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A QUATERNIONIC APPROACH to GEOMETRY of CURVES on SPACES of CONSTANT CURVATURE

TUNA BAYRAKDAR and A. A. ERGİN

Abstract. We construct the Frenet-Serret frame for a curve on four dimensional Euclidean space by means of quaternion algebra. To be able to generalize this approach to curves on the other model spaces of constant curvature we construct a quaternion algebra on each fiber of the tangent bundle compatible to the Riemannian metric, i.e. algebra with the multiplicative property with respect to the norm induced by the Riemannian metric. Via this construction we obtain Frenet-Serret frame and its derivative formulas for curves on three and four dimensional spaces of constant curvature in the language of the quaternion algebra.

1. Introduction

Any complete, simply connected Riemannian manifold with constant sectional curvature C is locally isometric to one of the spaces $\mathbb{R}^n(C = 0), \mathbb{S}^n(C > 0)$ and $\mathbb{H}^n(C < 0)$ for $n \geq 2$, which are model spaces for Euclidean and non-Euclidean geometries. The pullback metrics on these spaces are given locally by

$$
\begin{aligned}
ds^2_{\mathbb{E}} &= \sum_{i=1}^{n}(dx^i)^2 \\
ds^2_{\mathbb{S}} &= \frac{4}{(1+(x^1)^2+...+(x^n)^2)^2}\sum_{i=1}^{n}(dx^i)^2 \qquad (1)\\
ds^2_{\mathbb{H}} &= \frac{1}{(x^n)^2}\sum_{i=1}^{n}(dx^i)^2, \ x^n > 0,
\end{aligned}
$$

where $(x^1, ..., x^n)$ are coordinates on $\mathbb{R}^n$. The spaces of constant curvature has an importance in both theory and applications in differential geometry and mathematical physics.

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In the context of differential geometry, spaces of constant(sectional) curvature has a central role to study geometric and topological properties of a complete Riemannian manifold by comparing its sectional curvature with that of a model space [5]. Also in mathematical physics, a space of constant negative curvature appears (for example) as a solution of nonlinear partial differential equation of hyperbolic type, namely sine-Gordon equation, which naturally arises from the compatibility conditions on the Gauss equations for surfaces in 3-D with Gaussian curvature $K = -1$. Sine-Gordon equation describes a completely integrable Hamiltonian system with infinitely many degrees of freedom, see for example [3].

The theory of curves in classical differential geometry is of special interest especially in three and four dimensions. In three dimension, a smooth curve is a vector-valued function of a single variable which is determined by its invariants, called curvature and torsion, up to a position in the space. The curvature and torsion describes deviation of a curve from its tangent line and oscillating plane respectively. The motion of moving triad along a curve described by the well known system of Frenet-Serret equations. Frenet-Serret equations for curves on n-dimensional Riemannian space given firstly by Blaschke [2]. Quaternionic approach to the curve theory in three and four dimensional Euclidean spaces given in [1]. In that paper, authors achieved to obtain a frame for a curve in 4-D and its governing equations for a very special choice. One of the objectives of this work is to construct Frenet frame in a general way and represent its derivative formulas.

Quaternion algebra is an elegant tool to study local properties of an immersed geometric objects in a sense not only that makes the formulas simplified but also serves a relevance that reflects a simple interplay between geometry and algebra. Invariants of a curve for example, can be determined as a multiplication of two quaternions due to the fact that interesting relation between scalar product and quaternion multiplication. This serves that differential geometric properties determined by the metric can be treated also in quaternion algebraic sense. Since the invariants(curvatures) are habitants of the center of the quaternion algebra, the class of non-congruent curves in four space determined by taking quotient of quaternion algebra by its center, i.e curves obtained from non-congruent curves in three space can not be congruent in four dimensions. This can be thought as a functional dependence criteria on curvatures. In this point of view constructing a frame on a curve in four dimensions from the curve in three dimensions is geometrically meaningful.

In this paper we deals essentially with the construction of a quaternion algebra on each fiber of the tangent bundle of a space form of dimensions 3 and 4 such that, the norm induced by the metric on a base space has multiplicative property, i.e. satisfies $|pq| = |p||q|$, for $p, q \in T_xM$. It is so the isomorphisms between tangent spaces with algebraic structure on them determined by the metric structure on manifold. By means of this suitably arranged quaternion algebra we construct Frenet-Serret frame on curves in 3 and 4 dimensional spaces of constant curvature and we calculate Frenet-Serret formulas for them. This occurs naturally as the generalization of the procedure that we manage to frame a curve in three and four dimensional

Euclidean spaces and we see that identities that we use during the construction of the Frenet-Serret frame become more simplified thanks to the quaternion algebra.

2. Preliminaries

2.1. **(Euclidean) Quaternions.** Quaternions can be found in many texts [6],[9],[7]. The set $\mathbb{H}$ of quaternions $q = a + bi + cj + dk$ is an associative, non-commutative division algebra over the reals with respect to the multiplication

$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik,\ i^2 = j^2 = k^2 = -1, \tag{2}$$

where $1 = (1, 0, 0, 0), i = (0, 1, 0, 0), j = (0, 0, 1, 0), k = (0, 0, 0, 1)$ are the basic vectors for $\mathbb{R}^4$ and 1 stands for the unity of the algebra. The quaternion algebra is isomorphic to a subalgebra of two by two matrices with complex entries $M(2, \mathbb{C})$:

$$q = a + bi + cj + dk \longleftrightarrow \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \tag{3}$$

