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FOLIATIONS DEFINED BY CLOSED DIFFERENTIAL 1-FORM

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Abstract. Subject of present paper is the geometry of foliation defined by closed 1-form on compact manifold of constant curvature. In the paper everywhere smoothness of manifolds, foliations and maps is a class C^∞.

1. Introduction

The foliation theory is a branch of the geometry which is intensively developed, has wide applications in various areas of mathematics - such, as the optimal control theory, the theory of dynamic systems. There are numerous researches on the foliation theory. The review of the last scientific works on the foliation theory and very big bibliography is presented in work of Ph. Tondeur [11].On the applications of foliation theory in control theory one can read in[8].

Definition 1.1. *Let (M, A) be a smooth manifold of dimension n, where A is a C^r- atlas, $r \geq 1$, $0 < k < n$. A family $F = \{L_\alpha : \alpha \in B\}$ of path-wise connected subsets of M is called k-dimensional C^r - foliation if it satisfies to the following three conditions:*
$F_I : \bigcup_{\alpha \in B} L_\alpha = M$,
$F_{II} :$ *for every* $\alpha, \beta \in B$ *if* $\alpha \neq \beta$, *then* $L_\alpha \cap L_\beta = \emptyset$,
$F_{III} :$ *For any point $p \in M$ there exists a local chart (local coordinate system) $(U, \varphi) \in A$, $p \in U$ so that if $U \bigcap L_\alpha \neq \emptyset$ for some $\alpha \in B$ the components of $\varphi(U \bigcap L_\alpha)$ are following subsets of parallel affine planes*
$(x_1, x_2, ..., x_n) \in \varphi(U) : x_{k+1} = c_{k+1}, x_{k+2} = c_{k+2}, ..., x_n = c_n$,
where numbers $c_{k+1}, c_{k+2}, ..., c_n$ are constant on components [11].

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The most simple examples of a foliation are given by integral curves of a vector field and by level surfaces of differentiable functions. If the vector field X without singular points is given on manifold M under the theorem of existence of the solution of the differential equation integral curves generate one-dimensional foliation on M .If we will assume that differentiable function $f : M \to R$ has no critical points, partition of M into level surfaces of function f is a n - 1- dimensional foliation (codimension one foliation).

From a geometrical point of view, the important classes of foliations are totally geodesic foliation and Riemannian foliation. Foliation on a Riemannian manifold is a totally geodesic if every geodesic tangent to the leaf of the foliation at one point lies in this leaf, i.e each leaf is a totally geodesic submanifold. The geometry of totally geodesic foliation studied in [1], [2], [4].

Foliation F is called a riemannian foliation if every geodesic orthogonal at some point to a leaf of foliation F remains orthogonal to leaves of F at all points. Riemannian foliation without singularities were first introduced and studied by Reinhart in [9]. This class of Foliation naturally arise in the study of bundles and level surfaces.

Let us consider a foliation F on a manifold M, generated by the closed differential form ω. In the paper [2] necessary and sufficient conditions were obtained under which there is a Riemannian metric g on the manifold M, in relation to which the foliation F will be totally geodesic.

We consider a Riemannian foliation F on a Riemannian manifold (M, g) generated by the closed differential form ω.We show that if (M, g) is a compact manifold of constant nonnegative curvature, then the foliation F is totally geodesic foliation with mutually isometric leaves.

Let M be a smooth n-dimensional smooth manifold, T_xM is a tangent space to M in x and TM is the tangent bundle. Suppose that ω is a differential 1-form on M. Consider subspace

$$V_x = \{X \in T_xM : \omega_x(X) = 0\}$$

of tangent space T_xM for all $x \in M$. So we have distribution $V : x \to V_x$ of dimension $n - 1$. By Frobenius' Theorem this distribution is complete integrable iff $\omega \wedge d\omega = 0$, where $d\omega$ is differential of ω. Closed forms always generate complete integrable distributions, as so as $d\omega = 0$. In particular, exact 1-forms are completely integrable, since they are differentials of smooth function: $\omega = df$.

