V}_2 \cdot \frac{\widehat{A}_1 \times \widehat{A}_2}{\left|\widehat{A}_1 \times \widehat{A}_2\right|} = 0, \qquad \widehat{D} \cdot \frac{\widehat{A}_1 \times \widehat{A}_2}{\left|\widehat{A}_1 \times \widehat{A}_2\right|} = 0,$$

we can say that $\widehat{V}_2$, $\widehat{D}$ and perpendicular bisector of $\widehat{A}_1$ and $\widehat{A}_2$ vectors are formed a perpendicular trihedron. On the other hand, if dual angle between $\widehat{A}_1$ and $\widehat{V}_2$ is indicated with $\Gamma = \gamma + \varepsilon\gamma^*$ and with the aid of equation as below;

$$\widehat{A}_1 \cdot \widehat{V}_2 = \frac{1 + \cos\overset{\frown}{\theta}_{12}}{\sqrt{2\left(1 + \cos\overset{\frown}{\theta}_{12}\right)}} = \sqrt{\frac{1 + \cos\overset{\frown}{\theta}_{12}}{2}} = \cos\frac{\overset{\frown}{\theta}_{12}}{2} = \cos\Gamma, \tag{3.2}$$

$$\gamma = \frac{\theta_{12}}{2}, \ \gamma^* = \frac{t_{12}}{2} \tag{3.3}$$

are found, [9].

In Eq. (3.3), as the denominators are equal to two, the top of perpendicular trihedron is in the middle of the perpendicular bisector. And each $\widehat{V}_2$ and $\widehat{D}$ vectors are made the same angle between $\widehat{A}_1$ and $\widehat{A}_2$ as illustrated in Fig 2.

Now consider dual unit vectors $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3$ which are not parallel to each other and dual unit vectors $\widehat{D}, \widehat{E}, \widehat{F}$ which are dual external bisectors of $\widehat{A}_1, \widehat{A}_2$ and $\widehat{A}_3$ as below;

$$\widehat{A}_1 \cdot \widehat{A}_2 = \cos\overset{\frown}{\theta}_{12}, \qquad \widehat{A}_2 \cdot \widehat{A}_3 = \cos\overset{\frown}{\theta}_{23}, \qquad \widehat{A}_3 \cdot \widehat{A}_1 = \cos\overset{\frown}{\theta}_{31}, \tag{3.4}$$

(3.5)

$$\widehat{D} = \frac{\widehat{A}_1 - \widehat{A}_2}{\sqrt{2\left(1 - \cos\overset{\frown}{\theta}_{12}\right)}}, \ \widehat{E} = \frac{\widehat{A}_2 - \widehat{A}_3}{\sqrt{2\left(1 - \cos\overset{\frown}{\theta}_{23}\right)}}, \ \widehat{F} = \frac{\widehat{A}_3 - \widehat{A}_1}{\sqrt{2\left(1 - \cos\overset{\frown}{\theta}_{31}\right)}}.$$

Then we can say that $\widehat{D}, \widehat{E}, \widehat{F}$ belong to the same recticongruence because these bisectors are linearly dependent.

To prove this, let show that there are dual numbers x, y, z which are not pure dual and provide the following equation:

$$x\widehat{D} + y\widehat{E} + z\widehat{F} = 0. \tag{3.6}$$

When Eq. (3.5) and (3.6) are considered; an equation system is obtained.

To show vectors $\widehat{D}, \widehat{E}, \widehat{F}$ are linearly dependent, let us proof that this equation system have a solution system which are not consist of pure dual numbers. Namely, let us proof that the determinant of the coefficient of this equation system given as

$$\Delta = \frac{\begin{vmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix}}{\sqrt{8\left(1-\cos\widehat{\theta}_{12}\right)\left(1-\cos\widehat{\theta}_{23}\right)\left(1-\cos\widehat{\theta}_{31}\right)}} \tag{3.7}$$

is equal to zero. Then because of A_1, A_2, A_3 lines are not parallel to each other, Δ is equal to zero. And thus x, y, z dual numbers which are not pure dual can be found.