An operation of conjugation of a quaternion $q = a + bi + cj + dk$ is defined by

$$\bar{q} = a1 - bi - cj - dk.$$

The corresponding map $q \mapsto \bar{q}$ is an anti-isomorphism that is $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$. As in the case of complex numbers real(scalar) and imaginary(vectorial) part of a quaternion $q = a + bi + cj + dk$ is defined respectively by

$$Re(q) = \frac{q + \bar{q}}{2} = a,\ Im(q) = bi + cj + dk.$$

Thus any quaternion can be written in the form $q = Re(q)+Im(q)$. The set of quaternions with vanishing real parts is given by $Im\mathbb{H} = \{q \in \mathbb{H} : \bar{q} = -q\}$, which is identified with $\mathbb{R}^3$. But $Im\mathbb{H}$ is not subalgebra of $\mathbb{H}$. The Euclidean norm of a quaternion is given by

$$|q|^2 = q\bar{q} = a^2 + b^2 + c^2 + d^2,$$

which is induced by the scalar product

$$< p, q >= Re(p\bar{q}) = \frac{1}{2}\left(p\bar{q} + q\bar{p}\right). \tag{4}$$

With respect to this inner product any two quaternions are said to be orthogonal if $p\bar{q} + q\bar{p} = 0$. Any non-zero quaternion has an inverse

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

The set of all unit quaternions is just the 3-sphere S^3 which isomorphic to SU(2), set of all unitary matrices with determinant 1.

3. Frenet-Serret Equations on $\mathbb{R}^3$

To derive underlying equations for curves on $\mathbb{R}^4$ recall the three dimensional case. Classical version of Frenet-Serret equations on $\mathbb{R}^3$ can be found in [8]. Quaternionic approach to this equations given in [1] is the following theorem:

Theorem 3.1. *Let $q(s) = x(s)i + y(s)j + z(s)k$ be a unit speed regular curve on $\mathbb{R}^3$ with $\kappa > 0$. Then the following formulas hold:*

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n} \\ \mathbf{n}' &= -\kappa\mathbf{t} + \tau\mathbf{b} \\ \mathbf{b}' &= -\tau\mathbf{n} \end{aligned} \tag{5}$$

Here the functions κ and τ are curvature and torsion of given curve respectively. Using the identities $\mathbf{b} = \mathbf{tn}$, $\mathbf{n} = \mathbf{bt}$ $\mathbf{t} = \mathbf{nb}$, the linear system of equations turns out to be the following system:

$$\frac{d}{ds}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} \kappa\mathbf{b} & 0 & 0 \\ 0 & \boldsymbol{\omega} & 0 \\ 0 & 0 & \tau\mathbf{t} \end{pmatrix}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}, \tag{6}$$

where $\boldsymbol{\omega} = \kappa\mathbf{b} + \tau\mathbf{t}$ is Darboux quaternion. So the system of Frenet-Serret equations can be considered as a left quaternionic eigenvalue problem [4],[10]. Integration of these first order quaternion differential equations is another story and it is not in the scope of this paper.

4. Frenet-Serret Formulas on $\mathbb{R}^4$ via Quaternions

In this section we construct Frenet-Serret frame field along a curve in four space in virtue of a curve in three space. We also give explicit calculations of the procedure and then we state the theorem describing motion of this frame along aforementioned curve.

Let $q(s) = \alpha(s) + \beta(s)i + \gamma(s)j + \delta(s)k$ be a unit speed regular curve in $\mathbb{E}^4$, that is

$$\mathrm{T} = \frac{dq}{ds}, \ |\mathrm{T}(s)| = 1 \ forall s.$$

Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ denote respectively that the tangent, normal and binormal quaternions of a curve in three space parametrized by an arc-length parameter $l = l(s)$. The unit quaternion fields defined by $\mathbf{t}\mathrm{T}$, $\mathbf{n}\mathrm{T}$, $\mathbf{b}\mathrm{T}$ are perpendicular to the quaternion T with respect to the Euclidean inner product (4) and hence the set $\{\mathrm{T}, \mathbf{t}\mathrm{T}, \ \mathbf{n}\mathrm{T}, \ \mathbf{b}\mathrm{T}\}$, forms an orthonormal basis for $\mathbb{H} \backsimeq \mathbb{E}^4$. The principal normal to the curve $q = q(s)$ is defined by

$$\mathrm{N} := \frac{\mathrm{T}'}{|\mathrm{T}'|}, \ |\mathrm{N}| = 1, \ k_1 = |\mathrm{T}'| > 0, \tag{7}$$

where $k_1 = |\mathrm{T}'|$ is the first curvature of $q = q(s)$. Taking derivative of the both side of $\mathrm{T}\bar{\mathrm{T}} = 1$, we see that T is orthogonal to N . Since N is a unit quaternion and perpendicular to T , it can be written as

$$\mathrm{N} = c_1\mathbf{t}\mathrm{T} + c_2\mathbf{n}\mathrm{T} + c_3\mathbf{b}\mathrm{T}, \ \sum_{i=1}^{3} c_i^2 = 1, \ c_i = c_i(s). \tag{8}$$

Since $\sum_{i=1}^{3} c_i^2 = 1$, $\dot{\hat{c}}$ is orthogonal to the purely imaginary quaternion $\hat{c} = c_1 i + c_2 j + c_3 k$, where dot denotes the derivative with respect to coordinate s. Let us denote the normalized quaternion along $\dot{\hat{c}}$ by $\hat{\omega}$:

$$\dot{\hat{c}} = \lambda\hat{\omega},\ \ \lambda \in \mathbb{R},\ \ |\omega| = 1.$$

