INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 2 (2013), No. 2, 56 - 67

NEW SIGNS OF ISOSCELES TRIANGLES

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Abstract. In this paper we prove new signs of isosceles triangles by two equal one-type elements and investigate cases when a non-isosceles triangle may have two equal one-type elements. Our results generalize and extend well-known Lehmus-Steiner theorem and Botema's result about external bisectors.

1. Introduction

The signs of isosceles triangles by two equal medians, bisectors or symmedians are well-known [1]. The sign of isosceles triangles by two equal bisectors is known as the Steiner-Lehmus theorem. The analogous statement is false for external bisectors, since one can construct a non-isosceles triangle with two equal external bisectors and with angles 12°, 36°, 132° (this triangle is known as Botema's triangle).

Let ABC be an arbitrary triangle with sides $AB = c$, $AC = b$, $CB = a$.

Definition 1.1. *Let n be a nonzero real number. An internal (external) n-line from the vertex A is a segment AA_1 (AA_2) such that the point A_1 (A_2) divides internally (externally) the opposite side BC in proportion of the n^{th} powers of the adjacent sides, i.e. $BA_1 : CA_1 = c^n : b^n$, $BA_2 : CA_2 = c^n : b^n$. Denote the lengths of the segments AA_1 and AA_2 by $l^{int}(a, n)$ and $l^{ext}(a, n)$ respectively. The internal (external) $(-n)$-line is called the internal (external) n-antiline.*

Although the external n-line AA_2 is not well-defined for the case $b = c$, we won't exclude this case and will set $l^{ext}(a, n) = \infty$ for $b = c$. The definition of the internal n-line can be extended for all real n : in the case $n = 0$ the internal n-line is the median.

It's easy to see that the internal (external) 1-, 2-lines are the internal (external) bisectors and symmedians respectively. Also the internal (external) 1-, 2-antilines are the internal (external) antibisectors and antisymmedians respectively [2, p. 136].

Keywords and phrases: Isosceles triangles, Internal n-lines, External n-lines, $(n + 1)$-sector lines
(2010)Mathematics Subject Classification: 51M04, 51M25, 51M30
Received: 20.07.2013. In revised form: 13.08.2013. Accepted: 10.09.2013.

By Stewart's theorem [3, Ch. 1], one can obtain the following formulae for the lengths of the internal and external n-lines

$$l^{int}(a,n) = \sqrt{\frac{b^2c^n + c^2b^n}{b^n + c^n} - \frac{a^2b^nc^n}{(b^n + c^n)^2}}, \tag{1}$$

$$l^{ext}(a,n) = \sqrt{\frac{c^2b^n - c^nb^2}{b^n - c^n} + \frac{a^2b^nc^n}{(b^n - c^n)^2}}. \tag{2}$$

Definition 1.2. *Let n be an arbitrary positive real number. We consider two lines AA_3, CC_3 such that $\frac{\angle CAA_3}{\angle BAA_3} = \frac{\angle ACC_3}{\angle BCC_3} = n$ and the points A_3, C_3 belong to the sides BC and AB respectively (Figure 1). The segments AA_3 and CC_3 are called the $(n+1)$-sector lines from the vertices A and C.*

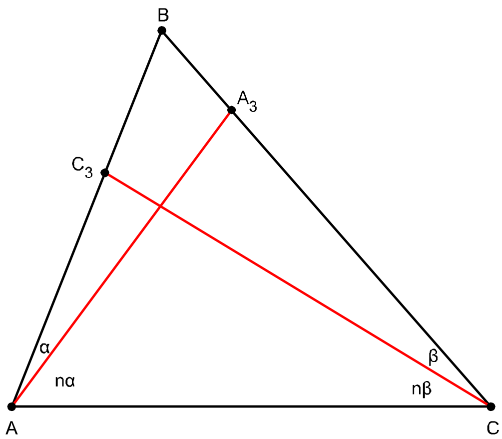

Figure 1. (*n+1*)-sector lines

The main part of the paper is divided on three subsections.

In the first part we prove that for any real n the equalities $l^{int}(a,n) = l^{int}(c,n)$, $l^{int}(a,-n) = l^{int}(c,-n)$ imply the equality of the sides a and c. Further, we show that for all quite small real n the equality $l^{int}(a,n) = l^{int}(c,n)$ implies the equality of the sides a and c generalizing well-known signs of isosceles triangles by two equal medians, bisectors or symmedians. Nevertheless, for all quite large n (positive or negative) one can find a non-isosceles triangle with two equal internal n-lines.

In the second part we show that for any real $n \neq 0$ the sides a and c are equal if and only if $l^{ext}(a,n) = l^{ext}(c,n)$, $l^{ext}(a,-n) = l^{ext}(c,-n)$. Further, we prove that for any real $n \neq 0$ there exists a non-isosceles triangle with two equal external n-lines.

