INTERNATIONAL JOURNAL OF GEOMETRY
Vol. 2 (2013), No. 2, 46 - 49

A NOTE ON THE TOTAL NUMBER OF LINES OF FINITE PROJECTIVE SPACE

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Abstract. In this paper we reformulate the well-known result in the literature on the total number of lines of a finite projective space and so we give a detailed proof of the result.

1. Introduction and Main result

First let we start by giving some basic knowledges about projective space from [1]. A *projective space* S is a linear space which any two-dimensional subspace is a projective plane. Moreover, any subspace of a projective space is a projective space. Any two lines of a projective space have the same number of points. Let the number of points per line be $k+1$, if the number is finite. Then we say S has order k. Clearly, k is the order of any projective plane in S. The *hyperplanes* of an n-dimensional projective space are precisely the $(n-1)$-dimensional subspaces.

Now it is time to give the following two lemmas we need to prove the main result from [1].

Lemma 1.1. *Let V be any subspace of the projective space S and let $p \notin V$. Then the subspace $\langle V \cup \{p\}\rangle$ is the set of points on all lines pq for $q \in V$.*

Lemma 1.2. *A proper subspace $\mathbf{H}$ of a projective space S is a hyperplane if and only if each line of S meets $\mathbf{H}$ in at least one point.*

Now, we can consecutively give two combinatorical results on $P(n,k)$ from [2].

Keywords and phrases: Finite projective space
(2010)Mathematics Subject Classification: 05B251
Received: 02.01.2013 In revised form: 13.05.2013 Accepted: 17.07.2013

Lemma 1.3. *Let $P(n,k)$ be a finite projective space of dimensional n and order k, and let U be a t-dimensional subspace of $P(n,k)$ $(1 \le t \le n)$. Then,*

i) the number of points of U is

$$k^t + k^{t-1} + ... + k + 1 = \frac{k^{t+1}-1}{k-1};$$

ii) the number of lines of U through a fixed point of U equals

$$k^{t-1} + ... + k + 1;$$

iii) the total number of lines of U equals

$$\frac{\left(k^t + k^{t-1} + ... + k + 1\right)\left(k^{t-1} + ... + k + 1\right)}{k+1}.$$

Lemma 1.4. *Let $P(n,k)$ be a finite projective space of dimensional n and order k. Then,*

i) the number of hyperplanes of $P(n,k)$ is exactly

$$k^n + k^{n-1} + ... + k + 1;$$

ii) the number of hyperplanes of $P(n,k)$ through a fixed point of $P(n,k)$ equals

$$k^{n-1} + ... + k + 1.$$

We know that we can obtain an n-dimensional projective space from an $(n+1)$-dimensional vector space over the skew-field F and that an affine space is a projective space less a hyperplane [1]. We are now ready to present the main result of this paper, which shows that it is possible to write as a polynomial to the statement in (iii) of Lemma 1.3.

Theorem 1.1. *The total number of lines of $P(n,k)$ equals to* $b = \sum_{i=0}^{2n-2} a_i k^{2n-2-i}$ *where* $a_i = \begin{cases} \left[\left|\frac{i}{2}\right|\right] + 1, & i < n \\ n - \left[\left|\frac{i+1}{2}\right|\right], & i \ge n \end{cases}$.

2. Proof of The Main Result

We will use induction on n to prove the claim. Let $n = 2$ then $P(2,k)$ is a projective plane with lines $b = k^2 + k + 1$. According to our theorem whilst $n = 2$, we have $b = \sum_{i=0}^{2} a_i k^{2-i} = a_0 k^2 + a_1 k + a_2 1 = 1 \cdot k^2 + 1 \cdot k + 1 \cdot 1 = k^2 + k + 1$ where $a_i = \begin{cases} \left[\left|\frac{i}{2}\right|\right] + 1, & i < 2 \\ n - \left[\left|\frac{i+1}{2}\right|\right], & i \ge 2 \end{cases}$. In this case, our claim is true.

Suppose that N is any point of $P(3,k)$ and that $\mathbf{H}$ is any hyperplane such that $N \notin \mathbf{H}$. Since $\mathbf{H}$ is $2-$dimensional projective space with order k, then $\mathbf{H}$ is a projective plane. The number of the lines through N equals to the number of points (and also lines) of the hyperplane, which is $k^2 + k + 1$. So, the number of lines through any point of $\mathbf{H}$ in $P(3,k)$ is also $k^2 + k + 1$. Of the lines, $k + 1$ intersect to $\mathbf{H}$ in more than one point (they belong to **H**) and k^2 intersect to **H** in one point. From Lemma 2, since every line of $P(3,k)$ must meet **H** in at least one point, the number of the lines intersecting **H** in one point is $k^2(k^2+k+1)$. If we consider the lines in **H**, the total number of lines of $P(3,k)$ is $b = k^2(k^2+k+1)+k^2+k+1 = k^4+k^3+2k^2+k+1$. According to our theorem whilst $n=3$, we obtain $b = \sum_{i=0}^{4} a_i k^{4-i} = a_0k^4 + a_1k^3 + a_2k^2 + a_3k^1 + a_4k^0 = 1k^4 + 1k^3 + 2k^2 + 1k + 1$ where $a_i = \begin{cases} \left[\left|\frac{i}{2}\right|\right]+1, & i<3 \\ n-\left[\left|\frac{i+1}{2}\right|\right], & i \geq 3 \end{cases}$. So, our claim is again true.