Let us suppose that the 1-form ω is closed. In such case distribution $V : x \to V_x$ generate foliation of codimension one.

Let N is an universal cover of manifold M and mapping $p : N \to M$ is cover map . This map induces differential 1-form $p^*\omega$ on N. Note that the form $p^*\omega$ also is closed.

As $\pi_1(N) = 1$, then differential 1-form $p^*\omega$ is exact, too. Actually, let's consider the function $f(x) = \int\limits_{x_0}^{x} p^*\omega$, as N is simple connected and $p^*\omega$ is closed form, function $f(x)$ is correctly defined. We took curve integral along any curve γ from x_0 to x, and it does not depend on the curve. So it takes place $p^*\omega = df$.

So on manifold N 1-differential form $p^*\omega$ is complete integrable and integrable submanifolds are level surfaces of $f(x)$. But there is a "smaller" covering $\widetilde{p} : \widetilde{M} \to M$ where 1-form $\widetilde{p}^*\omega$ is closed.

Let us $A \subset \pi_1(M)$ is maximal subgroup of $\pi_1(M)$ such as for all $z \in A$ it takes place $\int\limits_z \omega = 0$. It is clear that A include commutant of $\pi_1(M)$; in particular, A is a normal divisor.

It is known [3], there is such cover map $\widetilde{p} : \widetilde{M} \to M$, that

$$\pi_1(\widetilde{M}) = \widetilde{p}*\pi_1(\widetilde{M}) = A \subset \pi_1(M).$$

In this case the function $f(x) = \int\limits_{x_0}^{x} \widetilde{p}^*(\omega)$ is defined correctly and it takes place $\widetilde{p}^*(\omega) = df$.

Let us remember the notion of vertical-horizontal homotopy.

Let M be a smooth Riemannian manifold,F is foliation on M. Let's denote with $L(x)$ a leaf of foliation Fcontains the point x, $F(x)$ is a tangent space to $L(x)$ at x and $H(x)$ is an orthogonal complement of $F(x)$ in T_xM. Now we have two subbundles of TM,

$$TF = \{F(x) : \; x \in M\},$$

$$H = \{H(x) : \; x \in M\}$$

such as

$$TM = TF \oplus H.$$

So distribution H is orthogonal to foliation F.

Piecewise smooth curve $\gamma : [0,\, 1] \to M$ is called horizontal if

$$\frac{d\gamma(t)}{dt} \in H(\gamma(t))$$

for every $t \in [0, 1]$. Piecewise smooth curve which lies in a leaf of foliation F is called as vertical.

Let $I = [0,\, 1]$, $\nu : I \to M$ is vertical curve, $h : I \to M$ is horizontal curve and $h(0) = \nu(0)$. Piecewise smooth mapping $P : I \times I \to M$ as $(t, s) \to P(t, s)$ is called vertical-horizontal homotopy if it is vertical curve for every $s \in I$, $s \to P(t,\, s)$ and a horizontal curve for every $t \in I$, where $P(t, 0) = \nu(t)$ for $t \in I$ and $P(0, s) = h(s)$ for $s \in I$.

Distribution H is called Ehresmann's connection for foliation F if for every pair of vertical and horizontal curves ν, $h : I \to M$ with $h(0) = \nu(0)$ there exists corresponding a vertical-horizontal homotopy P. If distribution H is Ehresmann's connection then for every pair ν, h vertical curve and horizontal curve there exists unique corresponding vertical-horizontal homotopy P [1].

2. MAIN RESULT

It is known that leaves of foliation F on compact manifold M given by closed form ω, are mutually diffeomorphic [6]. Also it is known there is a riemannian metric g on manifold M, such that foliation F is a riemannian foliation of (M, g) [10].

The following theorem shows that riemannian foliation F defined by closed 1-form on smooth connected compact riemannian manifold (M, g) of constant nonnegative curvature is total geodesic foliation with isometric leaves.

