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RADII OF THE INSCRIBED AND ESCRIBED SPHERES OF A SIMPLEX

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Abstract. It is well-known that the reciprocal of the radius of the inscribed circle of a triangle is equal to the sum of the reciprocals of the radii of the three escribed circles. I generalize this theorem for an n-dimensional simplex. In three or higher dimensions, there are many types of spheres tangent to a simplex. I characterize the existence and uniqueness of such spheres and derive formulas connected with their radii.

1. Introduction

Consider a triangle and its inscribed and escribed circles (Figure 1). Let r_0 be the radius of the inscribed circle (incircle) and r_1, r_2, r_3 be the radii of the three escribed circles (excircles). It is well-known that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_0}. \tag{1}$$

According to [6], the formula (1) is due to Steiner, published in 1828. Considering the simplicity of the formula and its proof, it would not be surprising if the ancient Greek geometers knew about it. (1) also appears on p. 189 of [3] and as Exercise 6 of Section 1.4 of [1].

In this paper I generalize (1) to the case of an n-simplex in a Euclidean space $\mathbb{R}^n$. For an n-simplex, we can show that there exist an inscribed sphere and one escribed sphere opposite to the inscribed sphere about each face. Since there are $n+1$ faces, there are also $n+1$ escribed spheres. Letting r_0 be the radius of the inscribed sphere and $r_1, \dots, r_{n+1}$ be the radii of the escribed spheres, we can show the relation

$$\sum_{k=1}^{n+1} \frac{1}{r_k} = \frac{n-1}{r_0}. \tag{2}$$

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Clearly (1) is a special case of (2) by setting $n = 2$. Contrary to the two dimensional case, in Euclidean spaces with dimension three or higher, there are many types of spheres that are tangent to the faces of the n-simplex. For example, one can consider tangent spheres that are opposite to the inscribed sphere about two faces, three faces, and so on. However, such spheres may not always exist. In this paper I also characterize the existence and uniqueness of such spheres, derive their radii, and further generalize the formula (2).

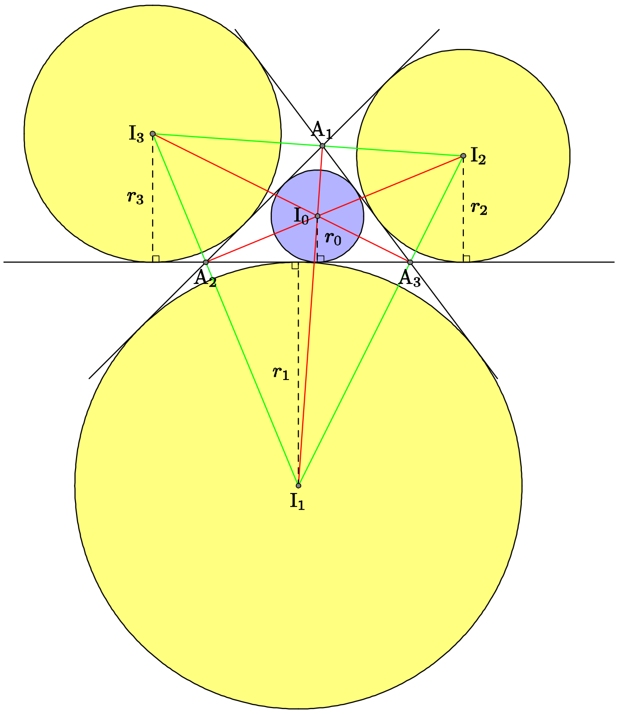

FIGURE 1. Incircle and excircles of a triangle. I_0 is the incenter of triangle $\mathrm{A}_1\mathrm{A}_2\mathrm{A}_3$ and $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3$ are excenters.

The rest of the paper is organized as follows. Section 2 provides a simple proof of (1) and an intuitive geometric proof of (2) that generalizes the two dimensional case. Section 3 introduces some notations and presents a few results in linear algebra suited for analyzing the geometry of the n-simplex. Section 4 provides an algebraic proof of (2) and further generalizes it.

