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PROTASOV LEMMA AND ITS APPLICATIONS

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Abstract. Protasov Lemma, its uses in the proof of generalization of Butterfly Theorem and solution of a locus problem are a discussed. A geometric inequality involving a line passing through the intersection of two circles is poved. New problems exploring other properties of the configuration are proposed for further study.

1. Introduction

According to the website www.cut-the-knot.org the first recorded mention of the Butterfly Theorem appears in a letter to William Wallace, dated 7 April 1805, by Sir William Herschel - the discoverer of Uranus and its two major moons, Titania and Oberon [4] or [3]. There are many proofs and generalizations of the Butterfly Theorem [4], [2], [1]. We shall discuss in the current paper its generalization as a geometric inequality which we named as Butterfly Inequality. We also study locus problem and geometric inequality which are solved using Protasov Lemma. V. Protasov used this lemma in the solution of his problem [5] (See Theorem 2.1 below).

Theorem 1.1. *(Butterfly Theorem) Let A be the midpoint of a chord BC of a circle, through which two other chords DF and EG are drawn; DG cuts BC at H and EF cuts BC at I. Then A is also the midpoint of HI.*

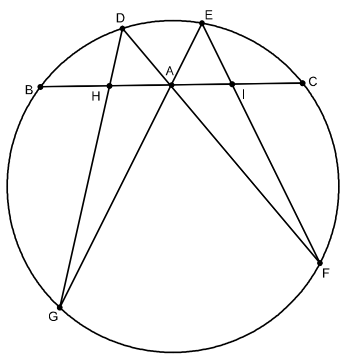

Figure 1

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Lemma 1.1. ([5]) *If AO_1OO_2 is a parallelogram, and H and I are points on a line through point A such that $|O_1H| = |O_1A|$, $|O_2I| = |O_2A|$, then $|OH| = |OI|$.*

2. MAIN RESULTS

Proof. (of Protasov Lemma) We first prove that triangles OO_1H and IO_2O are equal. Denoting angles $\angle H = y$ and $\angle AO_1O = z$ we calculate the remaining angles and see that $\angle HO_1O = \angle IO_2O = \pi - 2y + z$.

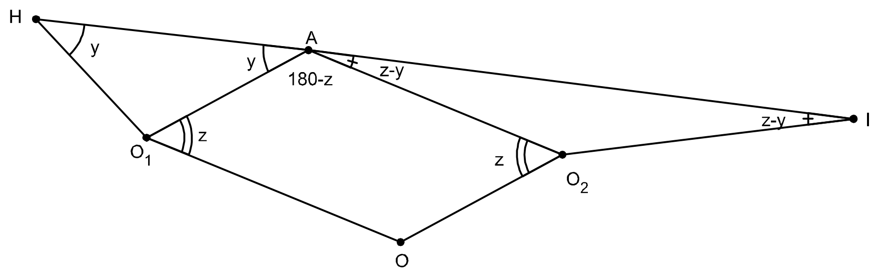

Figure 2

That is triangles OO_1H and IO_2O are equal by SAS principle. Note that the proof can easily be extended to include the cases when vertices of triangle OO_1H or triangle IO_2O are collinear. □

Remark 2.1. *The claim of Lemma 1.1 is true even in the case when vertices of parallelogram AO_1OO_2 are collinear.*

Theorem 2.1. *Let two chords DF and GE of a circle intersect in an interior point A, and let a line through A intersect the given circle in points B and C (as in Figure 3). If the line BC intersects the circle GAD again at H and the circle EAF again at I, then $|BH| = |CI|$.* ([5])

Proof. Let O be the center of the given circle, and O_1, O_2 be centers of circumscribed circles of triangles ADG and AEF, respectively. We will first prove that quadrilateral AO_1OO_2 is a parallelogram. It is obvious that $OO_1 \perp DG$, because DG is the common chord of two circles with centers O_1 and O. On the other hand if we denote $\angle O_2AF = x$ then $\angle D = \angle E = \frac{\pi}{2} - x$. Therefore $OO_1 \perp DG$, too. Consequently,

$$OO_2 || O_1A.$$

Similarly,

$$OO_2 || O_1A.$$

Hence the quadrilateral AO_1OO_2 is a parallelogram. By Lemma 1.1, $|OH| = |OI|$. Since $|OB| = |OC|$ the fact that $|OH| = |OI|$ will easily imply that $|BH| = |CI|$. □