In the third part for any real $n > 0$ there is proved the sign of isosceles triangles by two equal $(n+1)$-sector lines, this sign gives the other generalization of the Steiner-Lehmus theorem.

2. MAIN RESULTS

2.1. INTERNAL n-LINES, $n \in \mathbb{R}$.

Theorem 2.1. *Let n be a real number. Then a triangle ABC is isosceles ($a = c$) iff there holds $l^{int}(a, n) = l^{int}(c, n)$, $l^{int}(a, -n) = l^{int}(c, -n)$.*

Proof. The necessity follows from equality (1). Let's proof the sufficiency. If $n = 0$, then the result is well-known. Let $n \neq 0$. Suppose that $a > c$. Without loss of generality, we may assume that $n > 0$ and the length of the side AC is equal to 1.

STEP 1. Let's prove that $a > 1$ and $c < 1$.

From the equalities $l^{int}(a, n) = l^{int}(c, n)$, $l^{int}(a, -n) = l^{int}(c, -n)$, it follows that:

$$\frac{c^n + c^2}{c^n + 1} - \frac{a^2 c^n}{(c^n + 1)^2} = \frac{a^n + a^2}{a^n + 1} - \frac{c^2 a^n}{(a^n + 1)^2}\ , \tag{3}$$

$$\frac{c^{n+2} + 1}{c^n + 1} - \frac{a^2 c^n}{(c^n + 1)^2} = \frac{a^{n+2} + 1}{a^n + 1} - \frac{c^2 a^n}{(a^n + 1)^2}. \tag{4}$$

Equalities (3) and (4) imply the relation

$$\frac{(c^n - 1)(c^2 - 1)}{c^n + 1} = \frac{(a^n - 1)(a^2 - 1)}{a^n + 1}.$$

Consider the function

$$f(x) = \frac{(x^n - 1)(x^2 - 1)}{x^n + 1},$$

defined for $x > 0$. We'll find its stationary points. We have

$$f'(x) = \frac{2x(x^{2n} - 1) + 2nx^{n-1}(x^2 - 1)}{(x^n + 1)^2}.$$

For $x > 0$ there is only one stationary point $x_0 = 1$. Since $f(a) = f(c)$, so the derivative $f'(x)$ vanishes at some point from the interval (c, a), and hence, $c < 1 < a$.

STEP 2. Let's prove that $ac < 1$.

We'll use the following notation: $\delta = a^n, \beta = c^n, k = \frac{2}{n}$.

The equality $l^{int}(a, n) = l^{int}(c, n)$ may be rewritten in the form:

$$\begin{aligned} &\delta^k(\beta(\delta + 1)^2 + (\beta + 1)^2(\delta + 1)) - \beta^k(\delta(\beta + 1)^2 + (\beta + 1)(\delta + 1)^2) + \\ &+(\beta + 1)(\delta + 1)^2 - (\beta + 1)^2(\delta + 1) \ = \ 0. \end{aligned} \tag{5}$$

We denote the left-hand side of equality (5) by $g(\delta, \beta, k)$. Since $a > 1$, $c < 1$ and $n > 0$, so $\delta > 1$ and $\beta < 1$. It is obviously that for any fixed $\delta > 1$, $\beta < 1$ the function $g(\delta, \beta, k)$ is strictly increasing at $k > 0$.

Consequently, $g(\delta, \beta, 0) = (\delta - \beta)(\beta\delta - 1) < 0$. So, $\beta\delta < 1$, and therefore $ac < 1$.

STEP 3. Let's prove that $c < \sqrt[4]{1/2}$.

Let AA_1 and CC_1 be the internal n-lines, AA_2 and CC_2 be the internal n-antilines. Since $a > 1 > c$, $n > 0$, so the order of the points A_1, A_2, C_1, C_2 is the same as on Figure 2.

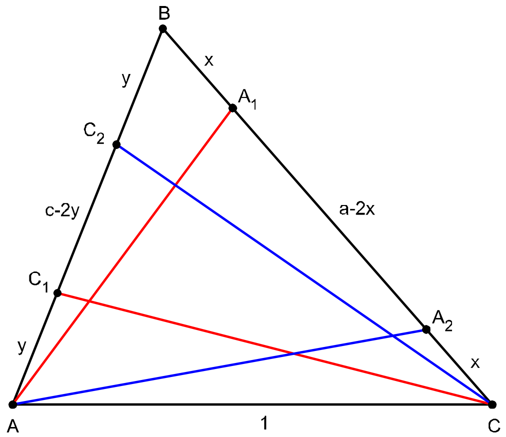

Figure 2. Internal n-lines and internal n-antilines

Denote $BA_1 = CA_2 = x$, $AC_1 = BC_2 = y$, then there hold the inequalities $0 < x < a/2, 0 < y < c/2$. Applying the cosine theorem for triangles ABA_1, CBC_1, ABA_2, CBC_2 and using the equalities $l^{int}(a,n) = l^{int}(c,n)$, $l^{int}(a,-n) = l^{int}(c,-n)$, we obtain the system:

$$(6) \qquad \begin{cases} c^2 + x^2 - 2cx\cos(B) &= a^2 + (c-y)^2 - 2a(c-y)\cos(B), \\ c^2 + (a-x)^2 - 2c(a-x)\cos(B) &= a^2 + y^2 - 2ay\cos(B). \end{cases}$$