Now, assume that the Theorem is true for $n-1$. Then, the total number of lines of $P(n-1,k)$ is $b = \sum_{i=0}^{2n-4} b_i k^{2n-4-i}$ where

$$b_i = \begin{cases} \left[\left|\frac{i}{2}\right|\right]+1, & i<n-1 \\ (n-1)-\left[\left|\frac{i+1}{2}\right|\right], & i \geq n-1 \end{cases}.$$

From this point we will only show that the Theorem is true for n. Suppose that N is any point of $P(n,k)$ and that **H** is any hyperplane such that $N \notin \mathbf{H}$. Then, **H** is $(n-1)$-dimensional projective space with order k and the number of points of **H** is $k^{n-1}+k^{n-2}+\cdots+k^2+k+1$. The number of the lines through N equals to the number of points of the hyperplane, which is $k^{n-1}+k^{n-2}+\cdots+k^2+k+1$. So, the number of lines through any point of **H** in $P(n,k)$ is also $k^{n-1}+k^{n-2}+\cdots+k^2+k+1$. Of the lines, $k^{n-2}+\cdots+k^2+k+1$ intersect to **H** in more than one point (they belong to **H**) and k^{n-1} intersect to **H** in one point. From Lemma 2, since every line of $P(n,k)$ must meet **H** in at least one point, the number of the lines intersecting **H** in one point is $k^{n-1}(k^{n-1}+k^{n-2}+\cdots+k^2+k+1)$. If we consider the lines in **H** (from our assumption), the total number of lines of $P(n,k)$ is

$$\begin{aligned} b &= k^{n-1}(k^{n-1}+k^{n-2}+..+k^2+k+1) + \sum_{i=0}^{2n-4} b_i k^{2n-4-i} \\ &= k^{2n-2}+k^{2n-3}+k^{2n-4}+k^{2n-5}+..+k^{n+1}+k^n+k^{n-1}+b_0k^{2n-4}+ \\ &\quad b_1k^{2n-5}+..+b_{n-5}k^{n+1}+b_{n-4}k^n+b_{n-3}k^{n-1}+b_{n-2}k^{n-2}+..+b_{2n-4}k^0 \\ &= k^{2n-2}+k^{2n-3}+(b_0+1)k^{2n-4}+(b_1+1)k^{2n-5}+..+(b_{n-5}+1)k^{n+1} \\ &\quad +(b_{n-4}+1)k^n+(b_{n-3}+1)k^{n-1}+b_{n-2}k^{n-2}+..+b_{2n-5}k^1+b_{2n-4}k^0. \end{aligned}$$

According to the claim of the theorem we will show that the total number of the lines $P(n,k)$ is equal to $b = \sum_{i=0}^{2n-2} a_i k^{2n-2-i}$. Firstly we show that

$a_0 = 1, a_1 = 1, a_2 = b_0+1, a_3 = b_1+1,..., a_{n-2} = b_{n-4}+1, a_{n-1} = b_{n-3}+1$, that is, $a_i = \left[\left|\frac{i}{2}\right|\right]+1$ whilst $i<n$: $a_i = b_{i-2}+1 = (\left[\left|\frac{i-2}{2}\right|\right]+1) = (\left[\left|\frac{i}{2}-1\right|\right]+1)+1 = \left[\left|\frac{i}{2}\right|\right]+1$. Secondly we show that $a_n = b_{n-2}$, $a_{n+1} = b_{n-1}$, $a_{n+2} = b_n,..., a_{2n-3} = b_{2n-5}$, $a_{2n-2} = b_{2n-4}$, that is, $a_i = n-\left[\left|\frac{i+1}{2}\right|\right]$, $i \geq n$: $a_i = b_{i-2} = ((n-1)-\left[\left|\frac{i-2+1}{2}\right|\right]) = n-1-(\left[\left|\frac{i}{2}-1+\frac{1}{2}\right|\right]) = n-1-((-1)+\left[\left|\frac{i+1}{2}\right|\right]) = n-\left[\left|\frac{i+1}{2}\right|\right]$. So, the proof is completed.

References

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[2] Beutelspacher, A. and Rosenbaum, U., *Projective Geometry: From Foundations to Applications*, Cambridge University Press, 1998.

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