Theorem 2.1. *Let (M, g) be a smooth connected compact riemannian manifold of constant nonnegative curvature. If F is riemannian foliation defined by closed 1-form ω then F is a total geodesic foliation with isometric leaves.*

Proof. Let $\widetilde{p} : \widetilde{M} \to M$ be a riemannian covering, such that $\pi_1(\widetilde{M}) = \widetilde{p}_* \pi_1(\widetilde{M}) = A \subset \pi_1(M)$, where $A \subset \pi_1(M)$ such maximal subgroup of $\pi_1(M)$ that for every $z \in A$ it takes place:

$$\int\limits_z \omega = 0.$$

In this case, as we remind before the lift $\widetilde{p}^*(\omega)$ of ω is differential of function $f(x) = \int\limits_{x_0}^{x} p^*(\omega)$, the lift $\widetilde{F}$ of foliation F is a riemannian foliation. Besides, as we considered the riemannian covering, the cover space $\widetilde{M}$ is a complete manifold of constant nonnegative curvature. In order to prove the theorem it is enough to show that foliation $\widetilde{F}$ is totally geodesic with isometric leaves. Let $\widetilde{g}$ is a riemannian metric on $\widetilde{M}$, $\widetilde{\nabla}$ is Levi-Civita connection, $\widetilde{g}(U, W)$-inner product of vector fields U and W, *gradf* is gradient vector field of function f. Since $\widetilde{F}$ is a riemannian foliation, from the results of [8] follows for every tangent vector field $\widetilde{X}$ to $\widetilde{F}$ it holds

$$\widetilde{X}\widetilde{g}\,(gradf, gradf) = 0.$$

It means that length of vector field *gradf* is constant along leaves. Furthermore integral curves of the vector field *gradf* are geodesic [7].It means that

$$\widetilde{\nabla}_{\widetilde{Z}}\widetilde{Z} = 0,$$

where $\widetilde{Z} = \frac{gradf}{|gradf|}$ is unite gradient vector field. Let's show the every leaf of foliation $\widetilde{F}$ is totally geodesic submanifold. Let $\widetilde{L}_0$ be some leaf of $\widetilde{F}$, $v : [0,\, l_0] \to \widetilde{L}_0$ is the shortest curve in $\widetilde{L}_0$, parameterized with arc length. Here L_0 is considered as riemannian manifold with induced riemannian metric from $\widetilde{M}$. Let $\gamma(t, s)$ be a gradient curve of function f starting from the point $v(t)$ at $s = 0$: $\gamma(t, 0) = v(t)$. As function f has no critical points and $\widetilde{M}$ is complete manifold $\gamma(t, s)$ is defined for all $s \in R^1$[2]. Following Ph. Tondeurs' Theorem flow of field Z translates the level surfaces into level surfaces ([11],p.107, Th 8.9). That's why if $\gamma(t, s)$ is a gradient curve starting from the point $v(t)$ at $s = 0$ and parameterized with arc length then curve $t \to \gamma(t, s)$ lies on the same level surface.It means that mapping

$$\gamma{:}[0,l_0] \times (\,-\,\infty, +\,\infty) \to M_1$$

is a vertical-horizontal homotype. Consider two dimensionally surface as $\Phi = \gamma(t, s)$:$t \in [0,\, l_0]$, $s \in (-\infty, +\infty)$. We'll show that gradient curves $\gamma_t : s \to \gamma(t, s)$ are straight lines on Φ.The surface Φ is considered with the restriction of riemannian metric $\widetilde{g}$. Restriction of riemannian metric $\widetilde{g}$ on