It follows from system (6) that there hold the equalities

$$(7) \qquad c^2 - 2ax + 2x^2 = a^2 - 2cy + 2y^2,$$

$$(8) \qquad x\frac{1-c^2}{a} = y\frac{a^2-1}{c} + \frac{2-a^2-c^2}{2}.$$

Let $p = \frac{a}{2} - x, q = \frac{c}{2} - y$, then equations (7) and (8) are equivalent to the system:

$$(9) \qquad \begin{cases} c(1-c^2)p = a(a^2-1)q, \\ p^2 - \frac{3a^2}{4} = q^2 - \frac{3c^2}{4}. \end{cases}$$

From system (9) we get the equations for p and q:

$$(10) \qquad p^2\frac{a^4 + a^2c^2 + c^4 + 1 - 2a^2 - 2c^2}{a^2(a^2-1)^2} = \frac{3}{4},$$

$$(11) \qquad q^2\frac{a^4 + a^2c^2 + c^4 + 1 - 2a^2 - 2c^2}{c^2(1-c^2)^2} = \frac{3}{4}.$$

Since $q < c/2$, so equality (11) implies the inequality:

(12) $$a^4 + a^2c^2 + c^4 + 1 - 2a^2 - 2c^2 > 3 - 6c^2 + 3c^4.$$

Rewrite inequality (12):

(13) $$2c^4 + 2a^2 + 2 < a^4 + a^2c^2 + 4c^2.$$

Since $ac < 1$, we can strengthen inequality (13):

(14) $$a^4 + 4c^2 > 2c^4 + 2a^2 + 1.$$

Inequality (14) is equivalent to the following inequality:

$$(a^2 - 1)^2 > 2(1 - c^2)^2.$$

Hence, we find that $a^2 > 1 + \sqrt{2} - c^2\sqrt{2}$. Since $1 > a^2c^2$, so we have $1 > c^2(1 + \sqrt{2} - c^2\sqrt{2})$.

Rewrite the last inequality: $(c^2 - 1)(c^2\sqrt{2} - 1) > 0$. Since $c < 1$, so $c^2\sqrt{2} - 1 < 0$. Hence, $c < \sqrt[4]{1/2}$.

STEP 4. Let's prove that $c \geq \frac{2\sqrt{2}}{3}$.

Since $p < \frac{a}{2}$, so from equation (10) one can get the inequality:

(15) $$a^4 + a^2c^2 + c^4 + 1 - 2a^2 - 2c^2 > 3 - 6a^2 + 3a^4.$$

Consider inequality (15) as a quadratic inequality with respect to a^2. Rewrite inequality (15):

$$2a^4 - a^2(c^2 + 4) + 2 + 2c^2 - c^4 < 0.$$

Obviously the discriminant must be nonnegative:

$$D = 9c^4 - 8c^2.$$

So $c^2 \geq \frac{8}{9}$.

The contradiction of the results obtained on steps 3 and 4 completes the proof of the theorem. □

Theorem 2.2. *Let $n \in [-0.5, 2]$. Then a triangle ABC is isosceles $(a = c)$ iff there holds $l^{int}(a, n) = l^{int}(c, n)$.*

Proof. If $n = 0$, then the statement is trivial. Let $n \neq 0$.

Consider the first case: $n \in (0, 2]$. Let $b = 1$, $\delta = a^n$, $\beta = c^n$, $k = \frac{2}{n}$, $a > c$, then we get

$$g(\delta, \beta, k) = \delta^k(\beta(\delta + 1)^2 + (\beta + 1)^2(\delta + 1)) -$$
$$- \beta^k(\delta(\beta + 1)^2 + (\beta + 1)(\delta + 1)^2) + (\beta + 1)(\delta + 1)(\delta - \beta) = 0.$$

Since $g(\beta, \beta, k) = 0$ for all $\beta > 0, k \geq 1$, so it is sufficient to prove that

(16) $$g'_\delta(\delta, \beta, k) > 0 \text{ for all } \delta \geq \beta > 0, k \geq 1.$$

