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ON SPECIAL INGARDEN MECHANICAL SYSTEM

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Abstract. In this paper we give a description of a particular case of Finslerian Mechanical System, called Special Ingarden Mechanical System, endowed with a special nonlinear connection.We determine the local coefficients of the canonical metrical d-connection.

1. Introduction

Let M be an n-dimensional C^{∞} manifold. Denote by (TM, τ, M) the tangent bundle of M .We consider a function $F : TM \to R_+$ verifying the following axioms:

i) F is a differentiable function on $\widetilde{TM} = TM - \{0\}$ and F continuous on the null section of the projection $\pi : TM \to M$;
ii) F is positively 1- homogeneous with respect to the variables y^i;
iii)$\forall (x, y) \in \widetilde{TM}$ the Hessian of F^2 with respect to y^i ,with the elements $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive defined and nondegenarate.

The space $F^n = (M, F(x, y))$ is called a Finsler space, F is the fundamental function and $g_{ij}(x, y)$ is the fundamental tensor field of the space F^n.

Let $F(x, y) = \alpha(x, y) + \beta(x, y)$ be a particular case of the fundamental function of the space F^n,where $\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta(x, y) = b_i(x) y^i$ is a 1-form. If we consider N the Lorentz nonlinear connection introduced by R. Miron[9] we obtain a particular case of Finsler space called Ingarden space. Insteed of N we consider a new special nonlinear connection N^* constructed from N , a given Lorentz nonlinear connection and we define a Special Ingarden Space, denoted SI^*.

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We construct a 4-uple $\sum_{SI^*} = \left(M, F^2, N^*, Fe\right)$ where M is the configuration space, $F(x,y) = \alpha(x,y) + \beta(x,y)$ is the fundamental function of the Ingarden space, N^* is the special nonlinear connection and Fe are the external forces.

2. Ingarden Spaces

Let $F^n = (M, F(x,y))$be a Finsler space with the fundamental function $F(x,y) = \alpha(x,y) + \beta(x,y)$ where $\alpha(x,y) = \sqrt{a_{ij}(x)\, y^i y^j}$ and $\beta(x,y) = b_i(x)\, y^i$; $a = a_{ij}(x)\, dx^i dx^j$ is a Riemannian metric on M and it gives the gravitational part of the metric F; $b_i(x)$ is an electromagnetic covector on M and $\beta(x, dx) = b_i(x)\, dx^i$ is the electromagnetic 1-form field on M.

We consider the integral of action of the Lagrangian $F^2(x,y)$ along a curve $c : t \in [0,1] \to c(t) \in M$:

$$I(c) = \int_0^1 F^2\left(x, \frac{dx}{dt}\right) dt = \int_0^1 \left[\alpha\left(x, \frac{dx}{dt}\right) + \beta\left(x, \frac{dx}{dt}\right)\right]^2 dt \tag{2.1}$$

The variational problem for $I(c)$ leads to the Euler-Lagrange equations:

$$E_i\left(F^2\right) := \frac{\partial(\alpha+\beta)^2}{\partial x^i} - \frac{d}{dt}\frac{\partial(\alpha+\beta)^2}{\partial y^i} = 0, y^i = \frac{dx^i}{dt}. \tag{2.2}$$

The energy of F^2 is

$$\varepsilon_{F^2} = y^i \frac{\partial F^2}{\partial y^i} - F^2. \tag{2.3}$$

The covector field $E_i\left(F^2\right)$ is expressed by

$$E_i\left(F^2\right) = E_i\left(\alpha^2\right) + 2\alpha E_i(\beta) + 2\frac{d\alpha}{dt}\frac{\partial \alpha}{\partial y^i}. \tag{2.4}$$

Theorem 2.1. *The Euler-Lagrange equations (1.2) are equivalent to the Lorentz equations:*

$$\frac{d^2x^i}{ds^2} + \gamma^i_{jk}(x)\frac{dx^j}{ds}\frac{dx^k}{ds} = \alpha \overset{\circ}{F^i_j}(x)\frac{dx^j}{ds}, \tag{2.5}$$

where $\overset{\circ}{F^i_j}(x) = a^{is} F_{sj}(x)$ *and* γ^i_{jk} *are the Christoffel symbols of the Riemannian metric tensor* $a_{ij}(x)$.

The Euler-Lagrange equations $E_i\left(F^2\right) = 0$ determines a canonical semispray S on the total space of the tangent bundle :

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \tag{2.6}$$

with the coefficients

(2.7) $$2G^i(x,y) = \gamma^i_{jk}(x)\,y^j y^k - \overset{\circ}{F}{}^i_j(x)\,y^j.$$

Now we consider the nonlinear connection N with the coefficients

(2.8) $$N^i_j = \gamma^i_{jk}(x)\,y^k - F^i_j(x).$$

where $F^i_j(x) = \frac{1}{2}\overset{\circ}{F}{}^i_j(x)$.

