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ON STRONG r-HELIX SUBMANIFOLDS AND SPECIAL CURVES

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Abstract. In this paper, we investigate special curves on a strong r-helix submanifold in Euclidean n-space E^n. Also, we give the important relations between strong r-helix submanifolds and the special curves such as line of curvature, geodesic and slant helix.

1. Introduction

In differential geometry of manifolds, an helix submanifold of $\mathbb{R}^n$ with respect to a fixed direction d in $\mathbb{R}^n$ is defined by the property that tangent planes make a constant angle with the fixed direction d (helix direction) in [5]. Di Scala and Ruiz-Hernández have introduced the concept of these manifolds in [5]. Besides, the concept of strong r-helix submanifold of $\mathbb{R}^n$ was introduced in [4]. Let $M \subset \mathbb{R}^n$ be a submanifold and let $H(M)$ be the set of helix directions of M. We say that M is a strong r-helix if the set $H(M)$ is a linear subspace of $\mathbb{R}^n$ of dimension greater or equal to r in [4].

Recently, M. Ghomi worked out the shadow problem given by H.Wente. And, He mentioned the shadow boundary in [8]. Ruiz-Hernández investigated that shadow boundaries are related to helix submanifolds in [12].

Helix hypersurfaces has been worked in nonflat ambient spaces in [6,7]. Cermelli and Di Scala have also studied helix hypersurfaces in liquid cristals in [3].

The plan of this paper is as follows. In section 2, we mention some basic facts in the general theory of strong r-helix, manifolds and curves. And, in section 3, we give the important relations between strong r-helix submanifolds and some special curves such as line of curvature, geodesic and slant helix.

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2. Preliminaries

Definition 2.1. *Let $M \subset \mathbb{R}^n$ be a submanifold of a euclidean space. A unit vector $d \in \mathbb{R}^n$ is called a helix direction of M if the angle between d and any tangent space T_pM is constant. Let $H(M)$ be the set of helix directions of M. We say that M is a strong r-helix if $H(M)$ is a r-dimensional linear subspace of $\mathbb{R}^n$.* [4]

Definition 2.2. *A submanifold $M \subset \mathbb{R}^n$ is a strong r-helix if the set $H(M)$ is a linear subspace of $\mathbb{R}^n$ of dimension greater or equal to r.* [4]

Definition 2.3. *A unit speed curve $\alpha : I \to E^n$ is called a slant helix if its unit principal normal V_2 makes a constant angle with a fixed direciton U.* [1]

Definition 2.4. *Let the $(n-k)$-manifold M be submanifold of the Riemannian manifold $\overline{M} = E^n$ and let $\overline{D}$ be the Riemannian connexion on $\overline{M} = E^n$. For C^∞ fields X and Y with domain A on M (and tangent to M), define D_XY and $V(X,Y)$ on A by decomposing $\overline{D}_XY$ into unique tangential and normal components, respectively; thus,*

$$\overline{D}_XY = D_XY + V(X,Y).$$

Then, D is the Riemannian connexion on M and V is a symmetric vector-valued 2-covariant C^∞ tensor called the second fundamental tensor. The above composition equation is called the Gauss equation. [9]

Definition 2.5. *Let the $(n-k)$-manifold M be submanifold of the Riemannian manifold $\overline{M} = E^n$, let $\overline{D}$ be the Riemannian connexion on $\overline{M} = E^n$ and let D be the Riemannian connexion on M. Then, the formula of Weingarten*

$$\overline{D}_XN = -A_N(X) + D^{\perp}_XN$$

for every X and Y tangent to M and for every N normal to M. A_N is the shape operator associated to N also known as the Weingarten operator corresponding to N and $D^{\perp}$ is the induced connexion in the normal bundle of M ($A_N(X)$ is also the tangent component of $-\overline{D}_XN$ and will be denoted by $A_N(X)$ =tang($-\overline{D}_XN$)). Specially, if M is a hypersurface in E^n, we have $\langle V(X,Y),N\rangle = \langle A_N(X),Y\rangle$ for all X, Y tangent to M. So,

$$V(X,Y) = \langle V(X,Y),N\rangle N = \langle A_N(X),Y\rangle N$$

and we obtain

$$\overline{D}_XY = D_XY + \langle A_N(X),Y\rangle N\ .$$

For this definition 2.5, note that the shape operator A_N is defined by the map $A_N : \varkappa(M) \to \varkappa(M)$, where $\varkappa(M)$ is the space of tangent vector fields on M and if $p \in M$, the shape operator A_N is defined by the map $A_p : T_p(M) \to T_p(M)$.The eigenvalues of A_p are called the principal curvatures (denoted by λ_i) and the eigenvectors of A_p are called the principal vectors. [10][11]