We calculate

$$g'_\delta(\delta,\beta,k) =$$
$$= k\delta^{k-1}(\beta(\delta+1)^2 + (\beta+1)^2(\delta+1)) + \delta^k(2\beta(\delta+1) + (\beta+1)^2) -$$
$$- \beta^k((\beta+1)^2 + 2(\beta+1)(\delta+1)) + 2(\beta+1)(\delta+1) - (\beta+1)^2.$$

Suppose that $g'_\delta(\delta,\beta,k) \le 0$ for some δ,β,k satisfying restriction (16).
Firstly, let's prove that $\beta \ge 1$. Since $\delta \ge \beta$, so

$$2(\beta+1)(\delta+1) - (\beta+1)^2 \ge (\beta+1)(\delta+1). \tag{17}$$

Moreover,

$$\delta^k(2\beta(\delta+1) + (\beta+1)^2) \ge \beta^k((\beta+1)^2 + 2\beta(\delta+1)). \tag{18}$$

Since $g'_\delta(\delta,\beta,k) \le 0$, so inequalities (17) and (18) imply the inequality:

$$2\beta^k(\delta+1) \ge (\beta+1)(\delta+1). \tag{19}$$

From relation (19) we get $2\beta^k \ge \beta+1$, that is impossible for $\beta < 1$ and $k \ge 1$. Hence, $\beta \ge 1$.

Since $\delta \ge \beta$, so

$$k\delta^{k-1}(\beta(\delta+1)^2 + (\beta+1)^2(\delta+1)) + \delta^k(2\beta(\delta+1) + (\beta+1)^2) -$$
$$- \beta^k((\beta+1)^2 + 2(\beta+1)(\delta+1)) \ \ge$$
$$\ge \ \delta^k\left(\frac{k}{\delta}\beta(\delta+1)^2 + \frac{k}{\delta}(\beta+1)^2(\delta+1) - 2(\delta+1)\right). \tag{20}$$

From the inequalities

$$2(\beta+1)(\delta+1) - (\beta+1)^2 \ge (\beta+1)(\delta+1),\ \ g'_\delta(\delta,\beta,k) \le 0$$

and relation (20) we conclude that

$$\frac{k}{\delta}\beta(\delta+1)^2 + \frac{k}{\delta}(\beta+1)^2(\delta+1) - 2(\delta+1) \le 0. \tag{21}$$

Let's simplify the inequality (21)

$$k\beta(\delta+1) + k(\beta+1)^2 \le 2\delta. \tag{22}$$

From (22) we conclude that $(2-k)\delta > 5k$. Hence, $k \in [1,2)$. Since a,b,c are the lengths of the triangle, so $a < b+c = 1+c$, that is equivalent to the inequality $\delta^{\frac{k}{2}} < \beta^{\frac{k}{2}}+1$. Consequently, $\delta < \left(\beta^{\frac{k}{2}}+1\right)^{\frac{2}{k}}$. Using the inequality $\left(\frac{|x_1|^s+|x_2|^s}{2}\right)^{\frac{1}{s}} \le \left(\frac{|x_1|^t+|x_2|^t}{2}\right)^{\frac{1}{t}}$, which is valid for all $x_1,x_2 \in \mathbb{R}$, $0 < s \le t$, and since $1 \le k < 2$ we get

$$\delta < 2^{\frac{2}{k}-1}(\beta+1) \le 2(\beta+1). \tag{23}$$

We obtain from (22) and (23) the following inequality:

$$3k\beta^2 + 5k\beta - 4\beta - 4 + k \le 0. \tag{24}$$

Since $3k\beta^2 \ge 3$, $5k\beta - 4\beta > 0$, $4-k \le 3$, so inequality (24) is impossible. We get a contradiction.

Consider the second case: $n \in [-0.5, 0)$.

We have

$$g'_\beta(\delta,\beta,-k) = \delta^{-k}[(\delta+1)^2 + 2(\beta+1)(\delta+1)]+$$
$$+k\beta^{-k-1}[\delta(\beta+1)^2 + (\beta+1)(\delta+1)^2] - \beta^{-k}[2\delta(\beta+1) + (\delta+1)^2]$$
$$+ + (\delta+1)^2 - 2(\beta+1)(\delta+1).$$

Since $g(\delta,\delta,-k) = 0$ for any δ, k, so it's enough to prove the inequality $g'_\beta(\delta,\beta,-k) > 0$ for any $\delta \geq \beta$ for all $k \geq 4$.

