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ABOUT A CONSTRUCTION PROBLEM

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Abstract. In this paper, we study the construction of a polygon if we known the midpoints of their sides.

1. Introduction

In this paper, we search the answer to the next problem: *in a plane there are given* n *distincts points,* $n \in \mathbb{N}$, $n \geq 3$*; let's built a polygon such that the midpoints of the polygon sides to be given points.*

In the second section of paper, we will demonstrate that this construction is possible, and for the odd values of n, it is also unique.

For the realization of our purpose, we use Varignon's Theorem and we will identify, in some cases, the point with their afixe. As demonstration, we will use the mathematical induction.

In the third section, we will demonstrate how the centroid of a polygon could be built and we will characterized the centroid of a polygon with the afixes of their vertexes.

Also, we will give a property which give a sufficient condition as the border of polygon to be closed if the polygon is constructed from the midpoints.

The following result is well-known.

Theorem 1.1 (Varignon's Theorem, 1731)**.** *Let* $ABCD$ *be a quadrilateral. If* M, N, P, Q *are the midpoints of the sides* AB, BC, CD, *respectively* DA, *then* $MNPQ$ *is a parallelogram and* $2T[MNPQ] = T[ABCD]$, *where* $T[ABCD]$ *is the area of quadrilateral* $ABCD$.

2. The construction of a polygon when we known the midpoints of their sides

We are going to study the next problem: *giving the midpoints of a polygon, construct the polygon.*

Proposition 2.1. *Giving three non collinear points, there exists a unique triangle with the property that the given points are also the midpoints of the triangle's sides.*

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Proof. Let M, N, P be the given points. We construct from P the parallel to MN, from M the parallel to NP and from N the parallel to MP (Figure 2.1). This parallels cross each other two by two, obtaining the seeked triangle ABC. This triangle was obtained in a unique way. □

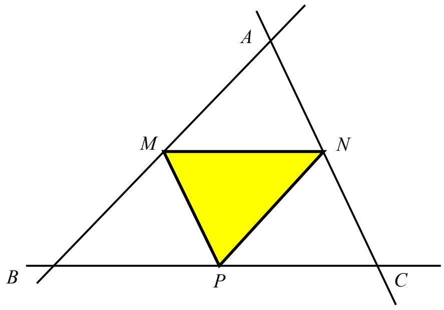

Figure 2.1

Proposition 2.2. *Giving non collinear points so that $MNPQ$ is a parallelogram and considering an arbitrary point A, there exist B, C, D so that M, N, P, Q are midpoints of sides AB, BC, CD, respectively DA.*

Proof. We note by z_A the afixe of point A. Because $MNPQ$ is a parallelogram, we have

$$z_M + z_P = z_N + z_Q. \tag{2.1}$$

By taking into account that M, N, P, Q are midpoints (Figure 2.2), we have that $z_B = 2z_M - z_A$, $z_C = 2z_N - z_B = 2z_N - 2z_M + z_A$, $z_D = 2z_P - z_C = 2z_P - 2z_N + 2z_M - z_A$. From (2.1), we obtain that $z_D = 2z_Q - z_A$ and then $\frac{z_A + z_D}{2} = z_Q$, so Q is the midpoint of side AD.

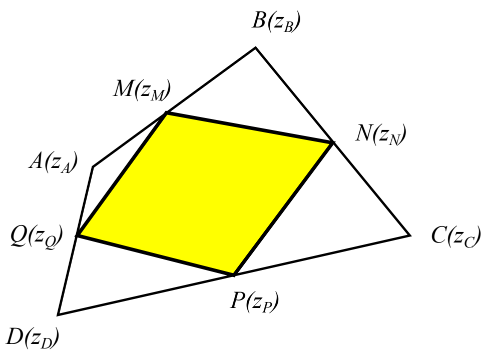

Figure 2.2

□

Proposition 2.3. *Giving five points, there exists a unique pentagon with the property that the given points are the midpoints of the pentagon's sides.*

Proof. We note by M, N, P, Q, S these five points (Figure 2.3).