Since the autoparallel curves of N are given by the Lorentz equations (2.5), we call it the Lorentz nonlinear connection of the metric $(\alpha+\beta)$.

The nonlinear connection N determines the horizontal distribution, denoted by N too, with the property $T_uTM = N_u \oplus V_u, \forall u \in TM, V_u$ being the natural vertical distribution on the tangent manifold TM.

The adapted basis to N is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_{i=1,\dots,n}$ with

(2.9) $$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^k_i\frac{\partial}{\partial y^k} = \frac{\partial}{\partial x^i} - \gamma^k_{is}(x)\,y^s\frac{\partial}{\partial y^k} + F^k_i\frac{\partial}{\partial y^k} = \frac{\overset{\circ}{\delta}}{\delta x^i} + F^k_i\frac{\partial}{\partial y^k},$$

where

(2.10) $$\frac{\overset{\circ}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \gamma^k_{is}(x)\,y^s\frac{\partial}{\partial y^k}$$

The adapted cobasis to N is $\left(dx^i, \delta y^i\right)_{i=1,\dots,n}$ with

(2.11) $$\delta y^i = dy^i + N^i_j dx^j = dy^i + \gamma^i_{jk}(x)\,y^k dx^j - F^i_j dx^j = \overset{\circ}{\delta}y^i - F^i_j dx^j,$$

where

(2.12) $$\overset{\circ}{\delta}y^i = dy^i + \gamma^i_{jk}(x)\,y^k dx^j$$

The weakly torsion of N is

(2.13) $$T^i_{jk} = \frac{\partial N^i_j}{\partial y^k} - \frac{\partial N^i_k}{\partial y^j} = 0.$$

The integrability tensor of N is

(2.14) $$R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}.$$

Definition 1. *The Finsler space $F^n = (M, F = \alpha+\beta)$ equipped with the Lorentz nonlinear connection N is called an Ingarden space. It is denoted IF^n.*

The fundamental tensor g_{ij} of IF^n is given by

(2.15) $$g_{ij} = \frac{F}{\alpha}(a_{ij} - \tilde{l}_i\tilde{l}_j) + l_il_j$$

where $\tilde{l}_i = \frac{\partial\alpha}{\partial y^i}$, $l_i = \frac{\partial F}{\partial y^i}$, $l_i = \tilde{l}_i + b_i$.

The following results holds [9].

Theorem 2.2. *There exists an unique N-metrical connection $I\Gamma\left(N\right) = \left(F^i_{jk}, C^i_{jk}\right)$ of the Ingarden space IF^n which verifies the following axioms:*

i) $\nabla^H_k g_{ij} = 0$; $\nabla^V_k g_{ij} = 0$;

ii) $T^i_{jk} = 0$; $S^i_{jk} = 0$.

The connection $I\Gamma\left(N\right)$ has the coefficients expressed by the generalized Christoffel symbols:

(2.16) $$\begin{cases} F^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{sk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^s}\right) \\ C^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right) \end{cases}$$

where $\frac{\delta}{\delta x^i}$ are given by (2.9).

3. Special Ingarden Spaces

Let IF^n be an Ingarden space and N the Lorenz nonlinear connection with the coefficients given by (2.11). Insteed of N we now consider a new nonlinear connection $\overset{*}{N}$ [8] with the coefficients

(3.1) $$\overset{*}{N}{}^i_j = N^i_j + \frac{F_{|j}y^i}{F},$$

where "|" denote the covariant differentiation with respect to $I\Gamma\left(N\right)$.

The nonlinear connection $\overset{*}{N}$ determines the horizontal distribution, denoted by $\overset{*}{N}$ too, with the property $T_uTM = \overset{*}{N}_u \oplus V_u$, $\forall u \in TM$, V_u being the natural vertical distribution on the tangent manifold TM.