Since $a = \delta^{-\frac{k}{2}}$, $c = \beta^{-\frac{k}{2}}$, $\alpha \geq \beta$, $a + c > 1$, so $\beta^{-\frac{k}{2}} > \frac{1}{2}$. Let's estimate the partial derivative $g'_\beta(\delta,\beta,-k)$:

$$g'_\beta(\delta,\beta,-k) \geq$$
$$\geq \beta^{-k}\left(\frac{k}{\beta}\delta(\beta+1)^2 + \frac{k}{\beta}(\beta+1)(\delta+1)^2 - 2\delta(\beta+1) - (\delta+1)^2\right) - (\delta+1)^2.$$

For any $k \geq 2$ we have $\frac{k}{\beta}\delta(\beta+1)^2 \geq 2\delta(\beta+1)$, then

$$g'_\beta(\delta,\beta,-k) \geq \beta^{-k}k(\delta+1)^2 - (\delta+1)^2.$$

Since $\beta^{-k} > \frac{1}{4}$, so for any $k \geq 4$ the inequality $g'_\beta(\delta,\beta,-k) > 0$ is true. The theorem is proved. □

Theorem 2.3. *There exists a real positive number n_0 such that for any real number n, with $|n| \geq n_0$, there exists a non-isosceles triangle ABC, for which $l^{int}(a,n) = l^{int}(c,n)$.*

Proof. Firstly, we find a positive real number n_1 and a negative real number n_2, for which one can find non-isosceles triangles with two equal internal n_1-lines or two equal internal n_2-lines.

We construct an example with a positive value of n. Let $b = 1, \delta = a^n, \beta = c^n, k = \frac{2}{n}$. Then we have the relation:

$$\delta^k(\beta(\delta+1)^2 + (\beta+1)^2(\delta+1)) - \beta^k(\delta(\beta+1)^2 + (\beta+1)(\delta+1)^2) +$$
$$+ (\beta+1)(\delta+1)^2 - (\beta+1)^2(\delta+1) = 0.$$

Choose $\delta = 2, \beta = 0.25$, then $k \approx 0,03599618$.

Hence,

$$n = \frac{2}{k} \approx 55.56145052, \tag{25}$$

$$a = \delta^{\frac{k}{2}} \approx 1.012553467, \quad c = \beta^{\frac{k}{2}} \approx 0.975358043.$$

Now we construct an example with a negative value of n.

Choose $\delta = 4, \beta = 0.5$, then $k \approx -0,05421121$.

Consequently,

$$n = \frac{2}{k} \approx -36.89274095, \tag{26}$$

$$a = \delta^{\frac{k}{2}} \approx 0.963120885, \quad c = \beta^{\frac{k}{2}} \approx 1.018965781.$$

Let n_0 be a positive value as in (25). We prove that for any $n \geq n_0$ there exists a non-isosceles triangle ABC, for which $l^{int}(a,n) = l^{int}(c,n)$. Let

$k_0 = \frac{2}{n_0}$. Consider the function $g(\delta, \beta, k)$, defined in the proof of Theorem 2.1. We consider new function

$$h(\varepsilon, k) = g(2, 0.25 + \varepsilon, k),$$

where $0 \le k \le k_0, 0 \le \varepsilon \le 0.25$. Let's prove that for any $k \in (0, k_0)$ there exists $\varepsilon \in (0, 0.25)$, such that $h(\varepsilon, k) = 0$. We fix an arbitrary $k_1 \in (0, k_0)$.

Since there hold the relations

$$h(0, k_1) = g(2, 0.25, k_1) < g(2, 0.25, k_0) = 0,$$

$$h(0.25, k_1) = g(2, 0.5, k_1) > g(2, 0.5, 0) = 0$$

and the function $h(\varepsilon, k_1)$ is continuous at $\varepsilon \in [0, 0.25]$, so we can find $\varepsilon_1 \in (0, 0.25)$, such that $h(\varepsilon_1, k_1) = 0$. It's evidently that a triangle with the sides $a = 2^{\frac{k_1}{2}}, b = 1, c = (0.25 + \varepsilon_1)^{\frac{k_1}{2}}$ exists, because $a < \sqrt{2}, c > 0.5$.

Let n_1 be a negative value as in (26). We'll prove that for any $n \le n_1$ there exists a non-isosceles triangle ABC, for which $l^{int}(a, n) = l^{int}(c, n)$. Let $k_2 = \frac{2}{n_1}$. We define the function

$$\varphi(\varepsilon, k) = g(4 - \varepsilon, 0.5, k),$$

where $k_2 \le k \le 0,\ 0 \le \varepsilon \le 2$. Let's prove that for any $k \in (k_2, 0)$ there exists $\varepsilon \in (0, 2)$, such that $\varphi(\varepsilon, k) = 0$. We fix an arbitrary $k_3 \in (k_2, 0)$.