2. Geometric proof of (2)

First I prove (1). In Figure 1, let $\overline{\mathrm{A}_1\mathrm{A}_2} = c$, $\overline{\mathrm{A}_2\mathrm{A}_3} = a$, $\overline{\mathrm{A}_3\mathrm{A}_1} = b$, and the radii of circles with centers $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3$ be r_1, r_2, r_3, respectively. Consider the quadrilateral $\mathrm{A}_1\mathrm{A}_2\mathrm{I}_1\mathrm{A}_3$. Since

$$\begin{aligned}\square \mathrm{A}_1\mathrm{A}_2\mathrm{I}_1\mathrm{A}_3 &= \triangle \mathrm{I}_0\mathrm{A}_1\mathrm{A}_2 \cup \triangle \mathrm{I}_0\mathrm{A}_2\mathrm{A}_3 \cup \triangle \mathrm{I}_0\mathrm{A}_3\mathrm{A}_1 \cup \triangle \mathrm{I}_1\mathrm{A}_2\mathrm{A}_3 \\ &= \triangle \mathrm{I}_1\mathrm{A}_1\mathrm{A}_2 \cup \triangle \mathrm{I}_1\mathrm{A}_3\mathrm{A}_1,\end{aligned}$$

Accounting for the area in two ways we obtain

$$\frac{1}{2}r_0(c+a+b)+\frac{1}{2}r_1 a=\frac{1}{2}r_1(c+b) \iff \frac{1}{r_1}=\frac{b+c-a}{a+b+c}\frac{1}{r_0}.$$

Similarly,

$$\frac{1}{r_2}=\frac{c+a-b}{a+b+c}\frac{1}{r_0},$$
$$\frac{1}{r_3}=\frac{a+b-c}{a+b+c}\frac{1}{r_0}.$$

Adding these three equations, we obtain (1).

The geometric proof of (2) is similar. Let $P=\{\mathrm{A}_1,\mathrm{A}_2,\dots,\mathrm{A}_{n+1}\}$ be a collection of $n+1$ points in $\mathbb{R}^n$ that do not lie on a common hyperplane and $K=\operatorname{co}P$ be the n-simplex with vertices $\{\mathrm{A}_k\}$. ("co" denotes the convex hull.) Let $F_k=\operatorname{co}(P\backslash \mathrm{A}_k)$ be the face of K that does not include A_k. Let I_0 be the incenter of K and I_k be the excenter that lies opposite to I_0 about F_k. (The existence and uniqueness of such points are proved in the next section.) Let $r_0,\dots,r_{n+1}$ be the radii of the tangent spheres, with the obvious indexation. Let $|F_k|$ be the area ($n-1$-dimensional volume) of F_k. We account for the volume of the polytope $\operatorname{co}(P\cup \mathrm{I}_k)$ in two ways. Since

$$\operatorname{co}(P\cup \mathrm{I}_k)=\left(\bigcup_{j=1}^{n+1}\operatorname{co}(\mathrm{I}_0\cup F_j)\right)\cup(\mathrm{I}_k\cup F_k)=\bigcup_{j\neq k}\operatorname{co}(\mathrm{I}_k\cup F_j),$$

it follows that

$$\sum_{j=1}^{n+1}\frac{1}{n}r_0\,|F_j|+\frac{1}{n}r_k\,|F_k|=\sum_{j\neq k}\frac{1}{n}r_k\,|F_j| \iff \frac{1}{r_k}=\frac{F-2\,|F_k|}{F}\frac{1}{r_0},$$

where $F=\sum_{k=1}^{n+1}|F_k|$. Summing over all k's, we obtain

$$\sum_{k=1}^{n+1}\frac{1}{r_k}=\frac{(n+1)F-2F}{F}\frac{1}{r_0}=\frac{n-1}{r_0},$$

which is (2).

3. Preliminary results

In this section I introduce some notations and a few preliminary results in order to prove (2) rigorously.

All vectors belong to $\mathbb{R}^n$, where $n\geq 2$. For $x,y\in\mathbb{R}^n$, $\langle x,y\rangle$ denotes the usual inner product and the $\|x\|=\sqrt{\langle x,x\rangle}$ is the Euclidean norm. The distance between a point x and a set S is defined by $\operatorname{dist}(x,y)=\inf_{y\in S}\|x-y\|$. We say that the $n+1$ points $a_1,a_2,\dots,a_{n+1}$ are in *generic position* if the matrix

$$A=[a_1-a_{n+1},\dots,a_n-a_{n+1}] \tag{3}$$

is regular (nonsingular). It is easy to show that the notion of the generic position does not depend on the order of $a_1,\dots,a_{n+1}$. The following lemma shows that if $a_1, \dots, a_{n+1}$ are in generic position, any vector in $\mathbb{R}^n$ can be uniquely represented as an affine combination of these points.

