A NOTE ON THE TOTAL NUMBER OF LINES OF FINITE PROJECTIVE SPACE

AYCA BAYRAKTAR, FATMA O. ERDOGAN and ATILLA AKPINAR

Abstract. In this paper we reformulate the well-known result in the literature on the total number of lines of a finite projective space and so we give a detailed proof of the result.

1. Introduction and Main result

First let we start by giving some basic knowledges about projective space from [1]. A *projective space* $S$ is a linear space which any two-dimensional subspace is a projective plane. Moreover, any subspace of a projective space is a projective space. Any two lines of a projective space have the same number of points. Let the number of points per line be $k + 1$, if the number is finite. Then we say $S$ has order $k$. Clearly, $k$ is the order of any projective plane in $S$. The *hyperplanes* of an $n$-dimensional projective space are precisely the $(n - 1)$-dimensional subspaces.

Now it is time to give the following two lemmas we need to prove the main result from [1].

**Lemma 1.1.** Let $V$ be any subspace of the projective space $S$ and let $p \notin V$. Then the subspace $\langle V \cup \{p\} \rangle$ is the set of points on all lines $pq$ for $q \in V$.

**Lemma 1.2.** A proper subspace $H$ of a projective space $S$ is a hyperplane if and only if each line of $S$ meets $H$ in at least one point.

Now, we can consecutively give two combinatorical results on $P(n,k)$ from [2].

**Keywords and phrases:** Finite projective space

(2010)Mathematics Subject Classification: 05B251

Received: 02.01.2013 In revised form: 13.05.2013 Accepted: 17.07.2013
Lemma 1.3. Let $P(n, k)$ be a finite projective space of dimensional $n$ and order $k$, and let $U$ be a $t$-dimensional subspace of $P(n, k)$ ($1 \leq t \leq n$). Then,

i) the number of points of $U$ is
$$k^t + k^{t-1} + \ldots + k + 1 = \frac{k^t+1 - 1}{k-1};$$
i) the number of lines of $U$ through a fixed point of $U$ equals
$$k^{t-1} + \ldots + k + 1;$$
iii) the total number of lines of $U$ equals
$$\frac{(k^t + k^{t-1} + \ldots + k + 1)(k^{t-1} + \ldots + k + 1)}{k + 1}.$$

Lemma 1.4. Let $P(n, k)$ be a finite projective space of dimensional $n$ and order $k$. Then,

i) the number of hyperplanes of $P(n, k)$ is exactly
$$k^n + k^{n-1} + \ldots + k + 1;$$
i) the number of hyperplanes of $P(n, k)$ through a fixed point of $P(n, k)$ equals
$$k^{n-1} + \ldots + k + 1.$$

We know that we can obtain an $n$-dimensional projective space from an $(n+1)$-dimensional vector space over the skew-field $F$ and that an affine space is a projective space less a hyperplane [1]. We are now ready to present the main result of this paper, which shows that it is possible to write as a polynomial to the statement in (iii) of Lemma 1.3.

Theorem 1.1. The total number of lines of $P(n, k)$ equals to $b = \sum_{i=0}^{2n-2} a_i k^{2n-2-i}$

where $a_i = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor + 1, & i < n \\ n - \left\lfloor \frac{i+1}{2} \right\rfloor, & i \geq n \end{cases}$.

2. Proof of the Main Result

We will use induction on $n$ to prove the claim. Let $n = 2$ then $P(2, k)$ is a projective plane with lines $b = k^2 + k + 1$. According to our theorem whilst

\[ n = 2, \text{we have} \quad b = \sum_{i=0}^{2} a_i k^{2-i} = a_0 k^2 + a_1 k + a_2 = 1 \cdot k^2 + 1 \cdot k + 1 = k^2 + k + 1 \]

where $a_i = \begin{cases} \left\lfloor \frac{i}{2} \right\rfloor + 1, & i < 2 \\ n - \left\lfloor \frac{i+1}{2} \right\rfloor, & i \geq 2 \end{cases}$. In this case, our claim is true.