The local adapted basis to the horizontal and vertical vector spaces $\overset{*}{N}_u$ and V_u is given by $\left(\frac{\overset{*}{\delta}}{\delta x^k}, \frac{\partial}{\partial y^k}\right)$, $k = 1, ..., n$, where

(3.2)
$$\begin{aligned} \frac{\overset{*}{\delta}}{\delta x^k} &= \frac{\partial}{\partial x^k} - \overset{*}{N}{}^r_k\frac{\partial}{\partial y^r} = \frac{\partial}{\partial x^k} - N^r_k\frac{\partial}{\partial y^r} - \frac{F_{|k}y^r}{F}\frac{\partial}{\partial y^r} \\ &= \frac{\delta}{\delta x^k} - \frac{F_{|k}y^r}{F}\frac{\partial}{\partial y^r} = \frac{\overset{\circ}{\delta}}{\delta x^k} + F^r_k\frac{\partial}{\partial y^r} - \frac{F_{|k}y^r}{F}\frac{\partial}{\partial y^r} \\ &= \frac{\overset{\circ}{\delta}}{\delta x^k} + \left(F^r_k - \frac{F_{|k}y^r}{F}\right)\frac{\partial}{\partial y^r} \end{aligned}$$

and $\frac{\overset{\circ}{\delta}}{\delta x^k}$ are given by (2.10).

The adapted cobasis to N is $\left(dx^i, \overset{*}{\delta} y^i\right)$, $i = 1, ..., n$ with

(3.3)

$$\begin{aligned} \delta y^i &= dy^i + \overset{*}{N}{}^i_j dx^j = dy^i + N^i_j dx^j + \frac{F_{|j} y^i}{F} dx^j \\ &= dy^i + \gamma^i_{jk}(x)\, y^k dx^j - F^i_j dx^j + \frac{F_{|j} y^i}{F} dx^j \\ &= \overset{\circ}{\delta} y^i - \left(F^i_j - \frac{F_{|j} y^i}{F}\right) dx^j \end{aligned}$$

where $\overset{\circ}{\delta} y^i$are given by (2.12).

Definition 2. *The Finsler space $F^n = (M, \alpha + \beta)$ equipped wih the special nonlinear connection $\overset{*}{N}$ is called a Special Ingarden space. It is denoted SI^*F^n.*

Theorem 3.1. *There exists an unique $\overset{*}{N}$- metrical connection $I\overset{*}{\Gamma}\left(\overset{*}{N}\right) = \left(\overset{*}{F}{}^i_{jk}, \overset{*}{C}{}^i_{jk}\right)$ of the Ingarden space IF^n which satisfies the following axioms:*

i) $\overset{*}{\nabla}{}^H_k g_{ij} = 0$; $\overset{*}{\nabla}{}^V_k g_{ij} = 0$;

ii) $\overset{*}{T}{}^i_{jk} = 0$; $\overset{*}{S}{}^i_{jk} = 0$.

The connection $I\overset{*}{\Gamma}\left(\overset{*}{N}\right)$ has the coefficients expressed by the generalized Christoffel symbols

(3.4)

$$\begin{cases} \overset{*}{F}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\overset{*}{\delta} g_{sj}}{\delta x^k} + \frac{\overset{*}{\delta} g_{sk}}{\delta x^j} - \frac{\overset{*}{\delta} g_{jk}}{\delta x^s}\right) \\ \overset{*}{C}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right) \end{cases}$$

where $\frac{\overset{*}{\delta}}{\delta x^i}$ are given by (3.2). From a direct calculation we get

(3.5)

$$\begin{cases} \overset{*}{F}{}^i_{jk} = F^i_{jk} \\ \overset{*}{C}{}^i_{jk} = C^i_{jk} \end{cases}$$

4. Special Ingarden Mechanical Systems

For a manifold M, that is the configuration space , let us consider the tangent bundle TM to which we refer to as the velocity space. Suppose that there is a metric $F = \alpha + \beta$ on $\widetilde{TM}$ and $F_i(x, y)\, dx^i$ is a globally defined d-covector field on the velocity space.

Definition 3. *A special Ingarden Mechanical System is a 4-uple* $\sum_{SI^*} = \left(M, (\alpha+\beta)^2, \overset{*}{N}, F_e\right)$ *with* $\overset{*}{N}$ *the special nonlinear connection (3.1) and*

$$F_e = F^i(x,y)\frac{\partial}{\partial y^i} \tag{4.1}$$

the external forces given as a vertical vector field on TM*.*
One consider $F_i(x,y) = g_{ij}F^j(x,y)$ the covariant components of the external forces F_e.