The quaternion defined by

$$\text{(9)} \qquad \mathrm{B}_1 := \omega_1 \mathbf{t}\mathrm{T} + \omega_2 \mathbf{n}\mathrm{T} + \omega_3 \mathbf{b}\mathrm{T},\ \sum_{i=1}^{3} \omega_i^2 = 1,\ \omega_i = \omega_i(s)$$

is then orthogonal to N and T. Let us define the quaternion B_2 as

$$\text{(10)} \qquad \mathrm{B}_2 := \mathrm{B}_1 \overline{\mathrm{N}} \mathrm{T}.$$

It is seen by a direct calculation that B_2 can be written explicitly as

$$\text{(11)} \qquad \mathrm{B}_2 = (c_2\omega_3 - \omega_2 c_3)\mathbf{t}\mathrm{T} + (c_3\omega_1 - c_1\omega_3)\mathbf{n}\mathrm{T} + (c_1\omega_2 - c_2\omega_1)\mathbf{b}\mathrm{T}$$

The components of the quaternion B_2 in the basis $\mathbf{t}\mathrm{T}, \mathbf{n}\mathrm{T}, \mathbf{b}\mathrm{T}$ are determined by the components of the purely imaginary quaternion

$$\hat{a} = \hat{c}\hat{\omega} = (c_2\omega_3 - \omega_2 c_3)i + (c_3\omega_1 - c_1\omega_3)j + (c_1\omega_2 - c_2\omega_1)k = a_1 i + a_2 j + a_3 k.$$

Since $\hat{c}$ and $\hat{\omega}$ unit quaternions, B_2 is also unit quaternion:

$$\mathrm{B}_2 = a_1 \mathbf{t}\mathrm{T} + a_2 \mathbf{n}\mathrm{T} + a_3 \mathbf{b}\mathrm{T},\ \sum_{i=1}^{3} a_i^2 = 1,\ a_i = a_i(s).$$

From definition of $\mathrm{T}, \mathrm{N}, \mathrm{B}_1, \mathrm{B}_2$ it is easily seen that $\mathrm{B}_2 \perp \mathrm{T}$, $\mathrm{B}_2 \perp \mathrm{N}$, $\mathrm{B}_2 \perp \mathrm{B}_1$, and thus the moving frame $\{\mathrm{T}, \mathrm{N}, \mathrm{B}_1, \mathrm{B}_2\}$ forms a orthonormal basis for $\mathbb{R}^4$. Consequently, their derivatives with respect to arc-length parameter represented by the skew-symmetric matrix due to the orthonormality relations:

$$\text{(12)} \qquad \begin{aligned} \mathrm{T}' &= k_1 \mathrm{N} \\ \mathrm{N}' &= -k_1 \mathrm{T} + \lambda_1 \mathrm{B}_1 + \lambda_2 \mathrm{B}_2 \\ \mathrm{B}_1' &= -\lambda_1 \mathrm{N} + \mu_2 \mathrm{B}_2 \\ \mathrm{B}_2' &= -\lambda_2 \mathrm{N} - \mu_2 \mathrm{B}_1. \end{aligned}$$

It is in fact more convenient to use the set moving quaternions $\{\mathrm{T}, \mathrm{N}, \hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2\}$, where

$$\begin{aligned} \hat{\mathrm{X}}_1 &= \frac{1}{k_2}(\lambda_1 \mathrm{B}_1 + \lambda_2 \mathrm{B}_2) \\ \hat{\mathrm{X}}_2 &= \frac{1}{k_2}(\lambda_2 \mathrm{B}_1 - \lambda_1 \mathrm{B}_2),\ k_2 = \sqrt{\lambda_1^2 + \lambda_2^2}. \end{aligned}$$

One can easily see that $\hat{X}_1$ is orthogonal to the subspace spanned by $\{\mathrm{T}, \mathrm{N}, \hat{\mathrm{X}}_2\}$ and $\hat{X}_2$ is orthogonal to the subspace spanned by $\{\mathrm{T}, \mathrm{N}, \hat{\mathrm{X}}_2\}$. Therefore, the set $\{\mathrm{T}, \mathrm{N}, \hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2\}$ forms a basis for $\mathbb{R}^4$, and consequently we obtain the

Frenet-Serret frame on a unit speed curve in $\mathbb{R}^4$ via quaternion algebra:

$$\begin{aligned} \mathrm{T} &= \frac{dq}{ds} \\ \mathrm{N} &= \frac{\mathrm{T}'}{k_1} \\ \hat{\mathrm{X}}_1 &= \frac{1}{k_2}\left(\lambda_1 \mathrm{B}_1 + \lambda_2 \mathrm{B}_1 \overline{\mathrm{N}} \mathrm{T}\right) \\ \hat{\mathrm{X}}_2 &= \frac{1}{k_2}\left(\lambda_2 \mathrm{B}_1 - \lambda_1 \mathrm{B}_1 \overline{\mathrm{N}} \mathrm{T}\right) \end{aligned}$$