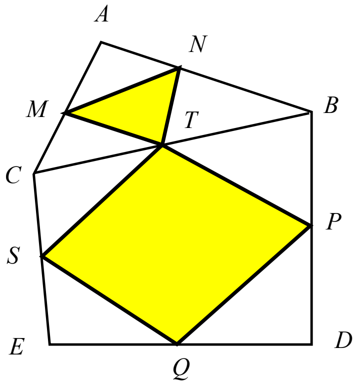

Figure 2.3

We construct the point T so that $SQPT$ is a parallelogram. Now, there exists a unique triangle ABC, so that the midpoints of its sides are also the points M, N, T. By taking Proposition 2.2 into account, starting from B, we obtain the points D and E. The seeked pentagon is $ABDEC$. □

Further, by using the idea from [4], we will extend the results demonstrated in the propositions above.

Theorem 2.1. *Let* $n \in \mathbb{N}^*$ *and* $2n+1$ *be given points. Then, there exists a unique polygon with* $2n+1$ *sides such that, the given points are the midpoints of the sides.*

Proof. We prove this theorem by mathematical induction. For $n = 1$ and $n = 2$, by taking Proposition 2.1 and Proposition 2.3 into account, the conclusions of Theorem 2.1 take place.

Let $k \in \mathbb{N}^*$, and we assume that the conclusions of Theorem 2.1 take place for $m \in \mathbb{N}^*$, $m \leq k$. We prove that the conclusion of Theorem 2.1 is also true for $k+1$. Let $M_1, M_2, \ldots, M_{2k+3}$ be $2k+3$ points (Figure 2.4).

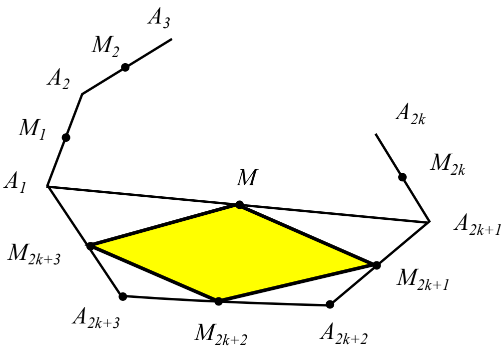

Figure 2.4

With the points M_{2k+1}, M_{2k+2}, M_{2k+3}, there exists a unique point M so that the quadrilateral $M_{2k+1}M_{2k+2}M_{2k+3}M$ is a parallelogram. Now, we have the points $M_1, M_2, \ldots, M_{2k}, M$, so according to the hypothesis of induction, there exists a unique polygon $A_1A_2 \ldots A_{2k+1}$ so that the points $M_1, M_2, \ldots, M_{2k}, M$ are the midpoints of the sides $A_1A_2, A_2A_3, \ldots, A_{2k}A_{2k+1}, A_{2k+1}A_1$. With the help of the points A_{2k+1} and A_1, by taking Proposition 2.2 into account, the points A_{2k+2} and A_{2k+3} are determined. □

3. The centroid of a polygon

In the following, using the idea from [4], we will give the afixe of the centroid for a polygon.

Let $A_1A_2A_3$ be a triangle, M_1, M_2, M_3 the midpoints of the sides A_2A_3, A_3A_1, respectively A_1A_2. Then A_1M_1, A_2M_2, A_3M_3 are called medians and are concurrent lines in a point G, called the centroid (Figure 3.1) and we have

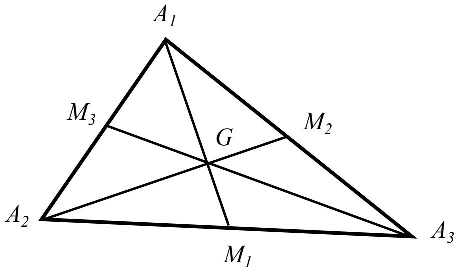

Figure 3.1

$$z_G = \frac{z_{A_1} + z_{A_2} + z_{A_3}}{3} \tag{3.1}$$

and

$$\frac{GA_1}{GM_1} = \frac{GA_2}{GM_2} = \frac{GA_3}{GM_3} = \frac{2}{1}, \tag{3.2}$$

where z_G is the afixe of point G.