Since

$$\varphi(0, k_3) = g(4, 0.5, k_3) > g(4, 0.25, k_2) = 0,$$

$$\varphi(2, k_3) = g(2, 0.5, k_3) < g(2, 0.5, 0) = 0$$

and the function $\varphi(\varepsilon, k_3)$ is continuous at $\varepsilon \in [0, 2]$, so we can find $\varepsilon_2 \in (0, 2)$, such that $h(\varepsilon_2, k_3) = 0$. It's evidently that a triangle with the sides $a = (4 - \varepsilon_2)^{\frac{k_3}{2}}, b = 1, c = 0.5^{\frac{k_3}{2}}$ exists, because $a > 0.5, c < \sqrt{2}$.

Thus, for any real $n, |n| \ge n_0$, there exists a non-isosceles triangle ABC, for which $l^{int}(a, n) = l^{int}(c, n)$. The theorem is proved. □

2.2. EXTERNAL n-LINES, $n \neq 0$.

Theorem 2.4. *Let $n \neq 0$ be a positive real number. Then a triangle ABC is isosceles ($a = c$) iff $l^{ext}(a, n) = l^{ext}(c, n),\ l^{ext}(a, -n) = l^{ext}(c, -n)$.*

Proof. The proof of the necessity is trivial. We prove that the sufficiency is true for all $n \in \mathbb{R} \setminus \{0\}$. It's enough to consider the case $n > 0$. From the conditions $l^{ext}(a, n) = l^{ext}(c, n)$, $l^{ext}(a, -n) = l^{ext}(c, -n)$ and equality (2) we get:

$$\frac{(1 + c^n)(1 - c^2)}{1 - c^n} = \frac{(1 + a^n)(1 - a^2)}{1 - a^n}. \tag{27}$$

Consider the function

$$f(x, n) = \frac{(1 + x^n)(1 - x^2)}{1 - x^n},$$

where $x \in (0,1) \cup (1,\infty),\ n > 0$. Denote $x^n = t,\ k = \frac{2}{n}$ and instead of the function $f(x,n)$ consider the function

$$g(t,k) = \frac{(1+t)(1-t^k)}{1-t},\ t \in (0,1) \cup (1,\infty),\ k > 0.$$

We prove that for each fixed $k > 0$ the function $g(t,k)$ is increasing on t. It's sufficient to show that $g'_t(t,k) \geq 0$ for any $t \in (0,1) \cup (1,\infty)$ and for each fixed $k > 0$ there exists a finite limit $\lim\limits_{t\to 1} g(t,k)$.

We find

$$g'_t(t,k) = \frac{2 - kt^{k-1} - 2t^k + kt^{k+1}}{(1-t)^2}.$$

Let us show that $2 - kt^{k-1} - 2t^k + kt^{k+1} \geq 0$. Consider the function

$$h_k(t) = 2 - kt^{k-1} - 2t^k + kt^{k+1},\ t \in (0,\infty),$$

for each fixed $k > 0$. Let $a_k = \inf\limits_{t\in(0,\infty)} h_k(t)$. We find the stationary points of the function $h_k(t)$.

We calculate the derivative

$$h'_k(t) = (k+1)kt^k - 2kt^{k-1} - k(k-1)t^{k-2} = kt^{k-2}((k+1)t^2 - 2t - (k-1)).$$

The derivative $h'_k(t)$ vanishes at the points $t_1 = 1, t_2 = \frac{1-k}{1+k}$. The point t_2 is the point of local maximum of the function $h_k(t)$ for $k \in (0,1)$, for other k the point t_2 does not belong to the domain of the function $h_k(t)$. The point t_1 is the point of local minimum of the function $h_k(t)$. So,

$$a_k = \min\left\{h_k(0), h_k(1), \lim_{t\to+\infty} h_k(t)\right\} = 0.$$

Hence, $g'_t(t,k) \geq 0$ for any $t \in (0,1) \cup (1,\infty),\ k > 0$.

Let us find the limit $\lim\limits_{t\to 1} g(t,k)$. We use L'Hôpital's rule:

$$\lim_{t\to 1} g(t,k) = \lim_{t\to 1} \frac{(1+t)(1-t^k)}{1-t} = 2\lim_{t\to 1} \frac{1-t^k}{1-t} = 2\lim_{t\to 1} \frac{-kt^{k-1}}{-1} = 2k < \infty.$$

Since the function $g(t,k)$ is increasing on t for each fixed $k > 0$, so relation (27) implies that $a = c$. The theorem is proved. □

Theorem 2.5. *For any real number $n \neq 0$, there exists a non-isosceles triangle ABC, for which $l^{ext}(a,n) = l^{ext}(c,n)$.*

Proof. Let $b = 1, \delta = a^n, \beta = c^n, k = \frac{2}{n}$. From the condition $l^{ext}(a,n) = l^{ext}(c,n)$ and from equality (2) it follows that

$$\delta^k(-\beta(\delta-1)^2 + (\beta-1)^2(\delta-1)) - \beta^k(-\delta(\beta-1)^2 + (\beta-1)(\delta-1)^2) + \tag{28}$$
$$+ (\beta-1)(\delta-1)(\delta-\beta) = 0.$$

Denote the left-hand side of equality (28) by $z(\delta,\beta,k)$. Consider $z(\delta,\beta,k)$ for fixed $k \in \mathbb{R}$. We have $z(\delta,1,k) = -\delta^k(\delta-1)^2 < 0\ \forall\delta > 0$ and $z(1,\beta,k) = \beta^k(\beta-1)^2 > 0\ \forall\beta > 0$.

