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THE EQUATIONS OF THE CONICS IN OBLIQUE COORDINATE SYSTEMS

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Abstract. In this paper we deduct the equation of hyperbola relative to its asymptotes. We will show that the equations of the conic sections (conics), in the frame of reference based on their conjugate directions suits formally the canonical equations, which are in fact one particular case of the former. Since the canonical coordinate system of the conics represents perpendicular conjugated directions.

1. Introduction

Every college student has learned how to transform the coordinates of a vector or a matrix of a linear transformation/quadratic form under a change of bases. These rules (for the simplicity: *SMS* rules in the sequel) have traditionally been taught in undergraduate courses on linear algebra. Most textbooks or exercise books on linear algebra do not, however, include any geometric application of these basic operations. By applications, I mean the use of *SMS* rules to solve problems applied in other branches of mathematics, mainly in elementary geometry. Note, Alexandrov's book [1] belongs to the bracing exceptions in some respects.

On the other hand, in most curricula geometry loses its ground. Often college lecturers have no explicit vision of what kind of geometry to teach to the students (see review [2]). I am going to point out in this paper that in a linear algebra problem solving course we can teach elementary geometry without any extra effort.

Integrating geometric applications in the teaching process of linear algebra has several advantages:
1) geometric applications strengthen the linear algebra skills,
2) a simple geometric idea helps in motivation of abstract reasoning,
3) the obtained geometric results are interesting as they are.

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The SMS rules. In all cases the "*old*" base is B, the "*new*" base is B', the matrix of change $B \to B'$ is S.

1. *Change of coordinates.*

The old coordinates of a vector are arranged in the column vector X, while the new coordinates in X':

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ X' = \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix}, \ X = SX' \Leftrightarrow X' = S^{-1}X.$$

2. *Change of the matrix of a quadratic form.*

The old matrix is M, the new matrix is M': $M' = S^T MS$.

2. The equations of a hyperbola relative to its asymptotes

In section 2 and 3 we consider only central conic sections, and we suppose that the origin of the used affine coordinates systems coincide with the center of this conics.

I think that the following exercise is a "must be" (at least for future teachers of mathematics) because in junior level (age=16), under the term hyperbola they often do not understand a conic section, but rather the graph of the function $x \to c/x$, $(c \neq 0, x \neq 0)$.

I. *Determine the equation of a hyperbola relative to the base, consisting of normed vectors which are parallel to the asymptotes of the hyperbola.*

First solution-based on the first SMS rule. To begin with the canonical setting, we conclude that the axes of the coordinate are the symmetry axes of hyperbola, and the origin is the center of symmetry. The canonical equation of the hyperbola is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Leftrightarrow (x, y) \begin{pmatrix} 1/a^2 & 0 \\ 0 & -1/b^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1. \tag{2.1}$$

Moreover $c = \sqrt{a^2 + b^2}$. Let $E = (\overrightarrow{e_1}, \overrightarrow{e_2})$ is the canonical base of the plan and $A = (\overrightarrow{a_1}, \overrightarrow{a_1})$ is the "asymptotic base" described in the text of the exercise. Denote by 2ϕ the acute angle between the asymptotes (see Figure 1). Then

$$cos\phi = \frac{a}{c}, \ \sin\phi = \frac{b}{c}. \tag{2.2}$$

Combining the new base vectors from the canonical base vectors we get:

$$\overrightarrow{a_1} = \cos\phi \cdot \overrightarrow{e_1} - \sin\phi \cdot \overrightarrow{e_2},$$
$$\overrightarrow{a_2} = \cos\phi \cdot \overrightarrow{e_1} + \sin\phi \cdot \overrightarrow{e_2}.$$

Therefore the matrix of the change $E \to A$ is:

$$S = \begin{pmatrix} \cos\phi & \cos\phi \\ -\sin\phi & \sin\phi \end{pmatrix} = \begin{pmatrix} \frac{a}{c} & \frac{a}{c} \\ -\frac{b}{c} & \frac{b}{c} \end{pmatrix} = \frac{1}{c} \begin{pmatrix} a & a \\ -b & b \end{pmatrix}.$$