Suppose that $N$ is any point of $P(3, k)$ and that $H$ is any hyperplane such that $N \notin H$. Since $H$ is 2–dimensional projective space with order $k$, then $H$ is a projective plane. The number of the lines through $N$ equals to the number of points (and also lines) of the hyperplane, which is $k^2 + k + 1$. So, the number of lines through any point of $H$ in $P(3, k)$ is also $k^2 + k + 1$. Of the lines, $k + 1$ intersect to $H$ in more than one point (they belong
to $H$) and $k^2$ intersect to $H$ in one point. From Lemma 2, since every line of $P(3,k)$ must meet $H$ in at least one point, the number of the lines intersecting $H$ in one point is $k^2(k^2 + k + 1)$. If we consider the lines in $H$, the total number of lines of $P(3,k)$ is $b = k^2(k^2 + k + 1) + k^2 + k + 1 = k^4 + k^3 + 2k^2 + k + 1$. According to our theorem whilst $n = 3$, we obtain

$$b = \sum_{i=0}^{4} a_i k^4 - i = a_0 k^4 + a_1 k^3 + a_2 k^2 + a_3 k^1 + a_4 k^0 = 1k^4 + 1k^3 + 2k^2 + 1k + 1$$

where $a_i = \begin{cases} \left[ \frac{i}{2} \right] + 1, & i < 3 \\ n - \left[ \frac{i+1}{2} \right], & i \geq 3 \end{cases}$. So, our claim is again true.

Now, assume the Theorem is true for $n - 1$. Then, the total number of lines of $P(n-1,k)$ is $b = \sum_{i=0}^{2n-4} b_i k^{2n-4-i}$ where

$$b_i = \begin{cases} \left[ \frac{i}{2} \right] + 1, & i < n - 1 \\ (n - 1) - \left[ \frac{i+1}{2} \right], & i \geq n - 1 \end{cases}$$

From this point we will only show that the Theorem is true for $n$. Suppose that $N$ is any point of $P(n,k)$ and that $H$ is any hyperplane such that $N \notin H$. Then, $H$ is $(n-1)$-dimensional projective space with order $k$ and the number of points of $H$ is $k^{n-1} + k^{n-2} + \cdots + k^2 + k + 1$. The number of the lines through $N$ equals to the number of points of the hyperplane, which is $k^{n-1} + k^{n-2} + \cdots + k^2 + k + 1$. So, the number of lines through any point of $H$ in $P(n,k)$ is also $k^{n-1} + k^{n-2} + \cdots + k^2 + k + 1$. Of the lines, $k^{n-2} + \cdots + k^2 + k + 1$ intersect to $H$ in more than one point (they belong to $H$) and $k^{n-1}$ intersect to $H$ in one point. From Lemma 2, since every line of $P(n,k)$ must meet $H$ in at least one point, the number of the lines intersecting $H$ in one point is $k^{n-1}(k^{n-1} + k^{n-2} + \cdots + k^2 + k + 1)$. If we consider the lines in $H$ (from our assumption), the total number of lines of $P(n,k)$ is

$$b = k^{n-1}(k^{n-1} + k^{n-2} + \cdots + k^2 + k + 1) + \sum_{i=0}^{2n-4} b_i k^{2n-4-i}$$

$$= k^{2n-2} + k^{2n-3} + k^{2n-4} + k^{2n-5} + \cdots + k^{n+1} + k^{n} + k^{n-1} + b_0 k^{2n-4} + b_1 k^{2n-5} + \cdots + b_{n-5} k^{n-1} + b_{n-4} k^{n} + b_{n-3} k^{n-1} + b_{n-2} k^{n-2} + \cdots + b_{2n-5} k^0$$

$$= k^{2n-2} + k^{2n-3} + (b_0 + 1) k^{2n-4} + (b_1 + 1) k^{2n-5} + \cdots + (b_{n-5} + 1) k^{n+1} + (b_{n-4} + 1) k^n + (b_{n-3} + 1) k^{n-1} + b_{n-2} k^{n-2} + \cdots + b_{2n-5} k^1 + b_{2n-4} k^0.$$
A note on the Total Number of Lines of Finite Projective Space

REFERENCES


DEPARTMENT OF MATHEMATICS
ULUDAĞ UNIVERSITY, FACULTY OF SCIENCE AND ART
16059 GORUKLE-BURSA, TURKEY
E-mails address:
ayca.byrktr@gmail.com,
fatmaozen@uludag.edu.tr,
aakpinar@uludag.edu.tr