From (12) and the definition of $\hat{\mathrm{X}}_1$ and $\hat{\mathrm{X}}_2$ we see that

$$\mathrm{N}' = -k_1 \mathrm{T} + k_2 \hat{\mathrm{X}}_1.$$

Since $\hat{\mathrm{X}}_1$ is unit quaternion $\hat{\mathrm{X}}_1' \perp \hat{\mathrm{X}}_1$ and from (12) $\hat{\mathrm{X}}_1' \perp \mathrm{T}$, so we have

$$\hat{\mathrm{X}}_1' = \rho_1 \mathrm{N} + \rho_2 \hat{\mathrm{X}}_2.$$

Similarly, since $\hat{\mathrm{X}}_2$ is unit quaternion $\hat{\mathrm{X}}_2' \perp \hat{\mathrm{X}}_2$ and from (12) $\hat{\mathrm{X}}_2' \perp \mathrm{T}$. It follows from here that

$$\hat{\mathrm{X}}_2' = \tau_1 \mathrm{N} + \tau_2 \hat{\mathrm{X}}_1.$$

Since $\hat{\mathrm{X}}_1 \perp \hat{\mathrm{X}}_2$, $\tau_2 = -\rho_2$. Taking also derivative of the both sides of

$$< \mathrm{N}, \hat{\mathrm{X}}_2 >= \frac{1}{2}\left(\mathrm{N}\overline{\hat{\mathrm{X}}_2} + \hat{\mathrm{X}}_2 \bar{\mathrm{N}}\right) = 0.$$

gives $\tau_1 = 0$. Taking again the derivative of the both sides of

$$< \mathrm{N}, \hat{\mathrm{X}}_1 >= \frac{1}{2}\left(\mathrm{N}\overline{\hat{\mathrm{X}}_1} + \hat{\mathrm{X}}_1 \bar{\mathrm{N}}\right) = 0.$$

results in $\rho_1 = -k_2$. Let us denote ρ_2 by k_3. Thus we have proved the following theorem:

Theorem 4.1. *Let $q(s) = \alpha(s) + \beta(s)i + \gamma(s)j + \delta(s)k$ be a unit speed curve in $\mathbb{R}^4$ with curvature $k_1 > 0$, then the following formulas hold:*

$$\begin{aligned} \mathrm{T}' &= k_1 \mathrm{N} \\ \mathrm{N}' &= -k_1 \mathrm{T} + k_2 \hat{\mathrm{X}}_1 \\ \hat{\mathrm{X}}_1' &= -k_2 \mathrm{N} + k_3 \hat{\mathrm{X}}_2 \\ \hat{\mathrm{X}}_2' &= -k_3 \hat{\mathrm{X}}_1. \end{aligned} \tag{13}$$

Here k_1, k_2 and k_3 denote the first, second and third curvatures respectively. In the proof of the theorem the functional dependence of curvatures with κ and τ have not appeared explicitly. To see this one should represent the derivatives of the frame $\{\mathrm{T}, \mathrm{N}, \hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2\}$ in the basis T, $\mathbf{t}$T, $\mathbf{n}$T, $\mathbf{b}$T and then compare the coefficients with k_2 and k_3. There is no condition on the tangent field T so on k_1 by reason of that T is defined independently from the frame $\mathbf{t}, \mathbf{n}, \mathbf{b}$ of the curve in $\mathbb{R}^3$. To see that the curvature k_2, for example, is a function of κ and τ calculate N$'$ directly as

$$\mathrm{N}' = \frac{d\mathrm{N}}{ds} = (c_1' - c_2\kappa\nu)\mathbf{t}\mathrm{T} + (c_2' + c_1\kappa\nu - c_3\tau\nu)\mathbf{n}\mathrm{T} + (c_3' + c_2\tau\nu)\mathbf{b}\mathrm{T} - k_1\mathrm{T}, \tag{14}$$

where $\nu = \frac{dl}{ds}$. Since N$'$ is perpendicular to N it can be written of the form

$$\mathrm{N}' = \lambda_1 \mathrm{B}_1 + \lambda_2 \mathrm{B}_2 + \lambda_3 \mathrm{T},$$

where $\lambda_1, \lambda_2, \lambda_3$ are defined respectively by

$$\begin{aligned} \lambda_1 &= \lambda + \kappa\nu a_3 + \tau\nu a_1 \\ \lambda_2 &= -\kappa\nu\omega_3 - \tau\nu\omega_1 \\ \lambda_3 &= -k_1, \end{aligned}$$

where $\lambda = |\dot{c}|$. Since k_2 defined as $k_2 = \sqrt{\lambda_1^2 + \lambda_2^2}$, it is written as $k_2 = f(\kappa, \tau)$. The condition on k_3 can be seen similarly.

5. Riemannian Quaternions: A Generalization of the Quaternion Algebra

In this part of the work, we try to construct quaternion algebra in such a way that scalar product of given to quaternions with respect to Riemannian metric on base space M arise naturally in the real part of the product of given two quaternions. More formally, we try to construct a quaternion algebra on each fiber of the tangent bundle of M, such that the arc-length of a curve α on M, with $\alpha(0) = x$, can be calculated as real part of the quaternion multiplication of the tangent quaternion to the curve α at point x with its conjugate, i.e.

$$\left(\frac{ds}{dt}\right)^2 = Re(p\bar{p}) = g_{ij}p^ip^j, \ p \in T_xM, \ \alpha'(0) = p. \tag{15}$$