Lemma 2.1. *For any fixed $k \in \mathbb{R}$ and for any $\varepsilon > 0$ there exist δ, β, such that $\delta \neq \beta, 0 < |\delta - 1| < \varepsilon, 0 < |\beta - 1| < \varepsilon$ and $z(\delta,\beta,k) = 0$.*

Proof. We fix $k \in \mathbb{R}$ and $\varepsilon > 0$. Since $z(1,2,k) > 0$, so there exists a positive number ρ, such that for any $1-\rho \le \delta \le 1+\rho$ the inequality $z(\delta,2,k) > 0$ holds. We set $\varepsilon_1 = \min\{\varepsilon,\rho\}$, then for any $s \in (0,\varepsilon_1)$ we have:

$$z(1-s,2,k) > 0 \text{ and } z(1-s,1,k) < 0.$$

Therefore, for any $s \in (0,\varepsilon_1)$ there exists number $\sigma_s \in (1,2)$, such that $z(1-s,\sigma_s,k) = 0$. We want to prove, that $\sigma_s \to 1$ as $s \to 0$. Suppose it is not true. Then there exists a sequence $s_m \to 0$ for $m \to \infty$, for which $\sigma_{s_m} \to A$ as $m \to \infty$, where $A \ne 1, A > 0$. Since the function $z(\delta,\beta,k)$ is continuous in a neighborhood of the point $(\delta,\beta) = (1,1)$ for any fixed $k \in \mathbb{R}$ and

$$z(1-s_m,\sigma_{s_m},k) = 0 \; \forall m \in \mathbb{N},$$

so we get

$$z(1,A,k) = 0. \tag{29}$$

But relation (29) is impossible for $A \ne 1, A > 0$. This contradiction proves the convergence $\sigma_s \to 1$ as $s \to 0$.

We conclude that there exists a positive ε_2, such that $1 < \sigma_s < 1+\varepsilon_1$ for any $s \in (0,\varepsilon_2)$. Let $\delta \in (1-\min\{\varepsilon_1,\varepsilon_2\},1)$ and $\beta = \sigma_{1-\delta}$. So, these values of β,δ are desired. The lemma is proved.

It follows from Lemma 2.1 that for any real k there exist positive real numbers δ,β such that one can find a non-isosceles triangle with sides $a = \delta^{\frac{k}{2}}, c = \beta^{\frac{k}{2}}, b = 1$. The theorem is proved. □

2.3. $(n+1)$-SECTOR LINES, $n > 0$.

Theorem 2.6. *Let $n > 0$ be a real number. A triangle ABC is isosceles ($a = c$) iff the lengths of its two $(n+1)$-sector lines AA_3, CC_3 are equal.*

Proof. The necessity is obvious.

Let's prove the sufficiency. Let $AC = b = 1$, $\angle BAA_3 = \alpha$, $\angle BCC_3 = \beta$ (fig. 1). Suppose that $AB = c > BC = a$ and there exists such $n > 0$ that the lengths of two $(n+1)$-sector lines AA_3, CC_3 are equal. Let $\xi_0 = (n+1)\alpha$, $\eta_0 = (n+1)\beta$. We have $\eta_0 > \xi_0$. By the law of sines:

$$\frac{AA_3}{\sin(n\beta+\beta)} = \frac{1}{\sin(n\beta+\beta+n\alpha)}, \tag{30}$$

$$\frac{CC_3}{\sin(n\alpha+\alpha)} = \frac{1}{\sin(n\alpha+\alpha+n\beta)}. \tag{31}$$

Conditions $AA_3 = CC_3$ and (30), (31) imply the equality

$$\sin(n\beta+\beta)\sin(n\alpha+\alpha+n\beta) = \sin(n\alpha+\alpha)\sin(n\beta+\beta+n\alpha). \tag{32}$$

Equation (32) may be rewritten as

$$\sin\xi_0 \sin(\eta_0 + x_0\xi_0) = \sin\eta_0 \sin(x_0\eta_0 + \xi_0), \tag{33}$$

where $x_0 = \frac{n}{n+1} \in (0,1)$.