Lemma 3.1. *If the $n+1$ points $a_1, \dots, a_{n+1}$ are in generic position, then any $x \in \mathbb{R}^n$ has a unique representation $x = \sum_{i=1}^{n+1} t_i a_i$, where $\sum_{i=1}^{n+1} t_i = 1$.*

Proof. Since by assumption A is regular, the vectors $\{a_i - a_{n+1}\}_{i=1}^n$ are linearly independent. Hence for all $x \in \mathbb{R}^n$, there exists a unique representation

$$x - a_{n+1} = \sum_{i=1}^n t_i(a_i - a_{n+1}) \iff x = \sum_{i=1}^n t_i a_i + \left(1 - \sum_{i=1}^n t_i\right) a_{n+1}.$$

The claim follows by setting $t_{n+1} := 1 - \sum_{i=1}^n t_i$.

An n-simplex K is the convex hull of $n+1$ points $a_1, \dots, a_{n+1}$ in generic position. For instance, a 2-simplex is a triangle in $\mathbb{R}^2$, and a 3-simplex is a tetrahedron in $\mathbb{R}^3$. With a slight abuse of language, I define the face of an n-simplex K by the hyperplanes that pass through all but one vertex of K. Let π_k be the face of K that does not contain a_k.

In order to define spheres that are tangent to all faces of K, we need to compute the distance between a point and a face of K. Let the matrix A be as in (3) and define vectors $b_1, \dots, b_{n+1}$ by $(A^{-1})' = B = [b_1, \dots, b_n]$ and

$$b_{n+1} = -\sum_{i=1}^n b_i. \tag{4}$$

The following proposition gives a formula for the distance between any point and π_k.

Proposition 3.1. *Let $\sum_{i=1}^{n+1} t_i = 1$ and $x = \sum_{i=1}^{n+1} t_i a_i$. Then the distance between x and the k-th face of K, π_k, is given by*

$$\operatorname{dist}(x, \pi_k) = \frac{|t_k|}{\|b_k\|}. \tag{5}$$

Proof. Since $B = [b_1, \dots, b_n]$ is a regular matrix, we have $b_k \neq 0$. Since $\{b_i\}_{i=1}^n$ are linearly independent, we have $b_{n+1} = -\sum_{i=1}^n b_i \neq 0$. Therefore the right-hand side of (5) is well-defined.

Let us show that for all $i, j, k = 1, 2, \dots, n+1$, we have

$$\langle a_i - a_j, b_k \rangle = \delta_{ik} - \delta_{jk}, \tag{6}$$

where δ is Kronecker's delta. (6) is trivial if $i = j$. If (6) holds for $j = n+1$, then (6) is true for any j because

$$\begin{aligned}\langle a_i - a_j, b_k \rangle &= \langle a_i - a_{n+1}, b_k \rangle - \langle a_j - a_{n+1}, b_k \rangle \\ &= (\delta_{ik} - \delta_{n+1,k}) - (\delta_{jk} - \delta_{n+1,k}) = \delta_{ik} - \delta_{jk}.\end{aligned}$$

Therefore, without loss of generality we may assume $i \leq n$ and $j = n+1$. If $k \leq n$, then computing the (i,k) element of $A'B = I_n$ we obtain

$$\langle a_i - a_{n+1}, b_k \rangle = \delta_{ik} = \delta_{ik} - \delta_{n+1,k},$$

so (6) holds. If $k = n+1$, then by (4) we get

$$\begin{aligned}\langle a_i - a_{n+1}, b_{n+1}\rangle &= -\sum_{j=1}^{n} \langle a_i - a_{n+1}, b_j\rangle \\ &= -\sum_{j=1}^{n}(\delta_{ij} - \delta_{n+1,j}) = -1 = \delta_{i,n+1} - \delta_{n+1,n+1},\end{aligned}$$

so (6) holds.