There are some obstructions to construct such an algebra that stems from the form of the metric on base space. Let us start with the general form of the metric and then modify it to go further for our objectives. Let (M, g) be a four dimensional Riemannian manifold with Riemannian connexion ∇ and let (x^1, x^2, x^3, x^4) be the local coordinates on M. In this coordinates distance element is written of the form

$$ds^2 = g_{ij}dx^idx^j. \tag{16}$$

To construct of quaternion algebra on each fiber of the tangent bundle with respect to quadratic form (16) we define the following multiplication rule on the coordinate basis $e_i = \frac{\partial}{\partial x^i}$ as

$$\begin{aligned} e_2e_2 &= -\frac{g(e_2,e_2)}{\sqrt{g(e_1,e_1)}}e_1, \ e_2e_3{=}\frac{\sqrt{g(e_2,e_2)g(e_3,e_3)}}{\sqrt{g(e_4,e_4)}}e_4, \ e_2e_4{=}-\frac{\sqrt{g(e_2,e_2)g(e_4,e_4)}}{\sqrt{g(e_3,e_3)}}e_3 \\ e_3e_2 &= -\frac{\sqrt{g(e_3,e_3)g(e_2,e_2)}}{\sqrt{g(e_4,e_4)}}e_4, \ e_3e_3{=}-\frac{g(e_3,e_3)}{\sqrt{g(e_1,e_1)}}e_1, \ e_3e_4{=}\frac{\sqrt{g(e_3,e_3)g(e_4,e_4)}}{\sqrt{g(e_2,e_2)}}e_2 \\ e_4e_2 &= \frac{\sqrt{g(e_4,e_4)g(e_2,e_2)}}{\sqrt{g(e_3,e_3)}}e_3, \ e_4e_3{=}-\frac{\sqrt{g(e_4,e_4)g(e_3,e_3)}}{\sqrt{g(e_2,e_2)}}e_2, \ e_4e_4{=}-\frac{g(e_4,e_4)}{\sqrt{g(e_1,e_1)}}e_1. \end{aligned}$$

Here e_1 is the unit element of the algebra and $g_{ij} = g(e_i, e_j)$. Accordingly, multiplication of two quaternions $p = \sum p^ie_i$ and $q = \sum q^ie_i$ can be written in the form

$$pq = \sum_{i,j} p^iq^je_ie_j,$$

where e_ie_j is defined as above. With respect to this multiplication rule conjugation operation and taking inverse are defined same as to classical quaternion algebra. Analogous to Euclidean case we should expect the scalar product of two quaternions $p = \sum p^i e_i$ and $q = \sum q^i e_i$ with respect to the Riemannian metric on M is to be defined by

$$< p, q >= Re(p\bar{q}).$$

If we calculate this with respect to the new quaternion algebra we find

$$< p, q >= Re(p\bar{q}) = \sum_i p^i q^i \frac{g_{ii}}{\sqrt{g_{11}}}. \tag{17}$$

We see from (17) that to define the scalar product of two tangent quaternions as $Re(p\bar{q})$ it is needed only the diagonal elements of the metric tensor. Consequently, if one wants study the geometric objects via this quaternionic approach, he should deal at least with the spaces on which metric has a diagonal form i.e. $ds^2 = g_{ii}dx^i dx^i$. For this type of metrics it is always possible to transform coordinates such that $g_{11} = 1$ i.e. there exist a local coordinate system $z^i = z^i(x^1, x^2, x^3, x^4)$ such that in this coordinates metric has the form

$$ds^2 = (dz^1)^2 + \sum_i g_{ii} dz^i dz^i. \tag{18}$$

This coordinate system is preferred coordinate system for us. In this sense, quaternion algebra which we want to construct is adaptable with the metric of the form (18). In this coordinates, quadratic form (18) can be represented by the quaternion multiplication as

$$ds^2 = Re(dq d\bar{q}) = \frac{1}{2}(dq d\bar{q} + d\bar{q} dq) = dq d\bar{q},$$

and the scalar product of two quaternions can be interpreted as

$$< p, q >= Re(p\bar{q}) = \frac{1}{2}(p\bar{q} + q\bar{p}) = \sum_i p^i q^i g_{ii}. \tag{19}$$

One can easily check that the norm induced by Riemannian metric (19) is compatible with quaternion algebra, i.e. it has multiplicative property. In this point of view, the spaces of constant curvature are legitimate candidates to apply the quaternion approach to them. In this sense, let us arrange the conformally flat metrics on space forms by a suitable change of coordinates to be able to investigate their submanifolds (only curves in this study) by the quaternion algebra which we constructed. Euclidean space is automatically applicable with standard coordinate. Consider four dimensional hyperbolic space with metric

$$ds^2_{\mathbb{H}} = \frac{1}{(x^4)^2}\big((dx^1)^2 + ... + (dx^4)^2\big).$$

Under the coordinate transformation

$$z^1 = \int^{x^1} \frac{dx^4}{\sqrt{(x^4)^2}}, z^2 = x^2, z^3 = x^3, z^4 = x^4,$$

metric turns the following one:

$$ds^2_{\mathbb{H}} = (dz^1)^2 + \frac{1}{(z^4)^2}\sum_{i=2}^{4}(dz^i)^2. \tag{20}$$

If we apply the similar coordinate transformation, metric on 4-sphere becomes

$$ds^2_{\mathbb{S}} = (dz^1)^2 + \frac{4}{(1+(z^1)^2+...+(z^4)^2)^2}\sum_{i=2}^{4}(dz^i)^2. \tag{21}$$

Thus the quaternion algebras constructed from the metrics (20) and (21) are defined respectively as

$$\begin{aligned} e_i e_j &= \frac{\sqrt{g(e_i,e_i)g(e_j,e_j)}}{\sqrt{g(e_k,e_k)}}\varepsilon_{ijk}e_k,\ i,j,k \neq 1 \\ e_i e_1 &= \sqrt{g(e_i,e_i)}e_1, \end{aligned}$$

where ε_{ijk} is Levi-Civita tensor, $g_{11} = 1$ and

$$g_{ii} = \frac{1}{(z^4)^2},\ for\ i = 2,3,4,$$

for the hyperbolic case and

$$g_{ii} = \frac{4}{(1+(z^1)^2+...+(z^4)^2)^2},\ for\ i = 2,3,4,$$

for the spherical case. Corresponding quadratic forms are then evaluated by the simple quaternion multiplication as

$$ds^2 = dqd\bar{q} = (dz^1)^2 + g_{22}(dz^2)^2 + g_{33}(dz^3)^2 + g_{44}(dz^2)^4, \tag{22}$$

where g_{ii}'s are defined as above. What the interesting part of this construction is that isomorphism between the algebras defined on the tangent spaces of corresponding manifolds determined by the metric structures on them.