Definition 2.6. *If α is a (unit speed) curve in M with C^∞ unit tangent T, then $V(T,T)$ is called normal curvature vector field of α and $k_T = \|V(T,T)\|$ is called the normal curvature of α.* [9]

3. Main Theorems

Theorem 3.1. *Let M be a strong r-helix hypersurface and $H(M) \subset E^n$ be the set of helix directions of M. If $\alpha : I \subset \mathbb{R} \to M$ is a (unit speed) line of curvature (not a line) on M, then $d_j \notin Sp\{N, T\}$ along the curve α for all $d_j \in H(M)$, where T is the tangent vector field of α and N is a unit normal vector field of M.*

Proof. We assume that $d_j \in Sp\{N, T\}$ along the curve α for any $d_j \in H(M)$. Then, along the curve α, since M is a strong r-helix hypersurface, we can decompose d_j in tangent and normal components:

$$d_j = \cos(\theta_j)N + \sin(\theta_j)T \tag{1}$$

where θ_j is constant. From (1),by taking derivatives on both sides along the curve α, we get:

$$0 = \cos(\theta_j)N' + \sin(\theta_j)T' \tag{2}$$

Moreover, since α is a line of curvature on M,

$$N' = \lambda\alpha' \tag{3}$$

along the curve α. By using the equations (2) and (3), we deduce that the system $\{\alpha', T'\}$ is linear dependent. But, the system $\{\alpha', T'\}$ is never linear dependent. This is a contradiction. This completes the proof.

Theorem 3.2. *Let M be a submanifold with $(n-k)$ dimension in E^n. Let $\overline{D}$ be Riemannian connexion (standart covariant derivative) on E^n and D be Riemannian connexion on M. Let us assume that $M \subset E^n$ be a strong r-helix submanifold and $H(M) \subset E^n$ be the space of the helix directions of M. If $\alpha : I \subset \mathbb{R} \to M$ is a (unit speed) geodesic curve on M and if $\langle V_2, \xi_j \rangle$ is a constant function along the curve α, then α is a slant helix in E^n, where V_2 is the unit principal normal of α and ξ_j is the normal component of a direction $d_j \in H(M)$.*

Proof. Let T be the unit tangent vector field of α. Then, from the formula Gauss in Definition (2.4),

$$\overline{D}_T T = D_T T + V(T, T) \tag{4}$$

According to the Theorem, since α is a geodesic curve on M,

$$D_T T = 0 \tag{5}$$

So, by using (4), (5) and Frenet formulas, we have:

$$\overline{D}_T T = k_1 V_2 = V(T, T)$$

That is, the vector field $V_2 \in \vartheta(M)$ along the curve α, where $\vartheta(M)$ is the normal space of M. On the other hand, since M is a strong r-helix submanifold, we can decompose any $d_j \in H(M)$ in its tangent and normal components:

$$d_j = \cos(\theta_j)\xi_j + \sin(\theta_j)T_j \tag{6}$$

where θ_j is constant. Moreover, according to the Theorem, $\langle V_2, \xi_j \rangle$ is a constant function along the curve α for the normal component ξ_j of a direction $d_j \in H(M)$. Hence, doing the scalar product with V_2 in each part of the equation (6), we obtain:

$$\langle d_j, V_2 \rangle = \cos(\theta_j)\left\langle V_2, \xi_j \right\rangle + \sin(\theta_j)\left\langle V_2, T_j \right\rangle \tag{7}$$

Since $\cos(\theta_j)\left\langle V_2, \xi_j \right\rangle$ =constant and $\langle V_2, T_j \rangle = 0$ ($V_2 \in \vartheta(M)$) along the curve α, from (7) we have:

$$\langle d_j, V_2 \rangle = \text{constant}.$$

along the curve α. Consequently, α is a slant helix in E^n.