Next, I show that the vector b_k is orthogonal to π_k. To see this, fix any $j \neq k$. Since π_k passes through a_j and is spanned by all vectors $\{a_i - a_j\}_{i\neq j,k}$, it suffices to show that $\langle a_i - a_j, b_k\rangle = 0$ for $i \neq j, k$. However, this is obvious by (6) since $i \neq k$ and $j \neq k$.

Finally, I show the distance formula (5). Again fix any $j \neq k$. Since b_k is orthogonal to π_k, it follows from (6) and $j \neq k$ that

$$\begin{aligned}\operatorname{dist}(x, \pi_k) &= \frac{|\langle x - a_j, b_k\rangle|}{\|b_k\|} = \frac{1}{\|b_k\|}\left|\sum_{i=1}^{n+1} t_i \langle a_i - a_j, b_k\rangle\right| \\ &= \frac{1}{\|b_k\|}\left|\sum_{i=1}^{n+1} t_i(\delta_{ik} - \delta_{jk})\right| = \frac{|t_k|}{\|b_k\|}.\end{aligned}$$

4. Algebraic proof of (2) and generalizations

In this section I characterize the existence of tangent spheres and rigorously prove a generalization of (2).

Let $a_1, \dots, a_{n+1}$ be points in generic position and $K = \operatorname{co}\{a_1, \dots, a_{n+1}\}$ be an n-simplex. Each face π_k of K divides $\mathbb{R}^n$ into two half spaces, one that contains a_k and the other not. In order to refer to the position of a point $x = \sum_{i=1}^{n+1} t_i a_i$ (with $\sum_i t_i = 1$) relative to π_k, let $\sigma = (\sigma_1, \dots, \sigma_{n+1})$ be a tuple of ± 1, which I call a *sign*. Since σ consists of $n+1$ elements which can take the value ± 1, there are in total 2^{n+1} possibilities. The sign with all 1 is denoted by $\mathbf{1} = (1, \dots, 1)$. For any sign σ, define the set

$$D(\sigma) = \left\{ x = \sum_{i=1}^{n+1} t_i a_i \,\middle|\, (\forall i)\sigma_i t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \right\}. \tag{7}$$

Clearly the n-simplex K is precisely $D(\mathbf{1})$.

The geometrical intuition of $D(\sigma)$ is that for all k, a point $x \in D(\sigma)$ lies on the same side as a_k with respect to π_k if $\sigma_k = 1$, and lies on the other side of a_k if $\sigma_k = -1$. The following lemma proves this fact and also shows that all but one $D(\sigma)$'s are nonempty.

Lemma 4.1. *Any $x \in \mathbb{R}^n$ belongs to at least one $D(\sigma)$. For any sign σ, $a_k \in D(\sigma)$ if and only if $\sigma_k = 1$. Furthermore, $D(-\mathbf{1}) = \emptyset$.*

Proof. That $x \in D(\sigma)$ for some σ follows by Lemma 3.1.

If $x \in D(-\mathbf{1})$, by definition there exist $t_1, \dots, t_{n+1}$ such that $t_i \le 0$ for all i and $\sum_{i=1}^{n+1} t_i = 1$, which is obviously impossible. Therefore, $D(-\mathbf{1}) = \emptyset$.

Again by Lemma 3.1, setting $t_k = 1$ and $t_i = 0$ for $i \ne k$ is the only way to express $a_k = \sum_i t_i a_i$ with $\sum_i t_i = 1$. Therefore $a_k \in D(\sigma)$ if and only if $\sigma_k = 1$.

If $\sigma \ne \sigma'$, there is some k such that $\sigma_k \ne \sigma'_k$. Hence by Lemma 4.1 one of $D(\sigma)$, $D(\sigma')$ contains a_k and the other not. Therefore $D(\sigma) \ne D(\sigma')$, and clearly they do not share interior points. Therefore $\mathbb{R}^n$ is divided into a total number of $2^{n+1} - 1$ nonempty $D(\sigma)$'s. ("-1" because $D(-\mathbf{1}) = \emptyset$.)