5.1. **Curves on Three Dimensional Spaces.** In this section we deal with Frenet-Serret equations for curves in three dimensional space which is considered as the submanifold M of four dimensional space of constant curvature $\tilde{M}$ determined locally by the inclusion map with vanishing first component ,say $x^1 = 0$, (we replaced z^i by x^i). This submanifold is obviously isometrically embedded. In this case, the tangent space of the submanifold is a subspace of purely imaginary quaternions constructed from the metric. In section 3, we touched on the Euclidean case, so we work only on non-Euclidean cases. On submanifold M, Riemannian metric reduces to

$$ds^2 = dqd\bar{q} = g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + g_{44}(dx^2)^4.$$

Let $x^i, i = 2,3,4$ denote the (preferred) local coordinates on M and let ∇ be the induced connection on the submanifold M, determined by projecting the Riemannian connection $\tilde{\nabla}$ to the subspace and suppose that the unit speed curve γ given in coordinates by $\big(x^2(s), x^3(s), x^4(s)\big)$. Since γ is unit speed curve its velocity vector $\mathbf{t} = t^i e_i$ satisfies

$$t^i = \frac{dx^i}{ds},\ \sum t^i t^i g_{ii} = 1.$$

Differentiating $\sum t^i t^i g_{ii} = 1$ gives

$$0 = \frac{d}{ds}(t^i t^i g_{ii}) = \frac{dq^k}{ds}\nabla_k(t^i t^i g_{ii}) = \left(\frac{dq^k}{ds}\nabla_k t^i\right) t^i g_{ii} = (\nabla_{\mathbf{t}} t^i) t^i g_{ii}$$

Thus the covariant derivative of quaternion field $\mathbf{t}$ is orthogonal to itself. (Here we use the notation $\frac{d}{ds} = \frac{dx^k}{ds}\nabla_k = \nabla_{\mathbf{t}}$ for the derivative of a vector field along a curve, where $\nabla_k = \nabla_{e_k}$). Define the quaternion

$$\frac{\nabla_{\mathbf{t}}\mathbf{t}}{|\nabla_{\mathbf{t}}\mathbf{t}|} := \mathbf{n}$$

where

$$|\nabla_{\mathbf{t}}\mathbf{t}| := \kappa$$

is the curvature of the curve γ. Since $\mathbf{n}^2 = -e_1$, curvature κ can be determined also by quaternion multiplication:

$$-(\nabla_{\mathbf{t}}\mathbf{t})\mathbf{n} := \kappa e_1. \tag{23}$$

Let us define the binormal quaternion as

$$\mathbf{b} := \mathbf{t}\mathbf{n}, \tag{24}$$

the quaternion multiplication is done with respect to the table given in the beginning of this section. It is explicitly found as

$$\mathbf{b} = (t^3n^4 - t^4n^3)\frac{\sqrt{g_{33}}\sqrt{g_{44}}}{\sqrt{g_{22}}}e_2 + (t^4n^2 - t^2n^4)\frac{\sqrt{g_{22}}\sqrt{g_{44}}}{\sqrt{g_{33}}}e_3 + (t^2n^3 - t^3n^2)\frac{\sqrt{g_{22}}\sqrt{g_{33}}}{\sqrt{g_{44}}}e_4,$$

where t^k and n^k's are the components of $\mathbf{t}$ and $\mathbf{n}$. With a tedious calculations one can find that $|\mathbf{b}| = 1$ and also $\mathbf{b}$ is orthogonal to $\mathbf{t}$ and $\mathbf{n}$. Since $\mathbf{n}$ and $\mathbf{b}$ are unit quaternions their covariant derivatives along the curve γ are orthogonal to theirselves:

$$\nabla_{\mathbf{t}}\mathbf{n} = c_1\mathbf{t} + c_2\mathbf{b},\ \nabla_{\mathbf{t}}\mathbf{b} = d_1\mathbf{t} + d_2\mathbf{n},\ c_i, d_i \in \mathbb{R}.$$

It follows from here and (23) that

$$\nabla_{\mathbf{t}}(\mathbf{t}\mathbf{n}) = (\nabla_{\mathbf{t}}\mathbf{t})\mathbf{n} + \mathbf{t}(\nabla_{\mathbf{t}}\mathbf{n}) = -\kappa - c_1 - c_2\mathbf{n}.$$

Since,

$$\nabla_{\mathbf{t}}(\mathbf{t}\mathbf{n}) = \nabla_{\mathbf{t}}\mathbf{b} = -\kappa - c_1 - c_2\mathbf{n}$$

has no $\mathbf{t}$ component, $d_1 = 0$. Besides that $\nabla_{\mathbf{t}}(\mathbf{t}\mathbf{n})$ is purely imaginary quaternion, $c_1 = -\kappa$ holds. Let us define the torsion as

$$(\nabla_{\mathbf{t}}\mathbf{b})\mathbf{n} := -\tau.$$

It follows from here and from $\nabla_{\mathbf{t}} < \mathbf{n}, \mathbf{b} >= \frac{1}{2}\nabla_{\mathbf{t}}(\mathbf{n}\bar{\mathbf{b}} + \mathbf{b}\bar{\mathbf{n}}) = 0$ that $c_2 = \tau$. Thus we have proved the following theorem:

Theorem 5.1. *Let $\gamma(s)$ be a unit speed curve on one of three dimensional Riemannian space of constant curvature with $\kappa \neq 0$, then the following formulas hold:*

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n} \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b} \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n} \end{aligned} \tag{25}$$

6. Curves on Four Dimensional Space

In this last section of the work we construct Frenet-Serret frame on a curve on four dimensional Riemannian spaces of constant curvature $\mathbb{H}^4$ and $\mathbb{S}^4$ analogous to technique that we have followed for a curve on $\mathbb{R}^4$. We construct Frenet-Serret frame on a unit speed curve in four dimensional space $\tilde{M}$ with the help of the Frenet frame of a curve in three-dimensional submanifold M, determined by $x^1 = 0$, with respect to the metric from which we construct a generalized quaternion algebra. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ denote respectively that the tangent, normal and binormal quaternions of a curve in M parametrized by its arc-length $l = l(s)$ and let T denotes the tangent of the unit speed curve $\gamma = \gamma(s)$, given locally by the $(x^1(s), x^2(s), x^3(s), x^4(s))$:

$$\mathrm{T}^i = \frac{dx^i}{ds}, \ \mathrm{T}^i\mathrm{T}^i g_{ii} = 1.$$

The unit quaternions $\mathbf{t}\mathrm{T}$, $\mathbf{n}\mathrm{T}$, $\mathbf{b}\mathrm{T}$ in $T\tilde{M}$ are perpendicular to the quaternion T with respect to the metric (22). Indeed, let $\mathrm{T} = \mathrm{T}^i e_i$ for $i = 1, 2, 3, 4$ and $\mathbf{t} = t^i e_i$ for $i = 2, 3, 4$ be given in coordinate basis. We will show that

$$g_{ii}\mathrm{T}^i(\mathbf{t}\mathrm{T})^i = 0.$$

One can find by a direct calculation that

$$\begin{aligned}
\mathbf{t}\mathrm{T} &= -\left(\sum_{i=2}^{4} t^i\mathrm{T}^i g_{ii}\right)e_1 + \left(t^2\mathrm{T}^1 + (t^3\mathrm{T}^4 - t^4\mathrm{T}^3)\frac{\sqrt{g_{33}}\sqrt{g_{44}}}{\sqrt{g_{22}}}\right)e_2 \\
&+ \left(t^3\mathrm{T}^1 + (t^4\mathrm{T}^2 - t^2\mathrm{T}^4)\frac{\sqrt{g_{22}}\sqrt{g_{44}}}{\sqrt{g_{33}}}\right)e_3 \\
&+ \left(t^4\mathrm{T}^1 + (t^2\mathrm{T}^3 - t^3\mathrm{T}^2)\frac{\sqrt{g_{22}}\sqrt{g_{33}}}{\sqrt{g_{44}}}\right)e_4.
\end{aligned}$$

Therefore, it is easily find that

$$g_{ii}\mathrm{T}^i(\mathbf{t}\mathrm{T})^i = Re\left(\mathrm{T}(\overline{\mathbf{t}\mathrm{T}})\right) = \frac{1}{2}\left(\mathrm{T}(\overline{\mathbf{t}\mathrm{T}}) + (\mathbf{t}\mathrm{T})\overline{\mathrm{T}}\right) = 0,$$

That is, $\mathbf{t}\mathrm{T} \perp \mathrm{T}$. We see here that the computations in coordinates become very simplified as a single quaternion equation. Repeating similar calculations for $\mathbf{n}\mathrm{T}$, $\mathbf{b}\mathrm{T}$ results in $\mathbf{n}\mathrm{T} \perp \mathrm{T}$ and $\mathbf{b}\mathrm{T} \perp \mathrm{T}$, and hence the set $\{\mathrm{T}, \mathbf{t}\mathrm{T}, \ \mathbf{n}\mathrm{T}, \ \mathbf{b}\mathrm{T}\}$ of quaternions, forms an orthonormal basis for tangent space of the Riemannian manifold $\tilde{M}$ at a given point. Analogous to Euclidean case we define the quaternions

$$\begin{aligned}
\mathrm{T} &= \frac{d\gamma}{ds} \\
\mathrm{N} &= \frac{\tilde{\nabla}_{\mathrm{T}}}{|\tilde{\nabla}_{\mathrm{T}}|} = c_1\mathbf{t}\mathrm{T} + c_2\mathbf{n}\mathrm{T} + c_3\mathbf{b}\mathrm{T} \\
\mathrm{B}_1 &= \omega_1\mathbf{t}\mathrm{T} + \omega_2\mathbf{n}\mathrm{T} + \omega_3\mathbf{b}\mathrm{T} \\
\mathrm{B}_2 &= \mathrm{B}_1\overline{\mathrm{N}}\mathrm{T} = a_1\mathbf{t}\mathrm{T} + a_2\mathbf{n}\mathrm{T} + a_3\mathbf{b}\mathrm{T},
\end{aligned}$$

where the coefficients are represented by the purely imaginary quaternions

$$\begin{aligned}\hat{c} &= c_2e_2 + c_3e_3 + c_4e_4 \\ \hat{\omega} &= \omega_2e_2 + \omega_3e_3 + \omega_4e_4, \ \omega_i = \frac{\dot{\hat{c}}_i}{|\dot{\hat{c}}|} \\ \hat{a} &= \hat{c}\hat{\omega} = \sum_{i<j,k\neq i,j} (c_i\omega_j - \omega_i c_j)\frac{\sqrt{g_{ii}}\sqrt{g_{jj}}}{\sqrt{g_{kk}}}e_k,\end{aligned}$$