Φ gives the metric $E(t,\, s)dt^2 + ds^2$, where $E(t,\, s) = |X(t,\, s)|^2$, $|X(t,\, s)|$ is the length of tangent vector $X(t,s)$ of curve $t \to \gamma(t,s)$ at $p = \gamma(t,\, s)$. Let,s note that each geodesic $\gamma_t : s \to \gamma(t,s)$ intersects every level surface only one time and each geodesic is a straight line. If $A = \gamma(t,s_1)$, $B = \gamma(t,s_2)$ and $s_2 > s_1$, then length of AB is equal to $s_2 - s_1$. Thus segment AB of geodesic $\gamma_t : s \to \gamma(t,s)$ is a shortest curve between points A and B. Let N be two dimensionally plane in tangent space $T_{\widetilde{x}}\widetilde{M}$ so as $\mathrm{N} = \mathrm{EXP}_{\mathrm{p}}^{-1}(\Phi)$, where $\mathrm{EXP_p}$ is exponential mapping at the point p, where $\mathrm{p} \in \Phi$. The Gaussian curvature of Φ at p is equal to the two dimensional sectional curvature of manifold $\widetilde{M}$ constructed by two dimensional direction N. By the conditions of the theorem M is the manifold of constant nonnegative curvature. Since the cover mapping $\widetilde{p} : \widetilde{M} \to M$ is Riemannian mapping $\widetilde{M}$ is the manifold of constant nonnegative curvature too [3]. So, the Gaussian curvature of surface Φ is constant. On the other hand as surface Φ contains the straight line so its curvature is zero [3]. It follows the all gradient curves of function f are parallel straight lines. That's why if the shortest curve on Φ orthogonal to the one gradient line then it's orthogonal to the all gradient lines as they are parallel ([3], Lemma 8,p.330). Now we denote by v_s the shortest curve in Φ from the point $\gamma(0,\, s)$ of gradient line $s \to \gamma(0,\, s)$ to gradient line $\gamma(l_0, s)$. It is the vertical curve and it's length is equal to the length of $v : [0,\, l_0] \to L_0$. Hence by uniqueness of the vertical-horizontal homotopy we have $v_s(t) = \gamma(t,s)$ for $t \in [0, l_0]$. It is easy to check that $[\widetilde{X}, \widetilde{Z}] = 0$, where$[\widetilde{X}, \widetilde{Z}]$ is the Lie bracket of vector fields $\widetilde{X}, \widetilde{Z}$. It follows that $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} = \widetilde{\nabla}_{\widetilde{Z}}\widetilde{X}$. Since $\widetilde{g}(\widetilde{X}, \widetilde{X}) = 1$, using the equality

$$W\widetilde{g}(U,V) = \widetilde{g}(\widetilde{\nabla}_W U, V) + \widetilde{g}(U, \widetilde{\nabla}_W V)$$