Let $A_1A_2A_3A_4$ be a quadrilateral and G_1, G_2, G_3, G_4 the centroids for the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, respectively $A_1A_2A_3$. The segments A_1G_1, A_2G_2, A_3G_3, A_4G_4 are called medians (Figure 3.2) and are concurrent in a point G, called the centroid of the quadrilateral (see [3]) and we have

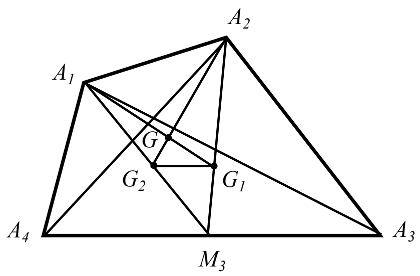

Figure 3.2

$$z_G = \frac{z_{M_1} + z_{M_2} + z_{M_3} + z_{M_4}}{4} \tag{3.3}$$

and

$$\frac{GA_1}{GG_1} = \frac{GA_2}{GG_2} = \frac{GA_3}{GG_3} = \frac{GA_4}{GG_4} = \frac{3}{1}. \tag{3.4}$$

We assume for every defined $k \le n-1$ the notion of median and centroid for polygon of k sides, so the median is a segment determined by a vertice of the given polygon and a centroid determined by other $k-1$ sides of the given polygon. These medians are concurrent in a point called centroid of the given polygon, and this point divides each median in the rate $\dfrac{k-1}{1}$.

We consider the polygon $A_1A_2\ldots A_n$ with n sides, and let $G_1, G_2, \ldots, G_n$ be the centroid of the polygons $A_2A_3\ldots A_n$, $A_1A_3\ldots A_n, \ldots$, respectively $A_1A_2\ldots A_{n-1}$ (Figure 3.3).

Let S be the centroid of polygon $A_1A_2\ldots A_{n-2}$. Then, the medians A_1G_1, A_2G_2, $\ldots$, A_nG_n are concurrent in a point G, called the centroid of polygon

$A_1A_2\ldots A_n$ and then, we have

$$z_G = \frac{z_{A_1} + z_{A_2} + \cdots + z_{A_n}}{n}. \tag{3.5}$$

and

$$\frac{GA_n}{GG_n} = \frac{GA_{n-1}}{GG_{n-1}} = \cdots = \frac{GA_1}{GG_1} = \frac{n-1}{1} \tag{3.6}$$

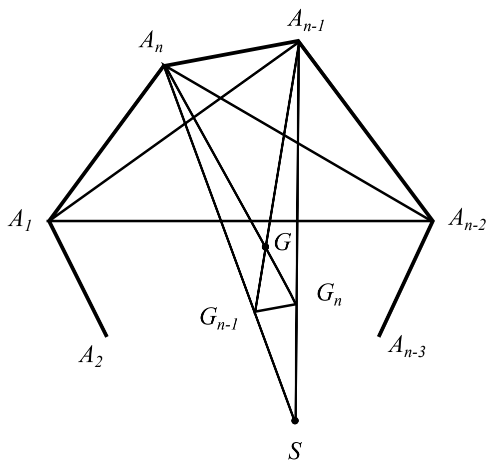

Figure 3.3

Theorem 3.1. *Let $n \in \mathbb{N}$, $n \geq 2$ and $A_1A_2\ldots A_{2n}$ be a polygon, $M_1, M_2, \ldots, M_{2n}$ are the midpoints of the sides A_1A_2, $A_2A_3, \ldots,$ respectively $A_{2n}A_1$. Then the polygons $M_1M_3\ldots M_{2n-1}$, $M_2M_4\ldots M_{2n}$ and $A_1A_2\ldots A_{2n}$ have the same centroid.*