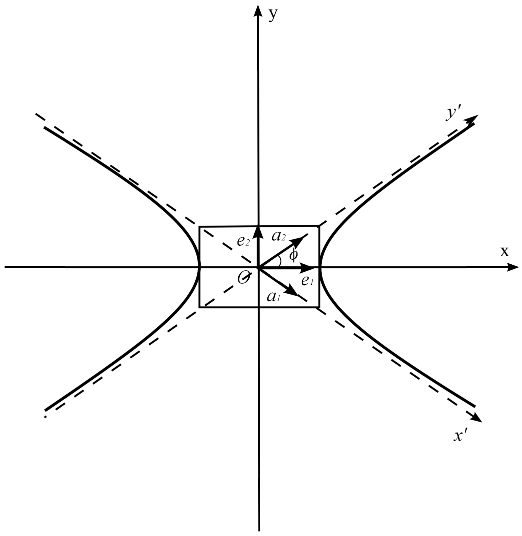

Figure 1

We note that $\det S = \sin 2\phi \neq 0$. By the first SMS rule the change of coordinates is

$$x = \cos\phi \cdot x' + \cos\phi \cdot y' = \frac{a}{c}\,(x' + y'),$$
$$y = -\sin\phi \cdot x' + \sin\phi \cdot y' = \frac{b}{c}\,(-x' + y').$$

It follows that the equation (2.1) of the hyperbola is equivalent to

$$\frac{\cos^2\phi(x'+y')^2}{a^2} - \frac{\sin^2\phi(x'-y')^2}{b^2} = 1. \tag{2.3}$$

Applying (2.2) we get

$$(x'+y')^2 - (x'-y')^2 = c^2 \Leftrightarrow 4x'y' = c^2. \tag{2.4}$$

Second solution-based on the second SMS rule. One can solve the problem directly from the second SMS rule: the new equation of the hyperbola is

$$(x', y')(S^T M S)\begin{pmatrix} x' \\ y' \end{pmatrix} = 1,$$

i.e.

$$(x', y')\frac{1}{c}\begin{pmatrix} a & -b \\ a & b \end{pmatrix}\begin{pmatrix} 1/a^2 & 0 \\ 0 & -1/b^2 \end{pmatrix}\frac{1}{c}\begin{pmatrix} a & a \\ -b & b \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1.$$

Executing the matrix multiplication, we get

$$(x', y')\begin{pmatrix} 0 & 2/c^2 \\ 2/c^2 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = 1.$$

The above line gives $4x'y' = c^2$ again.

After solving this easy exercise we may show students some geometrical applications. The following applications points out, how can the appropriate choice of coordinate-system simplify solutions.

II. *Fix a point P on the hyperbola with canonical equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and draw two lines through this point, each are parallel to one of the asymptotes. These lines with the asymptotes frame a parallelogram. Prove that the area of the parallelogram is ab/2, i.e. independent of the choice of the initial point.

Solution. We use the equation of the hyperbola relative to its asymptotes (see Figure 2):

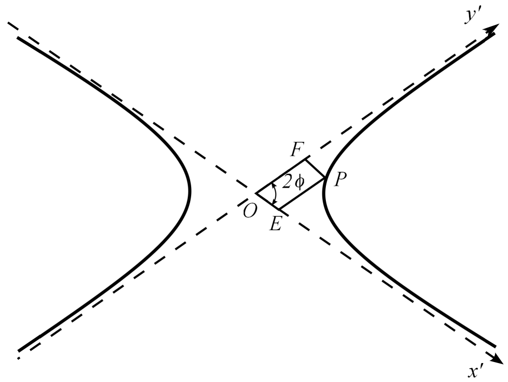

Figure 2

$$T_{OEPF} = |x'| \cdot |y'| \sin 2\phi = \frac{c^2}{4} \cdot \frac{2ab}{c^2} = \frac{ab}{2}\,.$$

There are several known problems where the application of the asymptote base proves to be effective.