quaternion multiplications are implemented by the metric. It is easily seen that the set $\{\mathrm{T}, \mathrm{N}, \mathrm{B}_1, \mathrm{B}_2\}$ of quaternions forms an orthonormal basis. So their derivatives with respect to the arc-length parameter s represented by skew-symmetric matrix due to the orthonormality, that is,

$$\begin{aligned}\tilde{\nabla}_{\mathrm{T}}\mathrm{T} &= k_1\mathrm{N} \\ \tilde{\nabla}_{\mathrm{T}}\mathrm{N} &= -k_1\mathrm{T} + \lambda_1\mathrm{B}_1 + \lambda_2\mathrm{B}_2 \\ \tilde{\nabla}_{\mathrm{T}}\mathrm{B}_1 &= -\lambda_1\mathrm{N} + \mu_2\mathrm{B}_2 \\ \tilde{\nabla}_{\mathrm{T}}\mathrm{B}_2 &= -\lambda_2\mathrm{N} - \mu_2\mathrm{B}_1.\end{aligned} \tag{26}$$

Define quaternions $\hat{\mathrm{X}}_1$ and $\hat{\mathrm{X}}_2$ as

$$\begin{aligned}\hat{\mathrm{X}}_1 &= \frac{1}{k_2}\left(\lambda_1\mathrm{B}_1 + \lambda_2\mathrm{B}_2\right) \\ \hat{\mathrm{X}}_2 &= \frac{1}{k_2}\left(\lambda_2\mathrm{B}_1 - \lambda_1\mathrm{B}_2\right), \ k_2 = g(\hat{\mathrm{X}}_1, \hat{\mathrm{X}}_1) = g(\hat{\mathrm{X}}_2, \hat{\mathrm{X}}_2),\end{aligned}$$

here $k_2 = g(\hat{\mathrm{X}}_i, \hat{\mathrm{X}}_i) = \sqrt{\lambda_1^2 + \lambda_2^2}$. It will again be more convenient to use the quaternions $\mathrm{T}, \mathrm{N}, \hat{\mathrm{X}}_1, \hat{\mathrm{X}}_2$ as an orthonormal frame along curve γ. From the orthogonality relations one can easily see that $\tilde{\nabla}_{\mathrm{T}}\hat{\mathrm{X}}_i$, for $i = 1, 2$ has no tangential component and $\tilde{\nabla}_{\mathrm{T}}\hat{\mathrm{X}}_2$ has no normal component. If we denote the component of $\tilde{\nabla}_{\mathrm{T}}\hat{\mathrm{X}}_2$ in the direction of $\hat{\mathrm{X}}_1$ by k_3, then the system of equations (26) turns out to be the following system:

$$\begin{aligned}\tilde{\nabla}_{\mathrm{T}}\mathrm{T} &= k_1\mathrm{N} \\ \tilde{\nabla}_{\mathrm{T}}\mathrm{N} &= -k_1\mathrm{T} + k_2\hat{\mathrm{X}}_1 \\ \tilde{\nabla}_{\mathrm{T}}\hat{\mathrm{X}}_1 &= -k_2\mathrm{N} + k_3\hat{\mathrm{X}}_2 \\ \tilde{\nabla}_{\mathrm{T}}\hat{\mathrm{X}}_2 &= -k_3\hat{\mathrm{X}}_1,\end{aligned}$$

thus we have proved the following theorem.

Theorem 6.1. *Let $\gamma = \gamma(s)$ be a unit speed curve in four dimensional space of constant curvature with non-vanishing first curvature, then the following formulas hold:*

$$\tilde{\nabla}_{\mathrm{T}}\begin{pmatrix}\mathrm{T}\\ \mathrm{N}\\ \hat{\mathrm{X}}_1\\ \hat{\mathrm{X}}_2\end{pmatrix} = \begin{pmatrix}0 & k_1 & 0 & 0\\ -k_1 & 0 & k_2 & 0\\ 0 & -k_2 & 0 & k_3\\ 0 & 0 & -k_3 & 0\end{pmatrix}\begin{pmatrix}\mathrm{T}\\ \mathrm{N}\\ \hat{\mathrm{X}}_1\\ \hat{\mathrm{X}}_2\end{pmatrix}. \tag{27}$$

where k_1, k_2 and k_3 defined as above

7. Concluding Remarks and the Following Objectives

By means of the quaternion algebra, we able to construct Frenet-Serret frame on a curve in $\mathbb{R}^4$. This construction is achieved by a curve in three space in a sense that second and third curvature of the curve in $\mathbb{R}^4$ determined by κ and τ. To be able to apply this approach to curves on a three and four dimensional Riemannian spaces of constant curvature, we construct a generalized quaternion algebra on each fiber of tangent bundle in such a way that multiplication rule determined by the components of the metric tensor. Identical to the standard quaternion algebra, inner product of two quaternions p, q in the tangent bundle appears as the real part of multiplication $p\bar{q}$. Eventually, we construct an orthonormal frame on a curve on the model spaces algebraically via this generalized quaternion algebra. We see that with this construction, calculations depending on the metric tensor are rather simplified.

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AKDENİZ UNIVERSITY
DUMLUPINAR BOULEVARD
ANTALYA, TURKEY.
E-mail address: `tunabayrakdar@akdeniz.edu.tr`
`aaergin@akdeniz.edu.tr`