we have that $\widetilde{g}(\widetilde{\nabla}_{\widetilde{Z}}\widetilde{X}, \widetilde{X}) = 0$. On the other hand,from the equality $\widetilde{g}(\widetilde{Z}, \widetilde{Z}) = 1$ it follows that $\widetilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z}, \widetilde{Z}) = 0$. According to the equality $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} = \widetilde{\nabla}_{\widetilde{Z}}\widetilde{X}$ we get that $\widetilde{\nabla}_{\widetilde{Z}}\widetilde{X} = 0$. Now from equality $\widetilde{g}(\widetilde{X}, \widetilde{Z}) = 0$ we get $\widetilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{X}, \widetilde{Z}) = 0$. Differentiating $\widetilde{g}(\widetilde{X}, \widetilde{X}) = 1$ in the direction of $\widetilde{X}$ gives equality $\widetilde{g}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{X}, \widetilde{X}) = 0$. Thus, $\widetilde{\nabla}_{\widetilde{X}}\widetilde{X} = 0$ and $\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} = 0$ at all points of the surface. This implies that the curves $t \to \gamma(t,\, s)$ for each s are geodesic length of l_0. So the every geodesic on $\widetilde{M}$ is tangent to the leaf of foliation $\widetilde{F}$ stay on this leaf. Besides, as $\nabla_{\widetilde{Z}}\widetilde{X} = 0$, the flow of $\widetilde{Z}$ sends the geodesics of the leaf to the geodesics the same length. It follows, the foliation $\widetilde{F}$ is totally geodesic foliation with isometric leaves. Now let show that, by using the fact that $\widetilde{p} : \widetilde{M} \to M$ is Riemannian covering we took the statement of the theorem. Let v be the geodesic on manifold M, tangent to the leaf L of the foliation F at the point $x \in M$. Consider the lift $\widetilde{v}$ of curve v starting at the point $\widetilde{x} \in \widetilde{p}^{-1}(x)$. Then the lift $\widetilde{v}$ is also the geodesic as the covering $\widetilde{p} : \widetilde{M} \to M$ is isometric at each point and $\widetilde{v}$ lies on the leaf of the foliation $\widetilde{F}$,passing through the point $\widetilde{x}$. Therefore, the geodesic $v = \widetilde{p}(\widetilde{v})$lies on the leaf L. Now let us consider a vector field $Z = d\widetilde{p}(\widetilde{Z})$, where $d\widetilde{p}$ is the differential of mapping $\widetilde{p}$. Vector field Z is orthogonal to the foliation F and $\omega(Z) = 1$. Since the foliation $F$is a Riemannian and manifold M is compact the distribution generated by the vector field Z is an Ehresmann connection for foliation F [1]. Therefore, for the shortest vertical curve $v(t)$ and horizontal curve $\gamma(s)$ (integral curve of the vector field Z) with common initial point there is vertical-horizontal homotopy $P(t,s)$. The lift $\widetilde{P}(t,s)$ of the vertical-horizontal homotopy $P(t,s)$ on $\widetilde{M}$ is a vertical-horizontal homotopy for the foliation $\widetilde{F}$. Then the vector field $\widetilde{Z}$ is the tangent vector of the curve $s \to \widetilde{P}(t,s)$, the tangent vector field of curves $t \to \widetilde{P}(t,s)$ we denote by $\widetilde{X}$. As shown above,it holds $\widetilde{\nabla}_{\widetilde{Z}}\widetilde{X} = 0$. Given that, the covering $\widetilde{p} : \widetilde{M} \to M$ is isometric at each point we find that, $\nabla_Z X = 0$, where the vector field X is a tangent vector field of the curve $t \to P(t,s)$, where ∇ is Levi-Civita connection on M defined by Riemannian metric g. This means that the flow of the vector field Z maps a leaf of F to a leaf of this foliation isometrically. □

Corrolary 2.1. *Under the conditions of the theorem M is the manifold of constant zero curvature.*

Corrolary 2.2. *Under the conditions of the theorem the fundamental group of each leaf of foliation F isomorphic to the group $A \subset \pi_1(M)$.*

Actually restriction of covering map $\widetilde{p} : \widetilde{M} \to M$ to the leaf $\widetilde{L}$ of foliation $\widetilde{F}$ is the covering map $\widetilde{p} : \widetilde{L} \to L$, where L is the leaf of F. The leaf $\widetilde{L}$ is homeomorphic to $L = \widetilde{p}(\widetilde{L})$. The homeomorphism defined as follows: let $x_0 \in L$ and $v : [0,1] \to L$ is a vertical path in L from x_0 to a point $x \in L$. If $\widetilde{v} : [0,1] \to \widetilde{L}$ is lift of v starting at $\widetilde{x}_0 \in \widetilde{p}^{-1}(x_0)$,then we map a point x to the point $\widetilde{v}(1)$. This map does not depend from the path v since closed paths lift to closed paths.

As the manifold $\widetilde{M}$ diffeomorphic to the direct product $\widetilde{L} \times R^1$, where $\widetilde{L}$ is any level surface of function f, fundamental group $\pi_1(\widetilde{L})$ is isomorphic to $\pi_1(\widetilde{M})$[1]. Here follows the statement of Corollary 2.

Example. Consider the differential form $\omega = a_1dx_1 + a_2dx_2 + ... + a_ndx_n$, where $a_1, a_2, ..., a_n$ are real numbers. This form induce the differential form on n-dimensional torus $T^n = R^n/Z^n$, where Z is the set of integers. Equation $\omega = 0$ defines the foliation F codimension one on T^n. If rang of numbers $\{a_1, a_2, \cdots a_n\}$ over the set of rational numbers is equal to k, then group $A \subset \pi_1(M)$ is the $Z + Z + \cdots + Z$, here $n - k$ summands.

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