III. *A line d cuts the hyperbola in two points. Prove that the segments between the asymptotes and hyperbola branches are congruent.*

Solution. Let S, T the two points of intersection of the line d with the hyperbola and M, N with the asymptotes. If $S = (u, v)$, $T = (\alpha, \beta)$ and $S \neq T$, then $uv = \dfrac{c^2}{4} = \alpha\beta$ and $u \neq \alpha$, $v \neq \beta$. The equation of the line d is $y - \beta = m(x - \alpha)$, where $m = \dfrac{v-\beta}{u-\alpha} \Leftrightarrow \beta - m\alpha = v - mu$. The coordinates of the points M and N are $M = \left(\alpha - \dfrac{\beta}{m}, 0\right)$, $N = (0, v - um)$. The length of the segments SN and TM:

$$SN^2 = u^2 + u^2m^2 + 2u^2m\cos 2\phi = u^2(1 + m^2 + 2m\cos 2\phi),$$
$$TM^2 = \frac{\beta^2}{m^2} + \beta^2 + 2\frac{\beta^2}{m}\cos 2\phi = \frac{\beta^2}{m^2}\,(1 + m^2 + 2m\cos 2\phi).$$

Since

$$SN^2 = TM^2 \Leftrightarrow u^2m^2 = \beta^2 \Leftrightarrow u^2(v-\beta)^2$$
$$= \beta^2(u-\alpha)^2 \Leftrightarrow \left(\frac{c^2}{4} - u\beta\right)^2 = \left(\beta u - \frac{c^2}{4}\right)^2$$

and the latest equality is true, for this $SN = TM$.

3. The equation of an ellipse or a hyperbola relative to a pair of conjugate directions

IV. *Determine the equation of an ellipse or a hyperbola relative to the base, consisting of normed vectors which are parallel to a pair of conjugate diameters.*

We set the same canonical position of the conic section Q as in the solution of exercise I, therefore the canonical equation of the ellipse and hyperbola is:

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \Leftrightarrow (x,y)\begin{pmatrix} 1/a^2 & 0 \\ 0 & \pm 1/b^2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1.$$

Let M be the matrix

$$\begin{pmatrix} 1/a^2 & 0 \\ 0 & \pm 1/b^2 \end{pmatrix}.$$

First solution. In this solution we derive the matrix S explicitly.

The case of the ellipse (see Figure 3).

If $V = (a\cos t, b\sin t)^T \in Q$, $(t \in [0, 2\pi))$ then the polar angle of the vector $W \in Q$ conjugated to V is $t + \pi/2$. Then $W = (-a\sin t, b\cos t)^T$, and S has the form

$$\begin{pmatrix} a\cos t & -a\sin t \\ b\sin t & b\cos t \end{pmatrix} (\det S = ab \neq 0).$$

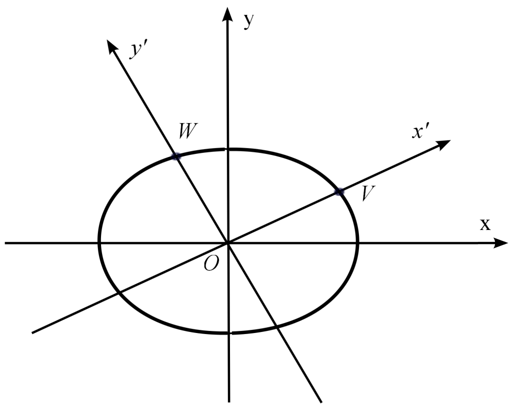

Figure 3

Then the transformed matrix of Q is the identity matrix:

$$S^TMS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The case of the hyperbola (see Figure 4). If a point V on the hyperbola Q is specified by the polar angle t, then

$$V = \left(\frac{a}{\cos t}, b\tan t\right)^T, \ t \in [0, 2\pi] - \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}.$$

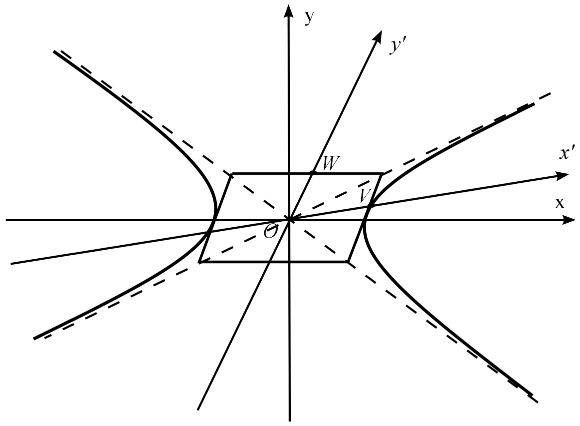

Figure 4

The conjugate point to V on the conjugated hyperbola $\overline{Q}$ is

$$W = \left(a\tan t, \frac{b}{\cos t}\right)^T.$$

In this case

$$S = \frac{1}{\cos t}\begin{pmatrix} a & a\sin t \\ b\sin t & b \end{pmatrix} \ (\det S = ab \neq 0).$$

Then the transformed matrix of Q by the second SMS rule will be written as following:

$$S^T MS = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Amazingly, the equation of the ellipse is $x^2 + y^2 = 1$, while the equation of the hyperbola is $x^2 - y^2 = 1$ in this coordinate system (we have not distinguished the new coordinates from the old ones).