Proof. Taking (3.5) into account, we have $z_{G_1} = \dfrac{z_{M_1} + z_{M_3} + \cdots + z_{M_{2n-1}}}{n}$, where G_1 is the centroid of the polygon $M_1M_3\ldots M_{2n-1}$. But M_1 is the midpoint of the side A_1A_2, so $z_{M_1} = \dfrac{z_{A_1} + z_{A_2}}{2}$, and similar

$$z_{M_3} = \frac{z_{A_3} + z_{A_4}}{2}, \ldots, z_{M_{2n-1}} = \frac{z_{A_{2n-1}} + z_{A_{2n}}}{2},$$

from where $z_{G_1} = \dfrac{z_{A_1} + z_{A_2} + \cdots + z_{A_{2n}}}{2n}$. If G_2 is the centroid of the polygon $M_2M_4\ldots M_{2n}$, similar we obtain that $z_{G_2} = \dfrac{z_{A_1} + z_{A_2} + \cdots + z_{A_{2n}}}{2n}$. The centroid of the polygon $A_1A_2\ldots A_{2n}$ have the afixe

$$z_G = \frac{z_{A_1} + z_{A_2} + \cdots + z_{A_{2n}}}{2n}.$$

From the remarks above, the conclusion of Theorem 3.1 follows. □

Theorem 3.2. *Let $n \in \mathbb{N}$, $n \geq 2$ and the given points $M_1, M_2, \ldots, M_{2n}$. For A_1 let A_2 be the symmetric to a point M_1, A_3 the symmetric A_2 to a point $M_2, \ldots, A_1'$ the symmetric A_{2n} to a point M_{2n}. Then, for any A_1 from the plan, A_1 coincide with A_1' if and only if the polygons $M_1M_3\ldots M_{2n-1}$ and $M_2M_4\ldots M_{2n}$ have the same centroid.*

Proof. We have

$$z_{A_2} = 2z_{M_1} - z_{A_1},\ z_{A_3} = 2z_{M_2} - z_{A_2} = 2z_{M_2} - 2z_{M_1} + z_{A_1},$$
$$z_{A_4} = 2z_{M_3} - z_{A_3} = 2z_{M_3} - 2z_{M_2} + 2z_{M_1} - z_{A_1}, \dots,$$
$$z_{A_{2n}} = 2z_{M_{2n-1}} - 2z_{M_{2n-2}} + 2z_{M_{2n-3}} - \dots + 2z_{M_1} - z_{A_1},$$

so

$$z_{A'_1} = 2z_{M_{2n}} - z_{A_{2n}} = 2z_{M_{2n}} - 2z_{M_{2n-1}} + 2z_{M_{2n-2}} - \dots - 2z_{M_1} + z_{A_1}.$$

Then A_1 and A'_1 coincide if and only if

$$2z_{M_{2n}} - 2z_{M_{2n-1}} + 2z_{M_{2n-2}} - 2z_{M_{2n-3}} + \dots + 2z_{M_2} - 2z_{M_1} + z_{A_1} = z_{A_1},$$

equivalent to

$$z_{M_{2n}} + z_{M_{2n-2}} + \dots + z_{M_2} = z_{M_{2n-1}} + z_{2n-3} + \dots + z_{M_1},$$

equivalent to

$$\frac{z_{M_1} + z_{M_3} + \dots + z_{M_{2n-1}}}{n} = \frac{z_{M_2} + z_{M_4} + \dots + z_{M_{2n}}}{n},$$

equivalent with the polygons $M_1M_2\dots M_{2n-1}$ and $M_2M_4\dots M_{2n}$ have the same centroid. □

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