On the other hand, let show that the perpendicular bisectors of D, E, F two by two is the same line.

$$\widehat{D}\times\widehat{E} = \frac{\widehat{A}_1\times\widehat{A}_2 + \widehat{A}_2\times\widehat{A}_3 + \widehat{A}_3\times\widehat{A}_1}{\left|\widehat{A}_1\times\widehat{A}_2 + \widehat{A}_2\times\widehat{A}_3 + \widehat{A}_3\times\widehat{A}_1\right|} \tag{3.8}$$

is the perpendicular bisector of D, E. Then because of $\widehat{D}, \widehat{E}, \widehat{F}$ bisectors are linearly dependent we can write

$$\widehat{F}\cdot\frac{\widehat{D}\times\widehat{E}}{\left|\widehat{D}\times\widehat{E}\right|} = 0. \tag{3.9}$$

Because of this, $\widehat{D}, \widehat{E}, \widehat{F}$ external bisectors belong to the same recticongruence such that the axes of recticongruence is

$$\widehat{N} = \frac{\widehat{M}}{M}, \tag{3.10}$$

where $\widehat{M} = \widehat{A}_1\times\widehat{A}_2 + \widehat{A}_2\times\widehat{A}_3 + \widehat{A}_3\times\widehat{A}_1, \qquad M = \left|\widehat{M}\right|.$

4. The Congruence Which Includes $\widehat{A}_1, \widehat{A}_2$ and $\widehat{A}_3$

Let us consider A_1, A_2, A_3 lines and with the aid of Eq. (3.10); Eq. (4.1), (4.2) and (4.3) are found as

(4.1)

$$\widehat{A}_1\cdot\widehat{N} = \frac{\widehat{A}_1\cdot\widehat{M}}{M} = \frac{\widehat{A}_1\cdot\left[\widehat{A}_1\times\widehat{A}_2 + \widehat{A}_2\times\widehat{A}_3 + \widehat{A}_3\times\widehat{A}_1\right]}{M} = \frac{\left(\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\right)}{M},$$

$$\widehat{A}_2\cdot\widehat{N} = \frac{\left(\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\right)}{M}, \tag{4.2}$$

$$\widehat{A}_3 \cdot \widehat{N} = \frac{\left(\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\right)}{M}. \tag{4.3}$$

And thus let us say

$$\frac{\left(\widehat{A}_1, \widehat{A}_2, \widehat{A}_3\right)}{M} = \cos\phi^*. \tag{4.4}$$

So

$$\widehat{A}_1 \cdot \widehat{N} = \widehat{A}_2 \cdot \widehat{N} = \widehat{A}_3 \cdot \widehat{N} = \cos\phi^* \% \tag{1}$$

is obtained.

As a result, any three lines such as A_1, A_2, A_3 are in the same line congruence, which have a constant dual slope, whose axis is the same with the axis of the recticongruence which dual external bisectors of A_1, A_2, A_3 lines are belong and finally which is consist of the lines which made the constant ϕ^* dual angle given in (4.5) with the axis of recticongruence.

5. Conclusion

However, previous researchers had deduced that the screws for displacing a line did not form a screw system by using different definitions of pitch, Huang and Wang [6] proved that the displacement screws can generated a screw system of the third order by using Dimentberg's definition of pitch. In this study; first, the formation of screw system with the help of Dimentberg's definition of pitch and the concept of a screw triangle was given. Second, the external bisectors of the lines which form the screw system and the recticongruence to which these bisectors were belong was determined. Finally, it was shown that the lines were in the same line congruence.

We hope that our study may contribute to the relation between theoretical and practical kinematics and also the relation between concepts of the screw system of a displacing line and the line congruence.

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