The base vectors are not normed so far. Denote by p the norm of V and by q the norm of W. In the base $(V/p, W/q)$ we get the following equations:

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1 \text{ (ellipse)}, \quad \frac{x^2}{p^2} - \frac{y^2}{q^2} = 1 \text{ (hyperbola)}.$$

Second solution. We may point out that one does not need to determine the explicit form of the matrix S. Therefore, let the column vectors V and W be conjugated with respect to the conic section Q:

$$V^T MW = 0,$$

moreover, suppose that

$$V^T MV = 1, \ W^T MW = \pm 1.$$

The new base is (V, W). We have to apply the second SMS rule:

$$S^T M S = \begin{pmatrix} V^T M V & V^T M W \\ W^T M V & W^T M W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

From this point we follow the first solution.

4. The equation of a parabola relative to a pair of conjugate directions

Considering the above exercises, the analogue question for the parabola naturally arises. Although, the parabola is not a central conic section, therefore instead of a simple base-change we need an affine coordinate transformation.

V. *Determine the equation of a parabola relative to an affine coordinate system* $(P; e_1', e_2')$*, where* P *is a point of the parabola,* e_1' *is parallel to the axis of the parabola,* e_2' *is parallel to the tangent line to the parabola in the point* P*, moreover the base vectors are normed, i. e.* $\|e_1'\| = \|e_2'\| = 1$ (see Figure 5).

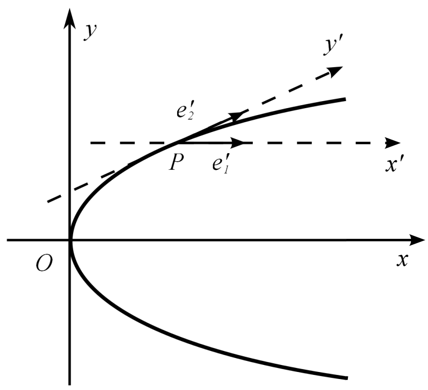

Figure 5

Solution. The canonical equation of the parabola is

$$y^2 = 2px \tag{4.1}$$

where $p > 0$ is the parameter of the parabola. Let $P = (x_0, y_0)$ and $y_0 > 0$. The normed tangent vector to the parabola (in direction of the velocity) in the point P is

$$\left(\frac{y_0}{\sqrt{p^2 + y_0^2}}, \frac{p}{\sqrt{p^2 + y_0^2}} \right).$$

Consequently, the matrix of the base-change $(e_1, e_2) \to (e_1', e_2')$ is

$$S = \begin{pmatrix} 1 & \dfrac{y_0}{\sqrt{p^2 + y_0^2}} \\ 0 & \dfrac{p}{\sqrt{p^2 + y_0^2}} \end{pmatrix}.$$

Therefore, the affine coordinate transformation is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{y_0}{\sqrt{p^2+y_0^2}} \\ 0 & \dfrac{p}{\sqrt{p^2+y_0^2}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{4.2}$$

We substitute (4.2) into the canonical equation (4.1):

$$\left(\frac{p}{\sqrt{p^2+y_0^2}}\, y' + y_0 \right)^2 = 2p \left(x' + \frac{y_0}{\sqrt{p^2+y_0^2}}\, y' + x_0 \right). \tag{4.3}$$

(4.3) is equivalent to

$$y'^2 = 2\frac{p^2+y_0^2}{p}\, x' = 2p'x' \tag{4.4}$$

where $p'/2$ is the distance of the point P from the directrix of the parabola.

The structure of (4.4) is the same as the canonical equation (4.1). What is the geometrical meaning of the parameter p'?

$$p' = \frac{p^2+y_0^2}{p} = \frac{p^2+2px_0}{p} = 2\left(\frac{p}{2} + x_0\right),$$

thus $p'/2$ is the distance of the point P from the directrix of the parabola.

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