A sphere with center x and radius r is said to be *tangent* to K if $r = \operatorname{dist}(x, \pi_k)$ for all $k = 1, 2, \dots, n+1$. The following proposition characterizes the existence and uniqueness of tangent spheres.

Proposition 4.1. *Let σ be a sign. A tangent sphere with center in $D(\sigma)$ exists if and only if $\sum_{i=1}^{n+1} \sigma_i \|b_i\| > 0$. Under this condition, the tangent sphere is unique and its radius is given by $r(\sigma) = 1/\sum_{i=1}^{n+1} \sigma_i \|b_i\|$.*

Proof. Suppose that a sphere with center $x = \sum_{i=1}^{n+1} t_i a_i \in D(\sigma)$ and radius r is tangent to K. Since by definition $r = \operatorname{dist}(x, \pi_k)$ for all k, by Proposition 3.1 we obtain

$$r = \frac{|t_1|}{\|b_1\|} = \dots = \frac{|t_{n+1}|}{\|b_{n+1}\|}. \tag{8}$$

Since $D(\sigma)$ is defined by (7), it follows that $\sigma_i t_i \ge 0$. Since $\sigma_i = \pm 1$, we get $|t_i| = |\sigma_i t_i| = \sigma_i t_i$. Hence by (8), t_i is uniquely determined such that $t_i = \frac{r\|b_i\|}{\sigma_i} = r\sigma_i \|b_i\|$. Summing over i and noting that $\sum_{i=1}^{n+1} t_i = 1$, we get $\sum_{i=1}^{n+1} \sigma_i \|b_i\| = \frac{1}{r} > 0$, and r is unique.

Conversely, if $\sum_{i=1}^{n+1} \sigma_i \|b_i\| > 0$, define $r > 0$ by $\frac{1}{r} = \sum_{i=1}^{n+1} \sigma_i \|b_i\| > 0$ and $t_i = r\sigma_i \|b_i\|$. Then these r and t_i's satisfy $\sigma_i t_i \ge 0$, $\sum t_i = 1$ and (8), so by definition there exists a sphere tangent to K with center in $D(\sigma)$.

Proposition 4.1 is essentially due to [7], although in their paper $\|b_i\|$ is replaced by the $(n-1)$-dimensional volume of the face but they do not compute the volume from the given vectors $\{a_1, \dots, a_{n+1}\}$.

By Proposition 4.1, if there is a tangent sphere with center in $D(\sigma)$, it is unique. Hence, it is legitimate to denote it by $S(\sigma)$. Let $r(\sigma)$ be the radius of $S(\sigma)$ if it exists. The following corollary is immediate from Proposition 4.1.

Corollary 4.1. *For each sign σ, exactly one of the followings hold: (i) $S(\sigma)$ exists, (ii) $S(-\sigma)$ exists, (iii) neither $S(\sigma)$ nor $S(-\sigma)$ exist.*

Proof. If $\sum \sigma_i \|b_i\| \ne 0$, then either $\sum \sigma_i \|b_i\|$ or $\sum(-\sigma_i)\|b_i\|$ is positive (but not both), so by Proposition 4.1 either $S(\sigma)$ or $S(-\sigma)$ exists (but not both). If $\sum \sigma_i \|b_i\| = 0$, then $\sum(-\sigma_i)\|b_i\| = 0$ also, so by Proposition 4.1 neither $S(\sigma)$ nor $S(-\sigma)$ exist.

I define the *escribed sphere* by a tangent sphere corresponding to signs σ of the form $\sigma_k = -1$ for some k, and $\sigma_i = 1$ for all $i \ne k$. (There are $n+1$ such signs.) The following theorem rigorously proves the formula (2).

Theorem 4.1. *For any n-simplex K, exactly one inscribed sphere and $n+1$ escribed spheres exist. Letting $r_0, r_1, \ldots, r_{n+1}$ be the radii of these spheres,* (2) *holds.*

Proof. Since $\sum_i \|b_i\| > 0$, the inscribed sphere $S(\mathbf{1})$ exists. Let σ^k be the sign with k-th element -1 and all other elements 1, so $\sigma^k_k = -1$ and $\sigma^k_i = 1$ for all $i \neq k$. Let us show that the k-th escribed sphere $S(\sigma^k)$ exists.