Theorem 3.3. *Let M be a submanifold with $(n-k)$ dimension in E^n. Let $\overline{D}$ be Riemannian connexion (standart covariant derivative) on E^n and D be Riemannian connexion on M. Let us assume that $M \subset E^n$ be a strong r-helix submanifold and $H(M) \subset E^n$ be the space of the helix directions of M. If $\alpha : I \subset \mathbb{R} \to M$ is a (unit speed) curve on M with the normal curvature function $k_T = 0$ and if $\langle V_2, T_j \rangle$ is a constant function along the curve α, then α is a slant helix in E^n, where V_2 is the unit principal normal of α and T_j is the tangent component of a direction $d_j \in H(M)$.*

Proof. Let T be the unit tangent vector field of α. Then, from the formula Gauss in Definition (2.4),

$$\overline{D}_T T = D_T T + V(T, T) \tag{8}$$

According to the Theorem, since the normal curvature $k_T = 0$,

$$V(T, T) = 0 \tag{9}$$

So, by using (8),(9) and Frenet formulas, we have:

$$\overline{D}_T T = k_1 V_2 = D_T T.$$

That is, the vector field $V_2 \in T_{\alpha(t)}M$, where $T_{\alpha(t)}M$ is the tangent space of M. On the other hand, since M is a strong r-helix submanifold, we can decompose any $d_j \in H(M)$ in its tangent and normal components:

$$d_j = \cos(\theta_j)\xi_j + \sin(\theta_j)T_j \tag{10}$$

where θ_j is constant. Moreover, according to the Theorem, $\langle V_2, T_j \rangle$ is a constant function along the curve α for the tangent component T_j of a direction $d_j \in H(M)$. Hence, doing the scalar product with V_2 in each part of the equation (10), we obtain:

$$\langle d_j, V_2 \rangle = \cos(\theta_j)\left\langle V_2, \xi_j \right\rangle + \sin(\theta_j)\left\langle V_2, T_j \right\rangle \tag{11}$$

Since $\sin(\theta_j)\langle V_2, T_j \rangle$ =constant and $\left\langle V_2, \xi_j \right\rangle = 0$ ($V_2 \in T_{\alpha(t)}M$) along the curve α, from (11) we have:

$$\langle d_j, V_2 \rangle = \text{constant}.$$

along the curve α. Consequently, α is a slant helix in E^n.

Definition 3.1. *Given an Euclidean submanifold of arbitrary codimension $M \subset \mathbb{R}^n$. A curve α in M is called a line of curvature if its tangent T is a principal vector at each of its points. In other words, when T (the tangent of α) is a principal vector at each of its points, for an arbitrary normal vector field $N \in \vartheta(M)$, the shape operator A_N associated to N says $A_N(T)$ =tang$(-\overline{D}_T N) = \lambda_j T$ along the curve α, where λ_j is a principal* *curvature and $\overline{D}$ be the Riemannian connexion(standart covariant derivative) on $\mathbb{R}^n$.* [2]

Theorem 3.4. *Let M be a submanifold with $(n-k)$ dimension in E^n and let $\overline{D}$ be Riemannian connexion (standart covariant derivative) on E^n. Let us assume that $M \subset E^n$ be a strong r-helix submanifold and $H(M) \subset E^n$ be the space of the helix directions of M. If $\alpha : I \to M$ is a line of curvature with respect to the normal component $N_j \in \vartheta(M)$ of a direction $d_j \in H(M)$ and if $N_j' \in \varkappa(M)$ along the curve α, then $d_j \in Sp\{T\}^{\perp}$ along the curve α, where T is the unit tangent vector field of α.*

Proof. We assume that $\alpha : I \to M$ is a line of curvature with respect to the normal component $N_j \in \vartheta(M)$ of a direction $d_j \in H(M)$. Since M is a strong r-helix submanifold, we can decompose $d_j \in H(M)$ in its tangent and normal components:

$$d_j = \cos(\theta_j)N_j + \sin(\theta_j)T_j$$

where θ_j is constant. So, $\langle N_j, d_j \rangle$ =constant and by taking derivatives on both sides along the curve α, we get $\left\langle N_j', d_j \right\rangle = 0$. On the other hand, since $\alpha : I \to M$ is a line of curvature with respect to the $N_j \in \vartheta(M)$,

$$A_{N_j}(T)=\text{tang}(-\overline{D}_T N_j) = \text{tang}(-N_j') = \lambda_j T$$

along the curve α. According to this Theorem, $N_j' \in \varkappa(M)$ along the curve α. Hence,

$$\text{tang}(-N_j') = -N_j' = \lambda_j T \tag{12}$$

Therefore, by using the equalities $\left\langle N_j', d_j \right\rangle = 0$ and (12), we obtain:

$$\langle T, d_j \rangle = 0$$

along the curve α. This completes the proof.