Theorem 4.1. *[10] For the special Ingarden mechanical system* $\sum_{SI^*} = \left(M, (\alpha+\beta)^2, N^*, F_e\right)$ *the following properties hold good:*

i) The operator S *defined by*

$$S = y^i\frac{\partial}{\partial x^i} - \left(2\overset{SI^*}{G^i} - \frac{1}{2}F^i\right)\frac{\partial}{\partial y^i} \tag{4.2}$$

is a vector field, global defined on the phase space TM*.*
ii) S *is a semispray which depends only on* $\sum_{SI^*}$ *and it is a spray if* F_e *are 2-homogeneous with respect to* y^i*.*
iii) The integral curves of the vector field S *are the evolution curves given by the Lagrange equations of* $\sum_{SI^*}$*:*

$$\frac{d^2x^i}{dt^2} + \overset{*}{\Gamma^i_{jk}}\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = \frac{1}{2}F^i\left(x, \frac{dx}{dt}\right). \tag{4.3}$$

The semispray S (4.2) has the coefficients $\overset{SI^*}{G^i}$ expressed by

$$2\overset{SI^*}{G^i} = 2\overset{*}{G^i} - \frac{1}{2}F^i(x,y) = \overset{*}{\Gamma^i_{jk}}(x,y)\,y^jy^k - \frac{1}{2}F^i(x,y). \tag{4.4}$$

Thus, the canonical nonlinear connection $\overset{SI^*}{N}$ of the *special Ingarden* mechanical system $\sum_{SI^*}$ has the coefficients

$$\overset{SI^*}{N^i_j} = \frac{\partial \overset{SI^*}{G^i}}{\partial y^j} = \overset{*}{N^i_j} - \frac{1}{4}\frac{\partial F^i}{\partial y^j}. \tag{4.5}$$

This nonlinear connection $\overset{SI^*}{N}$ determines a direct decomposition of the tangent space $\tilde{TM}$ into horizontal and vertical subspaces:

$$T_u\tilde{TM} = \overset{SI^*}{N_u} \oplus V_u, \forall u \in \tilde{TM}.$$

A local adapted basis to these decomposition is $\left(\frac{\overset{SI^*}{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_{i=\overline{1,n}}$ where

$$(4.6)\quad \frac{\overset{SI^*}{\delta}}{\delta x^i}=\frac{\partial}{\partial x^i}-\overset{SI^*}{N^i_j}\frac{\partial}{\partial y^j}=\frac{\partial}{\partial x^i}-\overset{*}{N^i_j}\frac{\partial}{\partial y^j}+\frac{1}{4}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}=\frac{\overset{*}{\delta}}{\delta x^i}+\frac{1}{4}\frac{\partial F^j}{\partial y^i}\frac{\partial}{\partial y^j}.$$

The adapted cobasis is $\left(dx^i, \overset{SI^*}{\delta}y^i\right)_{i=\overline{1,n}}$ with

$$(4.7)\quad \overset{SI^*}{\delta}y^i=dy^i+\overset{SI^*}{N^i_j}dx^j=dy^i+\overset{*}{N^i_j}dx^j-\frac{1}{4}\frac{\partial F^i}{\partial y^j}dx^j=\overset{*}{\delta}y^i-\frac{1}{4}\frac{\partial F^i}{\partial y^j}dx^j.$$

We determine the torsion $\overset{SI^*}{T^i_{jk}}$ and the curvature $\overset{SI^*}{R^i_{jk}}$of the canonical connection by a direct calculation:

$$(4.8)\quad \overset{SI^*}{T^i_{jk}}=\frac{\partial \overset{SI^*}{N^i_j}}{\partial y^k}-\frac{\partial \overset{SI^*}{N^i_k}}{\partial y^j}=0.$$

(4.9)

$$\overset{SI^*}{R^i_{jk}}=\frac{\overset{SI^*}{\delta}\overset{SI^*}{N^i_j}}{\delta x^k}-\frac{\overset{SI^*}{\delta}\overset{SI^*}{N^i_k}}{\delta x^j}=\overset{*}{R^i_{jk}}-\frac{1}{4}\left(\frac{\overset{*}{\delta}}{\delta x^k}\frac{\partial F^i}{\partial y^j}-\frac{\overset{*}{\delta}}{\delta x^j}\frac{\partial F^i}{\partial y^k}\right)+\frac{1}{4}\left(\frac{\partial F^j}{\partial y^k}\frac{\partial \overset{*}{N^i_j}}{\partial y^j}-\frac{\partial F^k}{\partial y^j}\frac{\partial \overset{*}{N^i_k}}{\partial y^k}\right).$$

Now we determine a canonical $\overset{SI^*}{N}$ - linear connection $SI^*\Gamma\left(\overset{SI^*}{N}\right)=\left(\overset{SI^*}{F^i_{jk}},\overset{SI^*}{C^i_{jk}}\right)$, metric with respect to g_{ij}.