Consider the function

$$g(x) = \frac{\sin(\eta_0 + x\xi_0)}{\sin(x\eta_0 + \xi_0)} - \frac{\sin\eta_0}{\sin\xi_0}, x \in (0,1).$$

We have

$$g(0) = 0,\ g(1) = 1 - \frac{\sin \eta_0}{\sin \xi_0} < 0.$$

Let's compute the derivative:

$$g'(x) = \frac{\xi_0 \cos(\eta_0 + x\xi_0)\sin(x\eta_0 + \xi_0) - \eta_0 \cos(x\eta_0 + \xi_0)\sin(\eta_0 + x\xi_0)}{\sin^2(x\eta_0 + \xi_0)}.$$

Denote

$$g'(0) = \frac{\xi_0 \cos \eta_0 \sin \xi_0 - \eta_0 \cos \xi_0 \sin \eta_0}{\sin^2 \xi_0} = Q.$$

Let us prove that $Q < 0$. We assume $Q \geq 0$, then

(34) $$\xi_0 \cos \eta_0 \sin \xi_0 \geq \eta_0 \cos \xi_0 \sin \eta_0.$$

Inequality (34) is equivalent to

(35) $$\frac{\cot \eta_0}{\eta_0} \geq \frac{\cot\ \xi_0}{\xi_0}.$$

Consider the function $z(x) = \frac{\mathrm{ctg} x}{x}, x \in (0, \pi)$.

We have

(36) $$z'(x) = \frac{\frac{-1}{\sin^2 x}x - \cot x}{x^2} = -\frac{x + \cos x \sin x}{x^2 \sin^2 x} = -\frac{2x + \sin 2x}{2x^2 \sin^2 x} < 0.$$

From relations (35) and (36) we get $\eta_0 \leq \xi_0$, this is a contradiction. Hence, $Q < 0$.

Lemma 2.2. *The function $g'(x)$ has at most one zero on $(0, 1)$.*

Proof. Suppose the contrary, then the equation

$$h(x) \overset{def}{=} \xi_0 \cot(\eta_0 + x\xi_0) - \eta_0 \cot(\xi_0 + x\eta_0) = 0$$

has at least two different roots $x_1 < x_2,\ x_1, x_2 \in (0, 1)$. Then there exists $x_3 \in (x_1, x_2)$, such that $h'(x_3) = 0$. We get

$$h'(x) = -\frac{\xi_0^2}{\sin^2(\eta_0 + x\xi_0)} + \frac{\eta_0^2}{\sin^2(\xi_0 + x\eta_0)}.$$

So,

(37) $$\xi_0 \sin(\xi_0 + x_3\eta_0) - \eta_0 \sin(\eta_0 + x_3\xi_0) = 0.$$

Consider the function

$$q(x) = \xi_0 \sin(\xi_0 + x\eta_0) - \eta_0 \sin(\eta_0 + x\xi_0),\ x \in [0, 1].$$

We show that the function $q(x)$ is monotonic at $[0, 1]$. Suppose the contrary, then we can find $x_4 \in (0, 1)$, such that $q'(x_4) = 0$.

We get

$$q'(x) = \xi_0\eta_0 \cos(\xi_0 + x\eta_0) - \eta_0\xi_0 \cos(\eta_0 + x\xi_0).$$

So, $\cos(\xi_0 + x_4\eta_0) = \cos(\eta_0 + x_4\xi_0)$.

Since $0 < (\xi_0 + x_4\eta_0) + (\eta_0 + x_4\xi_0) < 2\pi$, so $\xi_0 + x_4\eta_0 = \eta_0 + x_4\xi_0$, but then $\eta_0 = \xi_0$, a contradiction. So,

(38) $$q'(x) \neq 0 \text{ for any } x \in (0, 1).$$

From (38) and the continuity of $q'(x)$ it follows that $q'(x)$ has a constant sign in the interval $(0,1)$, it means that $q(x)$ is monotonic on the interval $[0,1]$. Since

$$q(0) = \xi_0 \sin\xi_0 - \eta_0 \sin\eta_0 < 0, q(1) = \xi_0 \sin(\xi_0 + \eta_0) - \eta_0 \sin(\eta_0 + \xi_0) < 0$$

and the function $q(x)$ is monotonic on the segment $[0,1]$, so there holds $q(x) < 0 \; \forall x \in [0,1]$. We get a contradiction with (37). Hence $g'(x)$ has at most one zero on the interval $(0,1)$. The lemma is proved.

Since $g(0) = 0, g(1) < 0, g'(0) < 0$ and the function $g(x)$ has at most one stationary point at the interval $(0,1)$, so, clearly, $g(x) < 0$ for all $x \in (0,1]$. The condition $g(x) < 0 \; \forall x \in (0,1]$ contradicts with (33). The theorem is proved. □

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