Theorem 3.5. *Let M be a submanifold with $(n-k)$ dimension in E^n and let $\overline{D}$ be Riemannian connexion (standart covariant derivative) on E^n. Let us assume that $M \subset E^n$ be a strong r-helix submanifold and $H(M) \subset E^n$ be the space of the helix directions of M. If $\alpha : I \to M$ is a curve in M and if the system $\left\{T_j', T\right\}$ is linear dependent along the curve α, where T_j' is the derivative of the tangent component T_j of a direction $d_j \in H(M)$ and T the tangent to the curve α, then α is a line of curvature in M.*

Proof. Since M is a strong r-helix submanifold, we can decompose $d_j \in H(M)$ in its tangent and normal components:

$$d_j = \cos(\theta_j)N_j + \sin(\theta_j)T_j \tag{13}$$

where θ_j is constant. If we take derivative in each part of the equation (13) along the curve α, we obtain:

$$0 = \cos(\theta_j)N_j' + \sin(\theta_j)T_j' \tag{14}$$

From (14), we can write

$$N_j' = -\tan(\theta_j)T_j' \tag{15}$$

So, for the tangent component of $-N_j^{\prime}$, from (15) we can write:

$$A_{N_j}(T)=\text{tang}(-\overline{D}_T N_j) = \text{tang}(-N_j^{\prime}) = \text{tang}(\tan(\theta_j)T_j^{\prime}) \tag{16}$$

along the curve α. According to the hypothesis, the system $\left\{T_j^{\prime}, T\right\}$ is linear dependent along the curve α. Hence, we get $T_j^{\prime} = \lambda_j T$. And, by using the equation (16), we have:

$$A_{N_j}(T) = \text{tang}(\tan(\theta_j)T_j^{\prime}) = \text{tang}(\tan(\theta_j)\lambda_j T)$$

and

$$A_{N_j}(T) = \text{tang}(\tan(\theta_j)\lambda_j T) \tag{17}$$

Moreover, since $T \in \varkappa(M)$, $\text{tang}(\tan(\theta_j)\lambda T) = (\tan(\theta_j)\lambda_j)T = k_j T$. Therefore, from (17), we have:

$$A_{N_j}(T) = k_j T.$$

It follows that α is a line of curvature in M for $N_j \in \vartheta(M)$. This completes the proof.

References

[1] Ali, A.T., Turgut, M., *Some characterizations of slant helices in the euclidean space E^n*, Hacettepe Journal of Mathematics and Statistics, **39(3)(2010)**, 327-336.

[2] Cartan, E., *Geometry of Riemannian spaces*, Brookline Mass., Math. Sci. Press, 1983.

[3] Cermelli, P., Di Scala, A.J., *Constant angle surfaces in liquid crystals*, Philos. Mag., **87(2007),** 1871-1888.

[4] Di Scala, A.J., *Weak helix submanifolds of Euclidean spaces*, Abh. Math. Semin. Univ. Hambg., **79(2009)**, 37-46.

[5] Di Scala, A.J., Ruiz-Hernández, G., *Helix submanifolds of euclidean spaces*, Monatsh Math, **157(2009),** 205-215.

[6] Dillen, F., Fastenakels, J., Van der Verken, J., Vrancken, L., *Constant angle surfaces in $S^2 \times IR$*, Monatsh. Math., **152(2007),** 89-96.

[7] Dillen, F., Munteanu, M.I., *Constant angle surfaces in $H^2 \times IR$*, Bull. Braz. Math. Soc., **40(1)(2009),** 85-97.

[8] Ghomi, M., *Shadows and convexity of surfaces*, Ann. Math., **155(1)(2002)**, 281-293.

[9] Hicks, N.J., *Notes on differential geometry*, Van Nostrand Reinhold Company, London, 1974.

[10] Lopez, R., Munteanu, M.I., *Constant angle surfaces in Minkowski space*, Bull. Belg. Math. Soc.-Simon Stevin, **18(2)(2011)**, 271-286.

[11] Munteanu, M.I., *From golden spirals to constant slope surfaces*, Journal of Mathematical Physics, **51(7)(2010)**, 1-9.

[12] Ruiz-Hernández, G., *Helix, shadow boundary and minimal submanifolds*, Illinois J.Math., **52(4)(2008)**, 1385-1397.

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