We denote $\overset{SI^*}{\nabla^H}$ and $\overset{SI^*}{\nabla^V}$ the h- and v- covariant derivative with respect to $SI^*\Gamma\left(\overset{SI^*}{N}\right)$:

$$\overset{SI^*}{\nabla^H_k}g_{ij}=\frac{\overset{SI^*}{\delta}g_{ij}}{\delta x^k}-\overset{SI^*}{F^s_{ik}}g_{sj}-\overset{SI^*}{F^s_{jk}}g_{si}$$
$$\overset{SI^*}{\nabla^V_k}g_{ij}=\frac{\partial g_{ij}}{\partial y^k}-\overset{SI^*}{C^s_{ik}}g_{sj}-\overset{SI^*}{C^s_{jk}}g_{si}$$

The tensor g_{ij} is covariant with respect to $SI^*\Gamma\left(\overset{SI^*}{N}\right)$if and only if $\overset{SI^*}{\nabla^H_k}g_{ij}=0$ and $\overset{SI^*}{\nabla^V_k}g_{ij}=0$ and we say that $SI^*\Gamma\left(\overset{SI^*}{N}\right)$ is a metrical $\overset{SI^*}{N}$-linear connection of the mechanical system $\sum_{SI^*}$ .The h- and v- torsions of $SI^*\Gamma\left(\overset{SI^*}{N}\right)$ are

$$\overset{SI^*}{T^i_{jk}}=\overset{SI^*}{F^i_{jk}}-\overset{SI^*}{F^i_{kj}}$$
and
$$\overset{SI^*}{S^i_{jk}}=\overset{SI^*}{C^i_{jk}}-\overset{SI^*}{C^i_{kj}}.$$

Theorem 4.2. *Let* $\sum_{SI^*} = \left(M, (\alpha+\beta)^2, \overset{SI^*}{N}, F_e\right)$ *be a special Ingarden mechanical system and* $\overset{SI^*}{N}$ *the canonical nonlinear connection of* $\sum_{SI^*}$. *There exists an unique d-connection* $SI^*\Gamma\left(\overset{SI^*}{N}\right)$ *determined by the following axioms:*

i) $\overset{SI^*}{\nabla}{}^H_k g_{ij} = 0$; $\overset{SI^*}{\nabla}{}^V_k g_{ij} = 0$;
ii) $\overset{SI^*}{T}{}^i_{jk} = 0$; $\overset{SI^*}{S}{}^i_{jk} = 0$.
We call this connection the canonical metrical d-connection of $\sum_{SI^*}$.

Theorem 4.3. *The local coefficients of the canonical metrical d-connection of* $\sum_{SI^*}$ *are*

$$\left\{\begin{array}{l} \overset{SI^*}{F}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\overset{SI^*}{\delta}\, g_{sj}}{\delta x^k} + \frac{\overset{SI^*}{\delta}\, g_{sk}}{\delta x^j} - \frac{\overset{SI^*}{\delta}\, g_{jk}}{\delta x^s}\right) \\ \overset{SI^*}{C}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right) \end{array}\right. . \tag{4.10}$$

In order to calculate these coefficients we take account of (3.5) and we get

$$\frac{\overset{*}{\delta} g_{sj}}{\delta x^k} = \overset{*}{\nabla}{}^H_k g_{sj} + F^i_{sk} g_{ij} + F^i_{jk} g_{si}. \tag{4.11}$$

and from (4.6) we obtain

$$\frac{\overset{SI^*}{\delta}\, g_{sj}}{\delta x^k} = \overset{*}{\nabla}{}^H_k g_{sj} + F^i_{sk} g_{ij} + F^i_{jk} g_{si} + \frac{1}{4}\frac{\partial F^j}{\partial y^k}\frac{\partial}{\partial y^j}.$$

Now we can state

Theorem 4.4. *The canonical metrical d-connection of* $\sum_{SI^*}$ *has the coefficients*

$$\left\{\begin{array}{l} \overset{SI^*}{F}{}^i_{jk} = F^i_{jk} + \frac{1}{2} g^{is}\left(\overset{*}{\nabla}{}^H_k g_{ij} + \overset{*}{\nabla}{}^H_j g_{sk} - \overset{*}{\nabla}{}^H_s g_{jk}\right) + \frac{1}{8} g^{is}\left(\frac{\partial F^h}{\partial y^k}\frac{\partial g_{sj}}{\partial y^h} + \frac{\partial F^h}{\partial y^j}\frac{\partial g_{sk}}{\partial y^h} - \frac{\partial F^h}{\partial y^s}\frac{\partial g_{jk}}{\partial y^h}\right) \\ \overset{SI^*}{C}{}^i_{jk} = C^i_{jk}. \end{array}\right. \tag{4.12}$$

Using the geometrical theory of the special Ingarden mechanical systems we can write the generalized Maxwell equations for the electromagnetic fields of $\sum_{SI^*}$.

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