Since by (4) we have $\sum_i b_i = 0$, we get $\sum_{i \neq k} b_i = -b_k$. Taking the norm of both sides and invoking the triangle inequality, we obtain $\sum_{i \neq k} \|b_i\| \geq \|b_k\|$. Equality does not hold because any n vectors of $b_1, \ldots, b_{n+1}$ are linearly independent since $B = [b_1, \ldots, b_n]$ is regular. Therefore

$$\sum_{i \neq k} \|b_i\| > \|b_k\| \iff 0 < \sum_{i=1}^{n+1} \sigma^k_i \|b_i\| = \frac{1}{r(\sigma^k)} = \frac{1}{r_k}, \tag{9}$$

so by Proposition 4.1 $S(\sigma^k)$ uniquely exists. Summing $\frac{1}{r_k}$ over k and using (9), since $\frac{1}{r_0} = \sum_i \|b_i\|$ and $\sigma^k_k = -1$, $\sigma^k_i = 1$ $(i \neq k)$, we obtain (2).

In two dimension, the only tangent circles to a triangle are the inscribed circle and the three escribed circles. In three dimension, in addition to the inscribed sphere and the four escribed spheres, in general there are other tangent spheres. Figure 2 shows the edges of the tetrahedron K corresponding to $a_1 = (0, 0, -1)'$, $a_2 = (1, 0, -1)'$, $a_3 = (0, 1, -1)'$, $a_4 = (0, 0, 0)'$ and tangent spheres associated with signs $\sigma = (1, 1, 1, 1)$ (inscribed sphere) and $\tau = (1, 1, -1, -1)$. The larger sphere is touching the faces of K below the "roof-like" region $D(\tau)$.

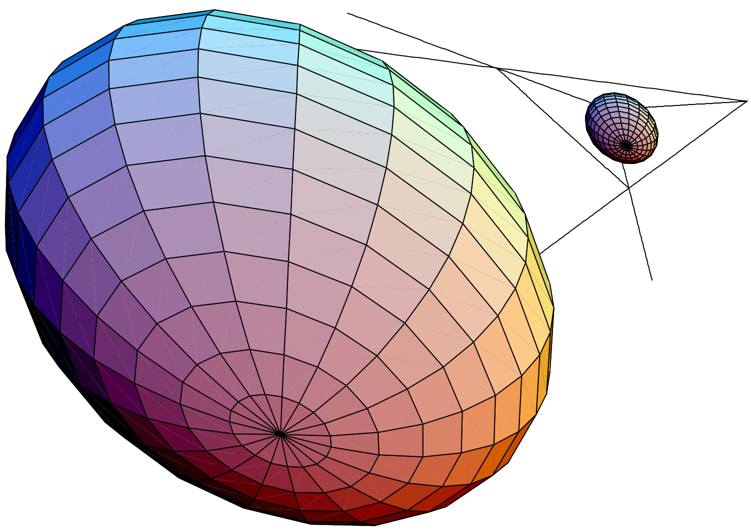

FIGURE 2. Two spheres tangent to the tetrahedron K.

In three or higher dimension, we can further generalize the formula (2). To this end, for each sign σ define the quantity

$$\chi(\sigma) = \begin{cases} \frac{1}{r(\sigma)}, & (S(\sigma) \text{ exists}) \\ -\frac{1}{r(-\sigma)}, & (S(-\sigma) \text{ exists}) \\ 0. & (\text{Neither } S(\sigma) \text{ nor } S(-\sigma) \text{ exist}) \end{cases} \tag{10}$$

Thus $\chi(\sigma)$ is either zero or the reciprocal of the radius of a tangent sphere, with either a positive or negative sign. This definition is unambiguous in view of Corollary 4.1. Furthermore, let $|\sigma|$ be the number of elements of σ equal to (-1), so $|\sigma| = \#\{i|\sigma_i = -1\}$. For example, $|\mathbf{1}| = 0$. Define the product $\sigma\tau$ of two signs σ, τ by the element-wise multiplication. Note that $\sigma\tau$ is again a sign because $\sigma_i = \pm 1$ and $\tau_i = \pm 1$.

With these definitions, we can prove the main result of this paper.