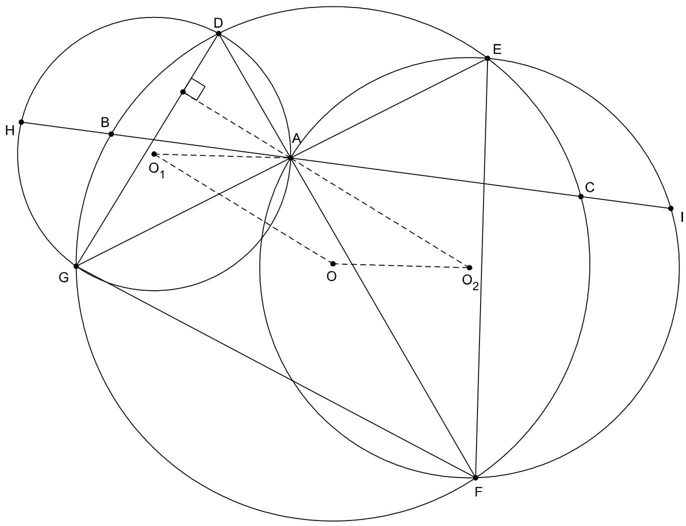

Figure 3

Theorem 2.2. *(Butterfly inequality) Let A be a point of a chord BC of a circle, through which two other chords DF and EG are drawn; DG cuts BC at H and EF cuts BC at I. Then the inequality $\frac{|AB|}{|AC|} \leq 1$ implies also that*

$$\frac{|AB|}{|AC|} \leq \frac{|AH|}{|AI|} \leq 1.$$

where either all the inequalities are strict or all the inequalities are actually equalities.

Proof. Let us make inversion with respect to circle with center at point A and radius equal to 1. We will denote the points in the new figure by the same letters. We obtain the configuration considered in Theorem 2.1. After the inversion all the lengths of segments AB, AC, AH, AI are replaced by their reciprocals. For example, the inversion transformed segment AB into a new segment with length $1/|AB|$. Therefore after the inversion the claim is transformed into the following: If $\frac{|AB|}{|AC|} \geq 1$ then $\frac{|AB|}{|AC|} \geq \frac{|AH|}{|AI|} \geq 1$. Noting that $|AH| = |AB| + a$ and $|AI| = |AC| + a$, where $|BH| = |CI| = a$ by Theorem 1, both of the above inequalities are easily proved:

$$\frac{|AH|}{|AI|} \geq 1 \Leftrightarrow \frac{|AB|+a}{|AC|+a} \geq 1 \Leftrightarrow \frac{|AB|}{|AC|} \geq 1,$$

$$\frac{|AB|}{|AC|} \geq \frac{|AH|}{|AI|} \Leftrightarrow \frac{|AB|}{|AC|} \geq \frac{|AB|+a}{|AC|+a} \Leftrightarrow \frac{|AB|}{|AC|} \geq 1.$$

□

Remark 2.2. *In particular if $|AB| = |AC|$ in Butterfly inequality then we obtain $|AH| = |AI|$. Note also that Butterfly Theorem and Butterfly inequality follows from A. Candy's result mentioned in p. 207 of Bankoff's encyclopedic article* [1]*(See Figure 1):*

$$\frac{1}{|AH|} - \frac{1}{|AI|} = \frac{1}{|AB|} - \frac{1}{|AC|}.$$

Proposition 2.1. *Let diagonals AC and BD of quadrilateral $ABCD$ intersect at point O. Let XY be a line through point O such that X and Y lies on sides AD and BC, respectively. Prove that if*

$$\frac{|AO|}{|OC|} \leq \frac{|DO|}{|OB|},$$

then

$$\frac{|AO|}{|OC|} \leq \frac{|XO|}{|OY|} \leq \frac{|DO|}{|OB|}.$$

The proof is left as an exercise.

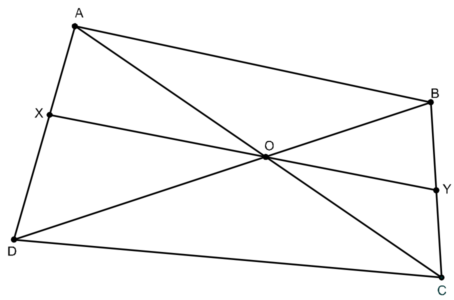

Figure 4

Proposition 2.2. *Let fixed circles w_1 and w_2 intersect at point A. A line through point A intersects circles w_1 and w_2 at points B and C, respectively. Let M be a fixed point of plane different from A. Let perpendiculars to MB and MC through points B and C, respectively, intersect at point D. Perpendicular to line BC through point D intersects line BC at point E. Show that locus of point E is a circle as BC rotates around point A.*

Proof. It is sufficient to show that line DE passes through a fixed point. To locate this point complete triangle O_1O_2A to parallelogram OO_1O_2A so that OA is a diagonal of this parallelogram. Here O_1, O_2 are centers of circles w_1 and w_2, respectively. Suppose that line MO intersects line DE at point F.

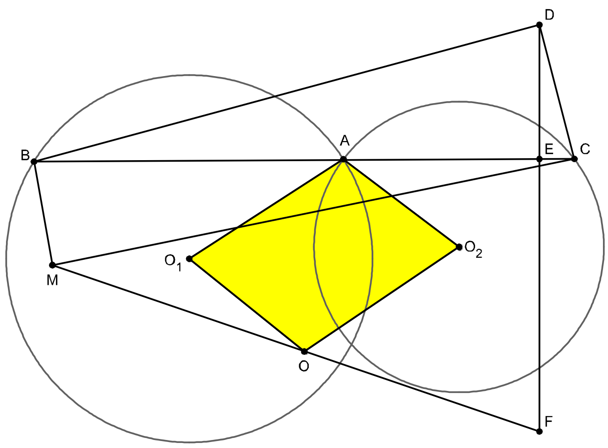

Figure 5

We shall prove that $|MO| = |OF|$. By Lemma 1.1, $|OB| = |OC|$. To complete the solution we drop perpendiculars MH and OG to line BC (See Figure 6).

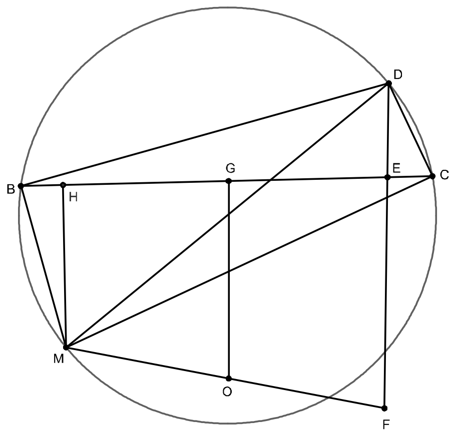

Figure 6

Since MD is the diameter of circumcircle of cyclic quadrilateral $MBDC$, the points H and E are equidistant from center of segment BC which coincides with point G. Therefore $|MO| = |OF|$. Since MO is constant, point F is fixed. □

Proposition 2.3. *The following properties of point F hold true:*

1) If point M coincides with center O_1 then $|MF| = 2 \cdot |O_2A|$.

2) If point M changes on a line parallel to line O_1O_2 then point F also changes on a line parallel to line O_1O_2.

3) If point M changes on a line parallel to line O_1O_2 then the circumcircle of triangle AEF intersects the line through intersection points of circles w_1 and w_2 at a fixed point.

The proof is left as an exercise.

Theorem 2.3. *Let circles w_1 and w_2 with centers O_1 and O_2 respectively, intersect at points A and A_1. Lines through points A and A_1 intersect circles w_1 and w_2 at points B, C and B_1, C_1, respectively (see Figure 7). Complete triangle O_1O_2A to parallelogram OO_1O_2A so that OA is a diagonal of this parallelogram. If segments BC and B_1C_1 do not intersect then*

$$|BB_1| + |CC_1| \leq 2 \cdot |AO|.$$

If segments BC and B_1C_1 intersect then

$$||BB_1| - |CC_1|| \leq 2 \cdot |AO|.$$

Proof. Denote midpoints of segments BC and B_1C_1 by D and D_1, respectively. By Lemma 1.1, $|OB| = |OC|$, hence OD is perpendicular line BC and therefore point D lies on circle with diameter AO (see Figure 7).

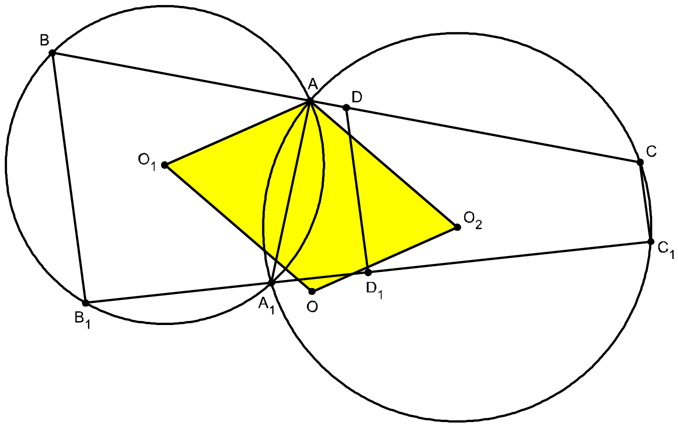

Figure 7

Since this circle passes through point A_1, it is easy to show that point D_1 lies on this circle, too. But if segments BC and B_1C_1 do not intersect then $|BB_1|+|CC_1| = 2\cdot|DD_1|$ and maximum of chord $|DD_1|$ is attained when it is a diameter whose length is equal to $|AO|$. Therefore, $|BB_1|+|CC_1| \leq 2\cdot|AO|$ holds true. The case when BC and B_1C_1 intersect can be considered in a similar way. □

Proposition 2.4. *The following propositions in the notations of Theorem 2.1 hold true:*

1) if $|BA| = |AC|$ then area of parallelogram OO_1O_2A is half the area of triangle COB;

2) $|AB| \cdot |AC| \leq |O_1O_2|^2 - (|O_1A| - |O_2A|)^2$.

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