Theorem 4.2. *Let σ be a sign and $1 \le m \le n$. Then*

$$\sum_{|\tau|=m} \chi(\sigma\tau) = \left[\binom{n}{m} - \binom{n}{m-1}\right] \chi(\sigma), \tag{11}$$

where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ is the binomial coefficient.

Proof. Let us first show that

$$\chi(\sigma) = \sum_{i=1}^{n+1} \sigma_i \|b_i\|. \tag{12}$$

If $\sum \sigma_i \|b_i\| = 0$, by Proposition 4.1 neither $S(\sigma)$ nor $S(-\sigma)$ exist, so by (10) we get $\chi(\sigma) = 0 = \sum \sigma_i \|b_i\|$. If $\sum \sigma_i \|b_i\| > 0$, by Proposition 4.1 and (10) we have $\sum_{i=1}^{n+1} \sigma_i \|b_i\| = \frac{1}{r(\sigma)} = \chi(\sigma)$. Finally, if $\sum \sigma_i \|b_i\| < 0$, $S(-\sigma)$ exists. Therefore, by (10) we obtain

$$\sum_{i=1}^{n+1} (-\sigma_i) \|b_i\| = \frac{1}{r(-\sigma)} = -\chi(\sigma) \iff \chi(\sigma) = \sum_{i=1}^{n+1} \sigma_i \|b_i\|.$$

By (12), the left-hand side of (11) is

$$\sum_{|\tau|=m} \chi(\sigma\tau) = \sum_{|\tau|=m} \sum_{i=1}^{n+1} \sigma_i\tau_i \|b_i\| = \sum_{i=1}^{n+1} \sigma_i \|b_i\| \sum_{|\tau|=m} \tau_i.$$

Since $\chi(\sigma) = \sum_i \sigma_i \|b_i\|$, it suffices to prove

$$\sum_{|\tau|=m} \tau_i = \binom{n}{m} - \binom{n}{m-1}. \tag{13}$$

Since there are $\binom{n}{m}$ cases for which $\tau_i = 1$ (we must have $\tau_j = -1$ for m choices of j's out of n) and $\binom{n}{m-1}$ cases for which $\tau_i = -1$ (we must have $\tau_j = -1$ for $m-1$ choices of j's out of n), (13) follows.

Theorem 4.2 states the following. Starting from the set $D(\sigma)$, cross exactly m faces of the simplex and go to $D(\sigma\tau)$, where $|\tau| = m$. Compute the radii of $S(\pm\sigma\tau)$ (at most one exists). By adding and subtracting the reciprocals of these tangent spheres, where adding if $S(\sigma\tau)$ exists and subtracting if $S(-\sigma\tau)$ exists, it will be equal to $\binom{n}{m} - \binom{n}{m-1}$ times either (i) the reciprocal of the radius of $S(\sigma)$, (ii) the negative of the reciprocal of the radius of $S(-\sigma)$, or (iii) zero, depending on whether $S(\sigma)$ exists, $S(-\sigma)$ exists, or neither exist.

Theorem 4.1 is a special case of Theorem 4.2 corresponding to $\sigma = \mathbf{1}$ (so $D(\mathbf{1}) = K$ and therefore $S(\mathbf{1})$ is the inscribed sphere) and $|\tau| = 1$ (so $S(\tau)$ is an escribed sphere). Although Theorem 4.1 is a natural extension of the well-known two dimensional case with a similar geometric proof, its generalization Theorem 4.2 is highly nontrivial.

5. Conclusion

In this paper I generalized a well-known theorem in plane geometry—that the sum of the reciprocals of the radii of the three escribed circles of a triangle equals the reciprocal of the radius of the inscribed circle—for an n-dimensional simplex. Considering the richness of the plane geometry, there would certainly be many other theorems on the Euclidean geometry of $\mathbb{R}^n$ that generalizes the corresponding theorems in the plane geometry. For example, [4] and [8] generalize the Euler inequality $R \geq 2r$, where R and r are the radii of the circumcircle and the incircle of a triangle. [5] and [2] generalize the Gergonne and Nagel points of a triangle for a simplex. Using the closed-form formula for the radii of tangent spheres given in Proposition 4.1, it might be possible to generalize some of the well-known equations and inequalities connected with the radii of tangent circles as reviewed in [6]. I leave this issue for future research.

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